ANNALES
POLONICI MATHEMATICI
XVIII (1966)

Indeterminate forms for multi-place functions

by A. I. FINE (Urbana) and S. KASS (Chicago)

§ 1. Introduction. A well-known theorem of Bernoulli, commonly called "l'Hospital's Rule" (cf. [1]), states that if a pair of differentiable 1-place functions f, g have a common zero or a common infinity at a point A, which is not a limit point of zeros of g', then

$$\lim_{A}\frac{f}{g}=L \quad \text{whenever} \quad \lim_{A}\frac{f'}{g'}=L.$$

In this article we extend this theorem to higher place functions by proving that if a pair of differentiable n-place functions f, g have a common zero or a common infinity at a point A, which is not a limit point of zeros of g_a, then

$$\lim_{A}\frac{f}{g}=L \quad \text{whenever} \quad \lim_{A}\frac{f_a}{g_a}=L,$$

provided that, in the case where $|g|\to\infty$, f and g are *externally bounded.* (Terminology and notation will be explained in § 2.) Just as in the classical case the theorem extends to the various "infinite cases" of L and A.

For simplicity results are stated for 2-place functions; however each result holds for n-place functions: simply replace "2" by "n" in each proof and read the summation signs accordingly.

§ 2. Terminology. The symbol $f(P)$ denotes the value of a function f at a point P: (p_1, p_2). Subscript notation will be used for partial derivatives. In particular,

$$f_a(P)=(\cos a)f_1(P)+(\sin a)f_2(P)=\frac{\sum(p_i-a_i)f_i(P)}{[\sum(p_i-a_i)^2]^{1/2}},$$

will be called the *directional derivative of f with respect to A.* Here A: (a_1, a_2) is a fixed point, P: (p_1, p_2) a point variable, and a the angle between the positively directed x-axis and the directed line determined by segment (AP). The symbol (AP) ($[AP]$) will be used to denote the open (closed) directed segment from A to P.

It is important for what follows to observe that if X: (x_1, x_2) and Y: (y_1, y_2) are any pair of distinct points on (AP), then $\cos\alpha = (x_1 - y_1)/\Delta$, and $\sin\alpha = (x_2 - y_2)/\Delta$, where $\Delta = \left[\sum (x_i - y_i)^2\right]^{1/2}$. Thus for functions f and g

$$(*) \qquad \frac{f_\alpha(P)}{g_\alpha(P)} = \frac{\sum (p_i - a_i) f_i(P)}{\sum (p_i - a_i) g_i(P)} = \frac{\sum (x_i - y_i) f_i(P)}{\sum (x_i - y_i) g_i(P)},$$

provided that these quotients are defined.

A point set $S \subseteq E^2$ is *starlike* with respect to point A if for each $P \in S$, $(AP) \subseteq S$.

A *neighborhood* $N(A, \delta)$ in S is the intersection in E^2 of S with the open sphere of center A and radius δ.

Relative to a fixed point A, we shall call a sequence of points $\{Q_i\}$ *external* to a sequence of points $\{Q_i'\}$ if for all but finitely many i, $Q_i' \in (Q_i A)$. We shall say that a function f is *externally bounded* with respect to A provided that the following condition holds for each neighborhood $N(A, \delta)$ of A: corresponding to each sequence $\{Q_i'\} \subseteq N(A, \delta)$ which converges to A, there exists a sequence $\{Q_i\} \subseteq N(A, \delta)$, external to $\{Q_i'\}$, on which f is bounded.

§ 3. Main results. We require the following extension of the Cauchy law of the mean to 2-place functions.

LEMMA 1. *Let f and g be 2-place functions defined on $S \subseteq E^2$. Suppose that S contains a line segment L directed from X: (x_1, x_2) to Y: (y_1, y_2) with α the angle between L and the positive x-axis.*

If both f and g are continuous on the closed segment and differentiable on the open segment, then there is some point $P \in (XY)$ such that

$$[f(X) - f(Y)] g_\alpha(P) = [g(X) - g(Y)] f_\alpha(P).$$

Proof. Form the function

$$h(T) = \frac{\begin{vmatrix} f(X) & g(X) & 1 \\ f(Y) & g(Y) & 1 \\ f(T) & g(T) & 1 \end{vmatrix}}{\left[\sum (x_i - y_i)^2\right]^{1/2}}$$

and apply the law of the mean for 2-place functions at X and Y.

THEOREM 1. *Let A: $(a_1, a_2) \in E^2$ and let f and g be functions whose domains include a set $S \subseteq E^2$ which is starlike with respect to A. Suppose that on S the functions are differentiable and that $g_\alpha(X)$, the directional derivative of g with respect to A, is never zero. With the understanding that all limits are taken from within S at A, there are two cases:*

(i) *$f(A) = g(A) = 0$ or*

(ii) *$|g| \to \infty$ and both f and g are externally bounded with respect to A. In either case we have that if*

(iii) $\lim\limits_{A} \dfrac{f_\alpha(X)}{g_\alpha(X)} = L \quad \left(i.e.\ \lim\limits_{A} \dfrac{\sum(x_i - a_i) f_i(x_1, x_2)}{\sum(x_i - a_i) g_i(x_1, x_2)} = L\right)$,

then

(iv) $\lim\limits_{A} \dfrac{f(X)}{g(X)} = L$.

Proof. We first note that if only $\lim\limits_{A} f = \lim\limits_{A} g = 0$ is assumed, then the hypotheses of case (i) are fulfilled if we define $f(A) = g(A) = 0$.

Case (i). Let $\varepsilon > 0$ be given. Then there exists a neighborhood $N(A, \delta)$ such that for every point U: (u_1, u_2) in $N(A, \delta)$

$$L - \varepsilon < \frac{\sum(u_i - a_i) f_i(U)}{\sum(u_i - a_i) g_i(U)} < L + \varepsilon . \tag{1}$$

Let X be any point $(\neq A)$ in $N(A, \delta)$ and consider the quotient

$$\frac{f(X) - f(A)}{g(X) - g(A)} \equiv \frac{f(X)}{g(X)} .$$

By the ordinary law of the mean, $g(X) \neq 0$ because $g_\alpha(U) \neq 0$ on $N(A, \delta)$. $N(A, \delta)$ is starlike with respect to A; therefore $[AX] \subseteq N(A, \delta)$. Hence, by the lemma there exists a point $P \in (AX)$ for which

$$\frac{f(X)}{g(X)} = \frac{\sum(x_i - a_i) f_i(P)}{\sum(x_i - a_i) g_i(P)} . \tag{2}$$

Now (1) certainly holds for $U = P$; therefore

$$L - \varepsilon < \frac{\sum(p_i - a_i) f_i(P)}{\sum(p_i - a_i) g_i(P)} < L + \varepsilon .$$

By (*) and (2), however

$$\frac{\sum(p_i - a_i) f_i(P)}{\sum(p_i - a_i) g_i(P)} = \frac{\sum(x_i - a_i) f_i(P)}{\sum(x_i - a_i) g_i(P)} = \frac{f(X)}{g(X)} .$$

Thus $L - \varepsilon < f(X)/g(X) < L + \varepsilon$ for all $X \in N(A, \delta)$, whence $\lim\limits_{A} \big(f(X)/g(X)\big) = L$.

Case (ii). $|g| \to \infty$. Let X be any point $(\neq A)$ in $N(A, \delta)$ and Y any point in (AX). We repeat the argument in case (i), with A replaced by Y, and find that

$$L - \varepsilon < \frac{f(X) - f(Y)}{g(X) - g(Y)} < L + \varepsilon , \tag{3}$$

for every X, $Y \in N(A, \delta)$, where $Y \in (AX)$. As before, $g(X) - g(Y) \neq 0$. We can assume $g \neq 0$ on $N(A, \delta)$ and so may divide in (3) to get

$$L - \varepsilon < \frac{\dfrac{f(Y)}{g(Y)} - \dfrac{f(X)}{g(Y)}}{1 - \dfrac{g(X)}{g(Y)}} < L + \varepsilon . \tag{4}$$

Now consider any sequence of points Q'_i of $N(A, \delta)$ which converges to A. By hypothesis there exists an external sequence of points Q_i of $N(A, \delta)$ on which both f and g are bounded. (4) is satisfied when X and Y are replaced by Q_i and Q'_i, respectively. Thus

$$L-\varepsilon<\frac{\dfrac{f(Q'_i)}{g(Q'_i)}-\dfrac{f(Q_i)}{g(Q'_i)}}{1-\dfrac{g(Q_i)}{g(Q'_i)}}<L+\varepsilon .$$

Since the $g(Q_i)$ are bounded, for large enough i, say in the neighborhood $N(A, \delta')$, we have that $1-\dfrac{g(Q_i)}{g(Q'_i)}>0$ for all $Q'_i \in N(A, \delta')$. Thus

$$(L-\varepsilon)\left[1-\frac{g(Q_i)}{g(Q'_i)}\right]+\frac{f(Q_i)}{g(Q'_i)}<\frac{f(Q'_i)}{g(Q'_i)}<(L+\varepsilon)\left[1-\frac{g(Q_i)}{g(Q'_i)}\right]+\frac{f(Q_i)}{g(Q'_i)}$$

for the $Q'_i, Q_i \in N(A, \delta^*)$, where $\delta^*=\min(\delta, \delta')$. If $i\to\infty$, then $1-\dfrac{g(Q_i)}{g(Q'_i)}\to 1$ and $\dfrac{f(Q_i)}{g(Q'_i)}\to 0$, from which it follows that

$$L-2\varepsilon<\frac{f(Q'_i)}{g(Q'_i)}<L+2\varepsilon$$

for all Q'_i in some subneighborhood of $N(A, \delta^*)$. Since the sequence Q'_i was chosen arbitrarily, $\lim\limits_{A}(f/g)=L$.

Corollary 1. *In the theorem, L may be replaced by either of the symbols ∞, $-\infty$.*

Proof. We consider only ∞. If $\lim\limits_{A}(f_a/g_a)=\infty$, then for given $M>0$ there exists $N(A, \delta)$ such that for all points P: $(p_1, p_2) \in N(A, \delta)$ we have

$$\frac{\sum(p_i-a_i)f_i(P)}{\sum(p_i-a_i)g_i(P)}>M .$$

As before, this gives

$$\frac{f(X)-f(Y)}{g(X)-g(Y)}>M$$

for all points X, $Y \in N(A, \delta)$ such that $Y \in [AX)$ in case (i) and $Y \in (AX)$ in case (ii).

Setting $Y=A$ in case (i), we obtain immediately that $f(X)/g(X)>M$ for all X in $N(A, \delta)$.

In case (ii), for any sequence $\{Q'_i\}$ converging to A we have for sufficiently large i

$$\frac{f(Q'_i)}{g(Q'_i)}>\left(1-\frac{g(Q_i)}{g(Q'_i)}\right)M+\frac{f(Q_i)}{g(Q'_i)}$$

where $\{Q_i\}$ is a sequence external to $\{Q'_i\}$ on which f and g are bounded. It follows that as $i \to \infty$, $\dfrac{f(Q'_i)}{g(Q'_i)} \to \infty$.

COROLLARY 2. *If A has coordinates (a_1, a_2) where either a_1 or a_2 is one of the symbols ∞, $-\infty$ then sufficient conditions that $f(X)/g(X) \to L$ as $X \to A$ are provided by using Theorem 1 to show that $f(X)/g(X) \to L$ as $X' \to A'$, where X' and A' are defined as follows. With X: (x_1, x_2), let X': (x'_1, x'_2) and A': (a'_1, a'_2) be given by having, for each i, $x'_i = x_i$ and $a'_i = a_i$ iff a_i is finite, while $x'_i = 1/x_i$ and $a'_i = 0$ iff a_i is either ∞ or $-\infty$.*

Proof. It suffices to show that for a 2-place function F, if $\lim\limits_{X' \to A'} F(X) = L$, then $\lim\limits_{X \to A} F(X) = L$. There are $3^2 - 1 = 8$ possibilities for A. Since the argument is similar for each, we shall just illustrate it for the case A: $(a, -\infty)$. Then, for given $\varepsilon > 0$, we have $\delta_1, \delta_2 > 0$ such that $|F(x_1, x_2) - L| < \varepsilon$ provided $0 < |x'_1 - a| < \delta_1$ and $0 < |x'_2| < \delta_2$. It follows, since $x'_1 = x_1$ and $x'_2 = 1/x_2$, that if $0 < |x_1 - a| < \delta$, and $x_2 < -1/\delta$ then $|F(x_1, x_2) - L| < \varepsilon$. Hence

$$\lim_{\substack{x_1 \to a \\ x_2 \to -\infty}} F(x_1, x_2) = L.$$

§ 4. **Remarks.** (i) It can be shown that if f/g has no unique value as $X \to A$ from within S, then $L_1 \subseteq L_2$, where L_1 is the set of all limit points of f/g and L_2 is the set of all limit points of f_a/g_a.

(ii) The operator D defined by $Df(x_1, x_2) = x_1 f_1(x_1, x_2) + x_2 f_2(x_1, x_2)$ is a derivation. When f is homogeneous of degree k then $Df = kf$. Thus if f and g are homogeneous functions of distinct (non-zero) degrees that meet the conditions of Theorem 1 with A at the origin, then either $\lim(f/g) = 0$ or else the limit does not exist.

(iii) The theorem takes on a particularly useful and simple form when reformulated in terms of polar coordinates $\{r, \theta_1, \ldots, \theta_{n-1}\}$ with point A set at the origin. Then $f_a = rF_r$, $g_a = rG_r$ and $f_a/g_a = F_r/G_r$. Here F and G are the functions in polar coordinates corresponding to f and g respectively, F_r and G_r have their usual meanings with respect to the polar coordinate r, and limits are taken as $r \to 0$ *uniformly* in the remaining polar coordinates θ_i.

(iv) If one knows that the limit of f/g exists, one can readily find it by applying the familiar one-dimensional form of the theorem along an appropriate path of approach. Thus in practice the theorem is useful not so much as a device in computing $\lim\limits_{A}(f/g)$ but to guarantee the existence of the limit within a given set S, which is starlike with respect to A. In some cases, one may actually find the largest set S for which the limit of f/g exists, as well as L_1 of (i), which is always an interval of the real line.

We give an example. Consider the quotient $(xy^3+2y^2)/(x^4+y^2)$ at the origin. This seems to fall under case (i) of our theorem, but the computation there yields an even more complicated quotient. Divide numerator and denominator by y^4 to get f/g with $f(x,y) = x/y+2/y^2$ and $g(x,y) = x^4/y^4+1/y^2$. This quotient comes under case (ii) and one easily finds that $f_a/g_a = 2$. Nevertheless, the quotient has no limit at $(0,0)$, as one can verify by trying the paths $y = x$ and $y = x^2$. One might suspect, however, that the limit is 2 if taken from within some "nice" region and indeed a routine working out of the hypotheses of Theorem 1 produces such a region: it is E^2 with certain open wedges excluded; viz., all points (x,y) between $y = \varepsilon_1 x$, $y = -\varepsilon_2 x$, for $\varepsilon_i > 0$, arbitrarily small.

References

[1] Gustav Eneström, *Sur le part de Jean Bernoulli dans la publication de l'Analyse des infiniment petits*, Bibliotheca Mathematica N. S., 8 (1894), pp. 65-72.

UNIVERSITY OF ILLINOIS
ILLINOIS INSTITUTE OF TECHNOLOGY

Reçu par la Rédaction le 27. 1. 1965