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Strong maximum principle for non-linear parabolic differential-functional inequalities in arbitrary domains

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Abstract. A diagonal system of second order differential-functional inequalities of the type

$$(0.1)\qquad u_t^i(t,x) \leqslant f^i(t,x,u(t,x),u_x^i(t,x),u_{xx}^i(t,x),u(t,\cdot)) \qquad (i = 1,\ldots,m)$$

is considered, where $x = (x_1,\ldots,x_n)$, $u = (u^1,\ldots,u^m)$, $u_x^i = (u_{x_1}^i,\ldots,u_{x_n}^i)$, and u_{xx}^i is the matrix of second order derivatives with respect to x. The function $u(t,x)$ is defined in an arbitrary open set $D \subset (t,x_1,\ldots,x_n)$ (bounded or not) and by $u(t,\cdot) = (u^1(t,\cdot),\ldots,u^m(t,\cdot))$ is meant an element of the space of continuous functions

$$u(t,\cdot)\colon\ S_t \ni x \to u(t,x) \in R^m,$$

where $S_{\tilde t}$ is the intersection of D with the plane $t = \tilde t$. Three theorems are proved: Theorem 1 on strong inequalities of type (0.1) (the sign $\leqslant$ being replaced by $<$), Theorem 2 (corresponding to the maximum principle) and Theorem 3 (corresponding to the strong maximum principle) on inequalities (0.1). Results obtained are a generalization of those given in [2]. Some ideas of this paper are patterned on those introduced by P. Besala in [1] and by W. Walter in [3]. In the above papers differential inequalities with f^i independent functionally on $u(t,\cdot)$ were dealt with.

1. Notations, definitions and assumptions.

ASSUMPTIONS H. *Let $D \subset (t,x_1,\ldots,x_n)$ be an arbitrary open set, contained in the zone $0 < t < T \leqslant +\infty$, and let its projection on the t-axis be the interval $(0,T)$.*

By D_p is denoted the subset of those points, $(\tilde t,\tilde x)$ belonging to the intersection of $\bar D = D \cup \partial D$ with the zone $0 < t < T$, for which there is a lower half neighbourhood

$$t < \tilde t, \qquad \sum_j (x_j-\tilde x_j)^2 + (t-\tilde t)^2 < r^2,$$

contained in D. It is clear that every point of D belongs to D_p.

We denote by $S_{\tilde t}$ $(0 < \tilde t < T)$ the projection on the space $(x_1,\ldots,x_n)$ of the intersection of D_p with the plane $t = \tilde t$. It is obvious that $S_{\tilde t}$ is open in the space $(x_1,\ldots,x_n)$.

We denote finally by Σ that part of ∂D which is disjoint with D_p and is contained in the zone $0 < t < T$ and by S_0 the subset of ∂D contained in the plane $t = 0$.

A point $(\tilde{t}, \tilde{x})$ being fixed in D_p, $S^-(\tilde{t}, \tilde{x})$ will stand for the set of points $(t, x) \in D_p$ that can be joined with $(\tilde{t}, \tilde{x})$ by a polygonal line contained in D_p along which the t-coordinate is increasing (weakly) from (t, x) to $(\tilde{t}, \tilde{x})$.

Assumptions G. *Let Σ_i^* $(i = 1, \ldots, m)$ be a subset (possibly empty) of Σ and let $a^i(t, x) > 0$, $b^i(t, x) \geqslant 0$ $(i = 1, \ldots, m)$ be functions defined on Σ_i^*. For $(t, x) \in \Sigma_i^*$ be given a direction $l^i(t, x)$, so that l^i is orthogonal to the t-axis and some open segment, with one extremity at (t, x), of the straight half line from (t, x) in the direction l^i is contained in D_p.*

For a function $u^i(t, x)$ defined on $D_p \cup \Sigma_i^*$ and for a point $(\tilde{t}, \tilde{x}) \in \Sigma_i^*$ we put

$$\underline{D}_{l^i} u^i(\tilde{t}, \tilde{x}) = \liminf_{\tau \to 0+0} \frac{u^i(\tilde{t}, \tilde{x} + \tau \text{ vers } l^i(\tilde{t}, \tilde{x})) - u^i(\tilde{t}, \tilde{x})}{\tau}.$$

Let $C_m(S_t)$ denote the space of continuous functions $z = (z^1(\cdot), \ldots \ldots, z^m(\cdot))$ from S_t in R^m. For the subspace of those z which are bounded in S_t introduce the norm

$$\|z\| = \max_i \sup\{|z^i(x)|: x \in S_t\}.$$

In $C_m(S_t)$ the following order is defined: for $z = (z^1(\cdot), \ldots, z^m(\cdot)) \in C_m(S_t)$, $\tilde{z} = (\tilde{z}^1(\cdot), \ldots, \tilde{z}^m(\cdot)) \in C_m(S_t)$ the inequality $z \leqslant \tilde{z}$ $(z < \tilde{z})$ means that

$$z^j(x) \leqslant \tilde{z}^j(x) \quad (z^j(x) < \tilde{z}^j(x)) \quad (j = 1, \ldots, m).$$

Let $f^i(t, x, u, q, r, z)$ $(i = 1, \ldots, m)$, where $q = (q_1, \ldots, q_m)$ and $r = (r_{jk})$ is a $n \times n$ real symmetric matrix, be defined for $(t, x) \in D_p$, u, q, r arbitrary and $z \in \tilde{C}_m(S_t)$, $\tilde{C}_m(S_t)$ being a subspace of $C_m(S_t)$ containing bounded functions.

A function $u(t, x) = (u^1(t, x), \ldots, u^m(t, x))$ is called *Σ-regular* in D if u^i is continuous in $D_p \cup \Sigma_i^*$ and u_t^i, u_x^i, u_{xx}^i are continuous in D_p.

A Σ-regular function $u(t, x)$ in D is called *Σ-regular solution* of (0.1) in D if $u(t, \cdot) \in \tilde{C}_m(S_t)$ and if it satisfies (0.1) for $(t, x) \in D_p$.

According to the definition given by P. Besala [1], a Σ-regular function $u(t, x)$ in D being given, the function $f^i(t, x, u, q, r, z)$ is said to be *uniformly parabolic* with respect to $u(t, x)$ in a subset $E \subset D_p$ if there is a constant $\varkappa > 0$ (depending on E) such that for any two real symmetric matrices $r = (r_{jk})$, $\tilde{r} = (\tilde{r}_{jk})$ and for $(t, x) \in E$ we have

$$(1.1)\qquad r \leqslant \tilde{r}\ (^1)\ \Rightarrow f^i\big(t, x, u(t, x), u^i_x(t, x), \tilde{r}, u(t, \cdot)\big) - \\ - f^i\big(t, x, u(t, x), u^i_x(t, x), r, u(t, \cdot)\big) \geqslant \varkappa \sum_{j=1}^{n} (\tilde{r}_{jj} - r_{jj}).$$

If (1.1) is satisfied with $\varkappa = 0$, then f^i is called *parabolic* with respect to $u(t, x)$ in E.

Sets Σ_i^*, functions $a^i(t, x)$, $b^i(t, x)$ and directions $l^i(t, x)$ satisfying Assumptions G being given, a function $u(t, x) = \big(u^1(t, x), \ldots, u^m(t, x)\big)$, with u^i continuous in $D_p \cup \Sigma_i^*$, is said to *satisfy strong* (*weak*) *initial and boundary inequalities of type* IB if

(I) For any sequence $(t^\nu, x^\nu) \in D_p$ such that t^ν is strictly decreasing to 0, $(t^\nu, x^\nu) \to (0, x_0) \in S_0$, or $|x^\nu| \to +\infty$, we have

$$(1.2)\qquad \liminf_{\nu\to\infty} u^i(t^\nu, x^\nu) < 0\ \ (\leqslant 0) \qquad (i = 1, \ldots, m).$$

(B) For any $\tilde{t} \in (0, T)$ and for any sequence $(t^\nu, x^\nu) \in D_p$, such that t^ν is strictly decreasing to $\tilde{t}$, $(t^\nu, x^\nu) \to (\tilde{t}, \tilde{x}) \in \Sigma$, or $|x^\nu| \to +\infty$, we have

$$(1.3)\qquad b^i(\tilde{t}, \tilde{x}) u^i(\tilde{t}, \tilde{x}) - a^i(\tilde{t}, \tilde{x}) \underline{D}_{l^i} u^i(\tilde{t}, \tilde{x}) < 0\ \ (\leqslant 0)\ \text{ if } (\tilde{t}, \tilde{x}) \in \Sigma_i^*,$$

and

$$(1.4)\qquad \liminf_{\nu\to\infty} u^i(t^\nu, x^\nu) < 0\ \ (\leqslant 0), \quad \text{ if } (\tilde{t}, \tilde{x}) \in \Sigma \setminus \Sigma_i^*, \text{ or } |x^\nu| \to +\infty.$$

2. LEMMA. *Let* $w(t, x) = \big(w^1(t, x), \ldots, w^m(t, x)\big)$, *with* w^i *continuous in* $D_p \cup \Sigma_i^*$, *satisfy strong initial and boundary inequalities of type* IB. *Then, we have the following assertions:*

1) *There is a* $t^* \in (0, T)$, *so that*

$$(2.1)\qquad w(t, x) < 0$$

in the intersection of D_p *with the zone* $0 < t < t^*$.

2) *If for a* $\tilde{t} \in (0, T)$ *we have inequalities* (2.1) *for* $t = \tilde{t}$ *and every* $x \in S_{\tilde{t}}$, *then there is a* $t^* \in (\tilde{t}, T)$, *so that inequalities* (2.1) *hold in the intersection of* D_p *with the zone* $\tilde{t} \leqslant t < t^*$.

Proof. Suppose 1) is not true. Then, there would exist a sequence $(t^\nu, x^\nu) \in D_p$ and an index j such that t^ν is strictly decreasing to 0, $(t^\nu, x^\nu) \to$ $\to (0, x_0) \in S_0$, or $|x^\nu| \to +\infty$, and

$$(2.2)\qquad w^j(t^\nu, x^\nu) \geqslant 0.$$

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(1) This inequality means that $\sum_{j,k} (\tilde{r}_{jk} - r_{jk}) \lambda_j \lambda_k \geqslant 0$.

From (2.2) follows

$$\liminf_{\nu\to\infty} w^j(t^\nu, x^\nu) \geqslant 0, \tag{2.3}$$

in contradiction with the assumption that $w(t,x)$ satisfies the strong initial inequality (1.2).

Now, suppose that 2) does not hold true; then there would exist a sequence $(t^\nu, x^\nu)\in D_p$ and an index j so that t^ν is strictly decreasing to $\tilde{t}$, $(t^\nu, x^\nu)\to(\tilde{t}, \tilde{x})\in \Sigma\cup(\tilde{t}\times S_{\tilde{t}})$ or $|x^\nu|\to+\infty$, and (2.2) is satisfied. Since for $t=\tilde{t}$ and $x\in S_{\tilde{t}}$ we have (see the assumption of 2)) inequalities (2.1), it follows, by (2.2) and continuity of $w^j(t,x)$ in D_p, that $(\tilde{t}, \tilde{x})\notin(\tilde{t}\times S_{\tilde{t}})$.

If $(\tilde{t}, \tilde{x})\in\Sigma\setminus\Sigma_j^*$ or $|x^\nu|\to+\infty$, we obtain (2.3) (like in 1)) in contradiction with the assumption that $w(t,x)$ satisfies the strong boundary inequality (1.4). If $(\tilde{t}, \tilde{x})\in\Sigma_j^*$, then by (2.2) and continuity of $w^j(t,x)$ in Σ_j^*, we would have

$$w^j(\tilde{t}, \tilde{x}) \geqslant 0. \tag{2.4}$$

On the other hand, by (2.4) and by the inequality $w^j(\tilde{t}, x)<0$, satisfied for $x\in S_{\tilde{t}}$, we have

$$\underline{D}_{lj} w^j(\tilde{t}, \tilde{x}) \leqslant 0. \tag{2.5}$$

From (2.4) and (2.5), and a^j, b^j being non-negative, it follows that

$$b^j(\tilde{t}, \tilde{x})w^j(\tilde{t}, \tilde{x}) - a^j(\tilde{t}, \tilde{x})\underline{D}_{lj} w^j(\tilde{t}, \tilde{x}) \geqslant 0,$$

in contradiction with the strong boundary inequality (1.3).

3. THEOREM 1 (strong inequalities). *Assume that*

1° $f^i(t,x,u,q,r,z)$ $(i=1,\ldots,m)$ *are defined for* $(t,x)\in D_p$, *where* D *is an open set satisfying Assumptions* H, *for* u, q, r *arbitrary and* $z\in\tilde{C}_m(S_t)$; *the function* f^i *is increasing with respect to* $u^1,\ldots,u^{i-1}, u^{i+1},\ldots,u^m, z$.

2° $u(t,x)=\big(u^1(t,x),\ldots,u^m(t,x)\big)$ *is a* Σ*-regular solution in* D *of the system* (0.1) *and* $v(t,x)=\big(v^1(t,x),\ldots,v^m(t,x)\big)$ *is a* Σ*-regular solution in* D *of the system*

$$v_t^i(t,x) > f^i\big(t,x,v(t,x),v_x^i(t,x),v_{xx}^i(t,x),v(t,\cdot)\big) \quad (i=1,\ldots,m). \tag{3.1}$$

3° *Sets* Σ_i^*, *functions* $a^i(t,x)$, $b^i(t,x)$ *and directions* $l^i(t,x)$ *satisfying Assumptions* G *being given, the difference* $u(t,x)-v(t,x)$ *satisfies strong initial and boundary inequalities of type* IB.

4° $f^i(t,x,u,q,r,z)$ $(i=1,\ldots,m)$ *are parabolic with respect to* $u(t,x)$ *in* D_p.

Under these assumptions we have

$$u(t,x) < v(t,x) \quad \textit{for} \ (t,x)\in D_p. \tag{3.2}$$

Proof. By 3° and lemma, the set

$$\{t^* \in (0, T);\ u(t, x) < v(t, x) \text{ for } (t, x) \in D_p,\ 0 < t < t^*\}$$

is non-void. Let $\tilde{t}$ be its least upper bound or $+\infty$, if it is unbounded. The assertion of our theorem is obviously equivalent with the equality

$$\tilde{t} = T. \tag{3.3}$$

Now, suppose the contrary holds true, i.e.

$$\tilde{t} < T. \tag{3.4}$$

Then, by the continuity of $u(t, x) - v(t, x)$ in D_p we would have

$$u(t, x) \leqslant v(t, x), \qquad (t, x) \in D_p, \qquad 0 < t \leqslant \tilde{t}. \tag{3.5}$$

By lemma and 3° and by the definition of $\tilde{t}$, there is an index j and a point $\tilde{x} \in S_{\tilde{t}}$ so that

$$u^j(\tilde{t}, \tilde{x}) = v^j(\tilde{t}, \tilde{x}). \tag{3.6}$$

From (3.5) and (3.6) it follows that the function $u^j(\tilde{t}, x) - v^j(\tilde{t}, x)$ attains its maximum in $S_{\tilde{t}}$ for $x = \tilde{x}$. Therefore, since $S_{\tilde{t}}$ is open and the function is of class C^2 in $S_{\tilde{t}}$ (see 2°), we have

$$u^j_x(\tilde{t}, \tilde{x}) = v^j_x(\tilde{t}, \tilde{x}), \tag{3.7}$$

$$u^j_{xx}(\tilde{t}, \tilde{x}) \leqslant v^j_{xx}(\tilde{t}, \tilde{x}). \tag{3.8}$$

By 2°, 4°, (3.8), 1°, (3.5), (3.6), (3.7) and (3.1), we get successively

$$\begin{aligned} u^j_t(\tilde{t}, \tilde{x}) &\leqslant f^j\big(\tilde{t}, \tilde{x}, u(\tilde{t}, \tilde{x}), u^j_x(\tilde{t}, \tilde{x}), u^j_{xx}(\tilde{t}, \tilde{x}), u(\tilde{t}, \cdot)\big) \\ &\leqslant f^j\big(\tilde{t}, \tilde{x}, u(\tilde{t}, \tilde{x}), u^j_x(\tilde{t}, \tilde{x}), v^j_{xx}(\tilde{t}, \tilde{x}), u(\tilde{t}, \cdot)\big) \\ &\leqslant f^j\big(\tilde{t}, \tilde{x}, v(\tilde{t}, \tilde{x}), v^j_x(\tilde{t}, \tilde{x}), v^j_{xx}(\tilde{t}, \tilde{x}), v(\tilde{t}, \cdot)\big) \\ &< v^j_t(\tilde{t}, \tilde{x}). \end{aligned} \tag{3.9}$$

On the other hand, by (3.5) and (3.6), the function $u^j(t, \tilde{x}) - v^j(t, \tilde{x})$, defined for t in some interval $(\hat{t}, \tilde{t})$, attains its maximum at $t = \tilde{t}$. Hence we have

$$u^j_t(\tilde{t}, \tilde{x}) \geqslant v^j_t(\tilde{t}, \tilde{x}),$$

in contradiction with (3.9). This completes the proof of (3.3).

4. Theorem 2 (weak inequalities, maximum principle). *Under assumptions* 1° *and* 4° *of Theorem* 1 *suppose that*

5° $u(t, x)$ *is a* Σ-*regular solution in* D *of system* (0.1), *while* $v(t, x)$ *is a* Σ-*regular solution in* D *of the system*

$$v^i_t(t, x) \geqslant f^i\big(t, x, v(t, x), v^i_x(t, x), v^i_{xx}(t, x), v(t, \cdot)\big) \qquad (i = 1, \ldots, m). \tag{4.1}$$

6° *Sets Σ_i^*, functions $a^i(t, x)$, $b^i(t, x)$ and directions $l^i(t, x)$ satisfying Assumptions* G *being given, the difference $u(t, x) - v(t, x)$ satisfies weak initial and boundary inequalities of type* IB.

7° *For $\tilde{u} \leqslant u$ and $z - \tilde{z}$ bounded we have*

$$(4.2)\quad f^i(t, x, u, q, r, z) - f^i(t, x, \tilde{u}, q, r, \tilde{z}) \leqslant \sigma\big(t, \max_j(u^j - \tilde{u}^j, \|z - \tilde{z}\|)\big) \quad (i = 1, \ldots, m),$$

where $\sigma(t, y)$ is continuous and non-negative for $t \geqslant 0$, $y \geqslant 0$ and such that $y(t) = 0$ is the unique solution of the ordinary differential equation $dy/dt = \sigma(t, y)$, satisfying the initial condition $y(0) = 0$.

Under these assumptions we have

$$(4.3)\quad u(t, x) \leqslant v(t, x) \quad \text{for } (t, x) \in D_p.$$

Proof. Fix $T^* < T$ and for $\varepsilon > 0$, denote by $y(t, \varepsilon)$ a solution of the equation

$$(4.4)\quad dy/dt = \sigma(t, y) + \varepsilon,$$

satisfying the initial condition

$$(4.5)\quad y(0, \varepsilon) = \varepsilon > 0.$$

Such a solution (not necessarily unique) exists and, for $\varepsilon > 0$ sufficiently small, is defined in the interval $[0, T^*)$. Moreover, we have

$$(4.6)\quad \lim_{\varepsilon \to 0} y(t, \varepsilon) = 0 \quad \text{for } t \in [0, T^*).$$

$\sigma(t, y)$ being non-negative, $y(t, \varepsilon)$ is increasing and hence, by (4.5), we get

$$(4.7)\quad y(t, \varepsilon) \geqslant \varepsilon > 0.$$

Now, we put

$$\tilde{v}(t, x) = v(t, x) + \tilde{y}(t, \varepsilon),$$

where $\tilde{y}(t, \varepsilon) = \big(y(t, \varepsilon), \ldots, y(t, \varepsilon)\big)$. Then, we have obviously $\tilde{v}_x = v_x$, $\tilde{v}_{xx} = v_{xx}$. By (4.1), 7°, (4.7) we get in D_p for $i = 1, \ldots, m$,

$$\begin{aligned}(4.8)\quad \tilde{v}_t^i(t, x) &= v_t^i(t, x) + dy(t, \varepsilon)/dt \\ &\geqslant f^i\big(t, x, v(t, x), v_x^i(t, x), v_{xx}^i(t, x), v(t, \cdot)\big) + \sigma\big(t, y(t, \varepsilon)\big) + \varepsilon \\ &\geqslant f^i\big(t, x, v(t, x) + \tilde{y}(t, \varepsilon), \tilde{v}_x^i(t, x), \tilde{v}_{xx}^i(t, x), v(t, \cdot) + \tilde{y}(t, \varepsilon)\big) - \\ &\qquad - \sigma\big(t, y(t, \varepsilon)\big) + \sigma\big(t, y(t, \varepsilon)\big) + \varepsilon \\ &> f^i\big(t, x, \tilde{v}(t, x), \tilde{v}_x^i(t, x), \tilde{v}_{xx}^i(t, x), \tilde{v}(t, \cdot)\big).\end{aligned}$$

From (4.7), (4.8) and from the assumptions of Theorem 2 it follows that for $u(t, x)$ and $\tilde{v}(t, x)$ all the assumptions of Theorem 1 are satisfied with T replaced by T^*. Hence

$$u(t,x) < \tilde{v}(t,x) = v(t,x) + \tilde{y}(t,\varepsilon) \quad \text{for } (t,x) \in D_p,\ 0 < t < T^*.$$

From the last inequality and from (4.6) we obtain in the limit inequality (4.3) for $(t,x) \in D_p$, $0 < t < T^*$. Hence, $T^* < T$ being arbitrary, (4.3) in D_p follows.

5. THEOREM 3 (strong maximum principle). *Let Assumptions* 1° *and* 4° *of Theorem* 1 *and* 5°, 6° *of Theorem* 2 *hold true. Assume, moreover, that*

8° $f^i(t,x,u,q,r,z)$ $(i = 1, \ldots, m)$ *are uniformly parabolic with respect to* $v(t,x)$ *in any compact subset of* D_p.

9° *If* $r_{jk} = \tilde{r}_{jk}$ *for* $j \neq k$ *and* $z - \tilde{z}$ *is bounded, then*

$$|f^i(t,x,u,q,r,z) - f^i(t,x,\tilde{u},\tilde{q},\tilde{r},\tilde{z})| \leqslant c(t)[\max_j(|u^j - \tilde{u}^j|, |q_j - \tilde{q}_j|, |r_{jj} - \tilde{r}_{jj}|, \|z - \tilde{z}\|)] \quad (i = 1, \ldots, m),$$

where $c(t) \geqslant 0$ *is continuous for* $t \geqslant 0$.

Under these assumptions we have inequalities (4.3) *and, moreover, if for some* $(\tilde{t},\tilde{x}) \in D_p$ *and some index* j *the equality*

$$u^j(\tilde{t},x) = v^j(\tilde{t},\tilde{x})$$

holds true, then

$$u^j(t,x) = v^j(t,x) \quad \text{for} \quad (t,x) \in S^-(\tilde{t},\tilde{x}).$$

Proof. Theorem 3 follows from Theorem 2 of this paper by the same argument which was used for the proof of Theorem 2 in paper [2] (see remark 2).

References

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