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The Ryll–Wojtaszczyk polynomials

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Abstract. This paper contains a proof of the existence of the Ryll–Wojtaszczyk polynomials that is simpler than the original one.

Let $P(k, n)$ be the vector space of homogeneous polynomials of degree k in the complex variables $z_1, \ldots, z_n$, regarded as functions on the unit sphere S^{2n-1} of C^n. The L^p-norms in the theorem below refer to the positive rotation-invariant measure σ_n on S^{2n-1} for which $\sigma_n(S^{2n-1}) = 1$.

The following was proved by Ryll and Wojtaszczyk [2].

THEOREM. *For every $n \geqslant 1$ there is a sequence $f_1, f_2, f_3, \ldots,$ with $f_k \in P(k, n)$, $\|f_k\|_2 = 1$, and $\|f_k\|_\infty < 2^n/\sqrt{\pi}$.*

The point is that the upper bound on $\|f_k\|_\infty$ is independent of k. They actually gave two proofs in [2]. The first relied on considerable acquaintance with Banach space theory, the second used geometric and probabilistic arguments (and gave a larger upper bound). Several applications of these polynomials f_k may be found in [2] and [4].

The present paper contains a proof whose main idea is similar to that of the first one but whose details are considerably more elementary. We begin with two propositions. The first of those is also used in [2]; the second yields our simplification.

We let $P_\infty(k, n)$ and $P_2(k, n)$ indicate $P(k, n)$, equipped with the sup-norm and the L^2-norm, respectively. The dimension of $P(k, n)$ will be denoted by $N = N(k, n)$. A simple combinatorial argument [3], p. 139, shows that

$$N = \frac{(n-1+k)!}{(n-1)!\,k!}. \tag{1}$$

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PROPOSITION 1. *The operator T defined by*

$$(Tf)(z) = N \int_{S^{2n-1}} f(w)\langle z, w\rangle^k \, d\sigma_n(w) \tag{2}$$

projects $C(S^{2n-1})$ onto $P_\infty(k, n)$, with norm

$$\|T\| = \frac{\Gamma(n+k)\Gamma(1+\frac{1}{2}k)}{\Gamma(1+k)\Gamma(n+\frac{1}{2}k)} \leqslant 2^{n-1}. \tag{3}$$

If T' is any other projection of $C(S^{2n-1})$ onto $P_\infty(k, n)$, then $\|T'\| \geqslant \|T\|$.

Here $\langle z, w\rangle = \sum z_j \bar{w}_j$, the usual hermitian inner product on C^n.

That T projects is perhaps most easily seen from the binomial expansion

$$(1-\langle z, w\rangle)^{-n} = \sum_{k=0}^{\infty} \frac{(n-1+k)!}{(n-1)!\,k!} \langle z, w\rangle^k \tag{4}$$

of the Cauchy kernel [1]. The norm of T is found by computing $\int |\langle z, w\rangle|^k \, d\sigma_n(w)$. For even k, this is done in [1], p. 17. The odd case can be done the same way.

Moreover, T is the *unique* projection of $C(S^{2n-1})$ onto $P_\infty(k, n)$ that commutes with the action of the unitary group $\mathscr{U} = \mathscr{U}(n)$ [1], Section 12.3. Hence

$$Tf = \int [T'(f \circ U)] \circ U^{-1} \, dU, \tag{5}$$

where dU is the Haar measure of the compact group $\mathscr{U}$. This gives $\|T\| \leqslant \|T'\|$.

PROPOSITION 2. *If $\{\varphi_1, \ldots, \varphi_N\}$ is an orthonormal basis of $P_2(k, n)$, then*

$$\frac{1}{N} \sum_{i=1}^{N} |\varphi_i(z)|^2 = 1 \tag{6}$$

for every $z \in S^{2n-1}$.

This is so because the reproducing kernel for $P_2(k, n)$, namely

$$K(z, w) = \sum_{i=1}^{N} \varphi_i(z)\overline{\varphi_i(w)}, \tag{7}$$

has $K(Uz, Uw) = K(z, w)$ for every $U \in \mathscr{U}$ [1], p. 257. Hence $K(z, z)$ is constant on S^{2n-1}. Since $\int |\varphi_i|^2 \, d\sigma = 1$, this constant is N.

Proof of the theorem. Let J be the identity map, regarded as an operator from $P_\infty(k, n)$ to $P_2(k, n)$. The conclusion of the theorem is that

$$\|J\| > 2^{-n}\sqrt{\pi}. \tag{8}$$

To prove (8), we embed J in the following diagram.

$$\begin{array}{ccc} C(S^{2N-1}) & \overset{W}{\dashrightarrow} & P_\infty(1, N) \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q^{-1}} \\ C(S^{2n-1}) & \xrightarrow{T} P_\infty(k, n) \xrightarrow{J} & P_2(k, n) \end{array}$$

Here T is the operator of Proposition 1. To define Q, let $\{\varphi_1, \dots, \varphi_N\}$ be an orthonormal basis of $P_2(k, n)$. Proposition 2 shows that the map $\Phi = (\varphi_1, \dots, \varphi_N)$ carries S^{2n-1} into $\sqrt{N}\, S^{2N-1}$. It therefore makes sense to define

(9) $$QF = \sqrt{N}\, F\left(\frac{1}{\sqrt{N}}\Phi\right).$$

Evidently, $\|Q\| = \sqrt{N}$.

The restriction of Q to $P_\infty(1, N)$ (i.e., to the linear functions) is a surjective isometry onto $P_2(k, n)$: if $F(w) = \sum_1^N c_i w_i$, then (9) shows that

(10) $$(QF)(z) = \sum_1^N c_i \varphi_i(z),$$

and

(11) $$\left\|\sum c_i \varphi_i\right\|_2^2 = \sum |c_i|^2 = \max\left\{\left|\sum c_i w_i\right| : w \in S^{2N-1}\right\}.$$

This explains the meaning of Q^{-1} as an operator from $P_2(k, n)$ to $P_\infty(1, N)$. Of course, $\|Q^{-1}\| = 1$.

Now define $W = Q^{-1} JTQ$, an operator from $C(S^{2N-1})$ to $P_\infty(1, N)$. It is easy to verify that the restriction of W to $P_\infty(1, N)$ is the identity map. Hence W projects $C(S^{2N-1})$ on $P_\infty(1, N)$, and Proposition 1 (with $1, N$ in place of k, n) shows that

(12) $$\|W\| \geqslant \frac{\Gamma(N+1)\Gamma(\frac{3}{2})}{\Gamma(2)\Gamma(N+\frac{1}{2})} > \Gamma(\tfrac{3}{2})\sqrt{N}$$

since $\Gamma(N+\frac{1}{2}) < \{\Gamma(N)\Gamma(N+1)\}^{1/2} = \Gamma(N+1)/\sqrt{N}$.

We conclude from all this that

$$\tfrac{1}{2}\sqrt{\pi N} < \|W\| \leqslant \|J\| \cdot \|T\| \cdot \|Q\| \leqslant \|J\| \cdot 2^{n-1} \cdot \sqrt{N}$$

which gives (8) and therefore proves the theorem.

References

[1] W. Rudin, *Function Theory in the Unit Ball of C^n*, Springer Verlag, 1980.

[2] J. Ryll and P. Wojtaszczyk, *The homogeneous polynomials on a complex ball*, Trans. Amer. Math. Soc. 276 (1983), 107–116.

[3] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.
[4] P. Wojtaszczyk, *On functions in the ball algebra*, Proc. Amer. Math. Soc. 85 (1982), 184–186.

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