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Local analytic solutions of the functional equation $\Phi(z) = H(z; \Phi[f_1(z)], \ldots, \Phi[f_m(z)])$

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The object of this paper is the functional equation of the form

$$\Phi(z) = H(z; \Phi[f_1(z)], \ldots, \Phi[f_m(z)]), \tag{1}$$

where $\Phi(z)$ is the unknown function and $f_1(z), \ldots, f_m(z)$ and $H(z; w_1, \ldots$ $\ldots, w_m)$ are given complex-valued functions of complex variables. This equation is a generalization of the equation

$$\Phi(z) = H(z, \Phi[f(z)]), \tag{2}$$

which has been investigated (under suitable assumptions) in the paper [2] We shall make use of Banach's fix-point theorem in order to prove the existence of local analytic solutions of equation (1). The proof is similar to that of the theorem on the existence of analytic solutions of equation (2) as given in [1]; and the latter is a shortened version of the proof of the some theorem in [2]. (The case where the functions f_i are the iterates of the same function f: $f_i(z) = f^i(z)$, has already been treated in [3].)

We assume that

(I) $f_k(z)$, $k = 1, \ldots, m$, are analytic functions at $z = 0$ and $f_k(z) = s_k z + \sum_{i=2}^{\infty} b_i^k z^i$ for $|z| \leqslant r_0$;

(II) $|s_k| < 1$, $k = 1, \ldots, m$;

(III) $H(z; w_1, \ldots, w_m)$ is an analytic function of $m+1$ variables $(z; w_1, \ldots, w_m)$ at $(z; w_1, \ldots, w_m) = (0; 0, \ldots, 0)$ and

$$H(z; w_1, \ldots, w_m) = \sum_{i=0}^{\infty} \sum_{j_1, \ldots, j_m = 0}^{\infty} a_{i;j_1\ldots j_m} z^i w_1^{j_1} \ldots w_m^{j_m}$$

for $|z| \leqslant r_0$ and $|w_k| \leqslant R_0$, $k = 1, \ldots, m$; where $a_{0;0\ldots 0} = 0$.

Evidently, the necessary condition of the existence of an analytic solution of equation (1) such that $\Phi(0) = 0$ is the existence of a formal solution of the form

$$\Phi(z) = \sum_{n=1}^{\infty} c_n z^n. \tag{3}$$

The coefficients of formal series (3) are given by the formulae

$$c_n = \frac{F_n}{1 - a_{0;1\ldots0}s_1^n - \ldots - a_{0;0\ldots1}s_m^n}, \quad n = 1, 2, \ldots, \tag{4}$$

where F_n are functions of $c_1, \ldots, c_{n-1}$ and the coefficients of the given functions. In virtue of assumption (II) the denominator of the fraction occurring in (4) is different from zero for almost all n. If $a_{0;1\ldots0}s_1^n + \ldots + a_{0;0\ldots1}s_m^n \neq 1$ for $n = 1, 2, 3, \ldots$, then there exists exactly one formal solution. It can happen, however, that for some n the numerator and denominator of (4) are equal zero. This may concern only finitely many such n's. In this case the formal solutions form a finite-parameter family. No formal solution exists if, for some n, the denominator is equal zero and the numerator is different from zero.

We prove the following

THEOREM. *Under hypotheses* (I)-(III) *every formal solution* (3) *of equation* (1) *is actual, i.e., series* (3) *has a positive radius of convergence.*

Proof. Suppose that (3) is a formal solution of equation (1). It follows from (II) that there exists a positive integer p such that

$$|a_{0;1\ldots0}|\,|s_1|^p + \ldots + |a_{0;0\ldots1}|\,|s_m|^p < 1. \tag{5}$$

We may write

$$\Phi(z) = P(z) + z^p\varphi(z), \tag{6}$$

where $P(z) = \sum_{n=1}^{p-1} c_n z^n$ and $\varphi(z) = c_p + \sum_{n=p+1}^{\infty} c_n z^{n-p}$.

In virtue of (II) there exists an r_1, $0 < r_1 \leqslant r_0$, such that for $|z| \leqslant r_1$ we have

$$|f_k(z)| \leqslant |z|, \quad k = 1, \ldots, m. \tag{7}$$

Since $P(0) = 0$, there exists an r_2, $0 < r_2 \leqslant r_1$, such that for $|z| \leqslant r_2$ we get $|P(z)| \leqslant M < R_0$.

We define the function

$$h(z; v_1, \ldots, v_m) = \frac{H\big(z; P[f_1(z)] + [f_1(z)]^p v_1, \ldots, P[f_m(z)] + [f_m(z)]^p v_m\big) - P(z)}{z^p}. \tag{8}$$

Let r_3, $0 < r_3 \leqslant r_2$, be such that

$$\frac{R_0 - M}{r_3^p} - |c_p| > 0.$$

The series (3) satisfies formally equation (1), so

$$\frac{\partial^q}{\partial z^q} H\big(z; P[f_1(z)]+[f_1(z)]^p v_1, \ldots, P[f_m(z)]+[f_m(z)]^p v_m\big)\Big|_{z=0} = q!\, c_q,$$

$q = 1, \ldots, p-1$. Thus the function h given by formula (8) is an analytic function of the variables $(z; v_1, \ldots, v_m)$ for $|z| \leqslant r_3$, $|v_k - c_p| < R_1$, where

$$0 < R_1 \leqslant \frac{R_0 - M}{r_3^p} - |c_p|.$$

$\varphi(z)$ formally satisfies the equation

$$\varphi(z) = h\big(z; \varphi[f_1(z)], \ldots, \varphi[f_m(z)]\big), \tag{9}$$

and consequently

$$c_p = h(0; c_p, \ldots, c_p). \tag{10}$$

Equations (1) and (9) are equivalent.

Differentiating (8) with respect to v_k we get

$$\begin{aligned} (11)\quad & h'_{v_k}(z; v_1, \ldots, v_m) \\ & = H'_{w_k}\big(z; P[f_1(z)]+[f_1(z)]^p v_1, \ldots, P[f_m(z)]+[f_m(z)]^p v_m\big)\left[\frac{f_k(z)}{z}\right]^p. \end{aligned}$$

Hence we obtain

$$h'_{v_k}(0; c_p, \ldots, c_p) = H'_{w_k}(0; 0, \ldots, 0)\, s_k^p.$$

Hence it follows by (5) that there exist $0 < \vartheta < 1$, $0 < r_4 \leqslant r_3$, $0 < R \leqslant R_1$ such that

$$|h'_{v_1}(z; v_1, \ldots, v_m)| + \ldots + |h'_{v_m}(z; v_1, \ldots, v_m)| < \vartheta \tag{12}$$

for every element of the set

$$A = \{(z; v_1, \ldots, v_m):\ |z| \leqslant r_4,\ |v_k - c_p| \leqslant R,\ k = 1, \ldots, m\}.$$

On account of the mean-value theorem, for any $\hat{v}_k$, $\hat{\hat{v}}_k$, $k = 1, \ldots, m$, such that $|\hat{v}_k - c_p| \leqslant R$, $|\hat{\hat{v}}_k - c_p| \leqslant R$, we have

$$\begin{aligned} (13)\quad & |h(z; \hat{v}_1, \ldots, \hat{v}_m) - h(z; \hat{\hat{v}}_1, \ldots, \hat{\hat{v}}_m)| \\ & \leqslant \sup_A |h'_{v_1}(z; v_1, \ldots, v_m)|\, |\hat{v}_1 - \hat{\hat{v}}_1| + \ldots + \sup_A |h'_{v_m}(z; v_1, \ldots, v_m)|\, |\hat{v}_m - \hat{\hat{v}}_m|, \end{aligned}$$

for $|z| \leqslant r_4$.

Let us fix

$$0 < K \leqslant R. \tag{14}$$

It follows from the continuity of $h(z; c_p, \ldots, c_p)$ that there exists a number $r > 0, r \leqslant r_4$, such that for $|z| \leqslant r$ we have

$$(15)\qquad |h(z; c_p, \ldots, c_p) - h(0; c_p, \ldots, c_p)| < (1-\vartheta)K.$$

We denote by $\mathscr{F}$ the set of all analytic functions $\varphi(z)$ which fulfil the conditions

$$(16)\qquad |\varphi(z) - c_p| \leqslant K \quad \text{for } |z| \leqslant r \quad \text{and} \quad \varphi(0) = c_p.$$

The set $\mathscr{F}$ with the metric

$$\varrho(\varphi_1, \varphi_2) = \sup_{|z|\leqslant r} |\varphi_2(z) - \varphi_1(z)|$$

is a complete metric space.

Now we consider the transformation $\psi = T(\varphi)$ defined in $\mathscr{F}$ by the formula

$$(17)\qquad \psi(z) = h\big(z; \varphi[f_1(z)], \ldots, \varphi[f_m(z)]\big).$$

We shall prove that $T(\mathscr{F}) \subset \mathscr{F}$. If $\varphi \in \mathscr{F}$, then we have by (17) and (10)

$$\begin{aligned}|\psi(z) - c_p| &= \big|h\big(z; \varphi[f_1(z)], \ldots, \varphi[f_m(z)]\big) - h(0; c_p, \ldots, c_p)\big| \\ &\leqslant \big|h\big(z; \varphi[f_1(z)], \ldots, \varphi[f_m(z)]\big) - h(z; c_p, \ldots, c_p)\big| + \\ &\quad + |h(z; c_p, \ldots, c_p) - h(0; c_p, \ldots, c_p)|.\end{aligned}$$

Hence in virtue of (7), (16), (13), (12) and (15) we get

$$|\psi(z) - c_p| < K\vartheta + (1-\vartheta)K = K.$$

From (11) we have $\psi(0) = c_p$, thus $\psi \in \mathscr{F}$.

Let $\varphi_1, \varphi_2 \in \mathscr{F}$, $\psi_1 \overset{\text{df}}{=} T[\varphi_1]$, $\psi_2 \overset{\text{df}}{=} T[\varphi_2]$. From the definition of the metric we infer by (17), (7), (16), (13) and (12),

$$\begin{aligned}&\varrho(\psi_1, \psi_2) \\ &= \sup_{|z|\leqslant r}\big|h\big(z; \varphi_2[f_1(z)], \ldots, \varphi_2[f_m(z)]\big) - h\big(z; \varphi_1[f_1(z)], \ldots, \varphi_1[f_m(z)]\big)\big| \\ &\leqslant \sup_A |h'_{v_1}(z; v_1, \ldots, v_m)| \sup_{|z|\leqslant r} |\varphi_2(z) - \varphi_1(z)| + \ldots + \\ &\quad + \sup_A |h'_{v_m}(z; v_1, \ldots, v_m)| \sup_{|z|\leqslant r} |\varphi_2(z) - \varphi_1(z)| \\ &< \vartheta \sup_{|z|\leqslant r} |\varphi_2(z) - \varphi_1(z)| = \vartheta\varrho(\varphi_1, \varphi_2).\end{aligned}$$

Consequently, T is a contraction map. Hence, on account of Banach's theorem it follows that $\varphi(z)$ is the solution of equation (9). $\Phi(z)$ given by (6) is the solution of equation (1). The proof of the Theorem is complete.

From the above considerations we have

COROLLARY. *There are as many actual solutions of equation* (1) *as formal solutions, thus either exactly one or a finite-parameter family, or none Equation* (1) *has no solution when it has no formal solution.*

It may actually happen that

$$a_{0;1\ldots0}s_1^n+\ldots+a_{0;0\ldots1}s_m^n = 1$$

for several values of n, and consequently equation (1) may have a solution containing several parameters. The equation

$$\varphi(z) = 2\varphi(\tfrac{3}{4}z)-2\varphi(\tfrac{1}{4}z) \tag{18}$$

provides an example of an equation of form (1) having a two-parameter family of analytic solutions.

In fact, let $\varphi(z) = \sum_{n=1}^{\infty} c_n z^n$ be an analytic solution of equation (18). Then

$$\sum_{n=1}^{\infty} c_n z^n = 2\sum_{n=1}^{\infty} c_n(\tfrac{3}{4})^n z^n - 2\sum_{n=1}^{\infty} c_n(\tfrac{1}{4})^n z^n = 2\sum_{n=1}^{\infty} \frac{3^n-1}{4^n} c_n z^n.$$

Thus

$$c_n = \frac{2(3^n-1)}{4^n}c^n, \quad n = 1, 2, \ldots \tag{19}$$

Hence c_1 and c_2 may be arbitrary. Since $2\cdot 3^n < 4^n$ for $n \geqslant 3$, we have

$$2(3^n-1) < 2\cdot 3^n < 4^n \quad \text{for } n \geqslant 3,$$

whence it follows according to (19) that $c_n = 0$ for $n \geqslant 3$. We thus obtain the two-parameter family of solutions

$$\varphi(z) = c_1 z + c_2 z^2. \tag{20}$$

It is readily verified that functions (20) actually satisfy equation (18).

References

[1] J. Matkowski, *Note on the functional equation*, Zeszyty Naukowe UJ, Prace Mat. (to appear).

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[3] — *Local analytic solutions of the functional equation* $\varphi(z) = h(z, \varphi[f(z)])$ *in multidimensional spaces*, Aequationes Math. 1 (1968), p. 20-36.

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