ANNALES
POLONICI MATHEMATICI
LII (1990)

On generic chaos of shifts in function spaces

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Abstract. The paper deals with the notion of generic chaos which was defined in [6]. It is shown that shifts in some function spaces considered as complete metric spaces are generically chaotic.

1. Let us recall that a set in a topological space is called a *set of the first category* iff it is a countable union of nowhere dense sets and that a set in a complete metric space with the complement of the first category is said to be a *generic set.*

This topological approach may be applied to characterize chaotic behaviour of dynamical systems acting on a metric space. (See [6]. Also see [1], for the other known definitions of chaos.) This way of describing dynamical systems has been proposed by A. Lasota.

2. Let (V, ϱ) be a metric space and let $\{S_t\}$ be a semigroup of transformations from V into V. The semigroup may be either discrete ($t \in \boldsymbol{N}$) or continuous ($t \in \boldsymbol{R}$). Let G be the set of all chaotic pairs (u, v) in V^2, i.e., of all (u, v) such that

$$\liminf_{t \to +\infty} \varrho(S_t u, S_t v) = 0 \quad \text{and} \quad \limsup_{t \to +\infty} \varrho(S_t u, S_t v) > 0.$$

DEFINITION. We call the dynamical system $\{S_t\}$ *generically chaotic* iff the set G is generic in V^2.

INTERSECTION PROPOSITION. *Let $\{S_t\}$ be a semigroup of transformations of a metric space (V, ϱ). Let us fix some $\alpha > 0$ and suppose that for every $T > 0$ and every $\varepsilon > 0$ the sets*

$$L_{T,\varepsilon} = \{(u, v) \in V^2 | \inf_{t \geq T} \varrho(S_t u, S_t v) < \varepsilon\},$$

$$U_T = \{(u, v) \in V^2 | \sup_{t \geq T} \varrho(S_t u, S_t v) > \alpha\}$$

are open and dense in V^2. Then $\{S_t\}$ is generically chaotic.

To prove the proposition it suffices to observe that the "chaotic set" G contains the intersection of the sets $L_{n,1/n}$ and U_n when the index n runs over the set of positive integers.

In the sequel, it will also be useful to denote by h the function $t \mapsto t/(t+1)$ for $t \geqslant 0$. Observe that h is continuous, strictly increasing, subadditive and maps $[0, +\infty)$ onto $[0, 1)$. Hence composing any metric with h we can replace the metric with a bounded one.

3. Chaos in the space $C(\boldsymbol{R}, \boldsymbol{R})$. Let $V = C(\boldsymbol{R}, \boldsymbol{R})$ be the set of all continuous real functions of the real variable. For $u, v \in V$ let us put

$$\varrho_n(u, v) = \max_{|x| \leqslant n} |u(x) - v(x)|, \qquad n = 1, 2, \ldots,$$

$$\varrho(u, v) = \sum_{n=1}^{\infty} 2^{-n} (h \circ \varrho_n)(u, v).$$

Then (V, ϱ) is a complete metric space and the convergence in ϱ is the convergence on compact subsets of $\boldsymbol{R}$. Now given $u \in V$ and $t \in \boldsymbol{R}$, let $(S_t u)(x) = u(x-t)$ for $x \in \boldsymbol{R}$. $\{S_t\}_{t \in \boldsymbol{R}}$ is a group of transformations of V. We shall prove

THEOREM 1. *The dynamical system $\{S_t\}$ is generically chaotic.*

Proof. We shall use the Intersection Proposition with a fixed $\alpha \in (0, \frac{1}{4})$.

(i) For every $t \geqslant 0$, $S_t \colon V \to V$ is a Lipschitzian function. Indeed, for $n \in N$, $t \geqslant 0$ and $u, v \in V$ we have

$$\varrho_n(S_t u, S_t v) \leqslant \varrho_{n+[t]+1}(u, v) \quad \text{ and } \quad \varrho(S_t u, S_t v) \leqslant 2^{[t]+1} \varrho(u, v)$$

(here $[t]$ stands for the integer part of t). Thus we have

$$(*) \qquad \forall t \geqslant 0 \; \forall \eta > 0 \; \exists \delta > 0 \; \forall u, v \in V \colon \; \varrho(u, v) < \delta \Rightarrow \varrho(S_t u, S_t v) < \tfrac{1}{2}\eta.$$

(ii) $L_{T,\varepsilon}$ are open sets in V^2. Indeed, let us fix $T > 0$, $\varepsilon > 0$ and $(u_0, v_0) \in L_{T,\varepsilon}$. There exist $t \geqslant T$ and $\eta > 0$ such that $\varrho(S_t u_0, S_t v_0) < \varepsilon - \eta$. We choose δ as in $(*)$. Then

$$\varrho(S_t u, S_t v) \leqslant \varrho(S_t u, S_t u_0) + \varrho(S_t u_0, S_t v_0) + \varrho(S_t v_0, S_t v) < \tfrac{1}{2}\eta + \varepsilon - \eta + \tfrac{1}{2}\eta = \varepsilon$$

for $(u, v) \in V^2$ such that $\varrho(u_0, u) < \delta$, $\varrho(v_0, v) < \delta$.

(iii) U_T are open sets in V^2. Indeed, let us fix $T > 0$ and $(u_0, v_0) \in U_T$. There exist $t \geqslant T$ and $\eta > 0$ such that $\varrho(S_t u_0, S_t v_0) > \alpha + \eta$. We choose δ as in $(*)$. Then

$$\begin{aligned} \alpha + \eta < \varrho(S_t u_0, S_t v_0) &\leqslant \varrho(S_t u_0, S_t u) + \varrho(S_t u, S_t v) + \varrho(S_t v, S_t v_0) \\ &< \eta + \varrho(S_t u, S_t v), \end{aligned}$$

thus $\varrho(S_t u, S_t v) > \alpha$ for $(u, v) \in V^2$ such that $\varrho(u_0, u) < \delta$, $\varrho(v_0, v) < \delta$.

(iv) $L_{T,\varepsilon}$ are dense sets. Indeed, let us fix $T > 0$, $\varepsilon > 0$, $(u_0, v_0) \in V^2$ and $\delta > 0$ There is $n_0 \in N$ such that

$$\sum_{n=n_0+1}^{\infty} 2^{-n} < \min\{\delta, \varepsilon\}.$$

Let $u, v \in V$ be such that

$$u(x) = \begin{cases} u_0(x), & |x| \leqslant n_0, \\ 0, & |x| \geqslant n_0+1, \end{cases}$$

$$v(x) = \begin{cases} v_0(x), & |x| \leqslant n_0, \\ 0, & |x| \geqslant n_0+1. \end{cases}$$

Functions like these exist by Urysohn's Lemma. We have

$$\varrho(u_0, u) = \sum_{n=n_0+1}^{\infty} 2^{-n}(h \circ \varrho_n)(u_0, u) \leqslant \sum_{n=n_0+1}^{\infty} 2^{-n} < \delta$$

and similarly

$$\varrho(v_0, v) < \delta.$$

Let $t \geqslant \max\{T, 2n_0+2\}$. Then $(S_t u)(x) = (S_t v)(x) = 0$, $|x| \leqslant n_0$ and

$$\varrho(S_t u, S_t v) = \sum_{n=n_0+1}^{\infty} 2^{-n}(h \circ \varrho_n)(S_t u, S_t v) \leqslant \sum_{n=n_0+1}^{\infty} 2^{-n} < \varepsilon.$$

(v) U_T are dense sets. Indeed, let us fix $T > 0$, $(u_0, v_0) \in V^2$ and $\delta > 0$. There is $n_0 \in N$ such that

$$\sum_{n=n_0+1}^{\infty} 2^{-n} < \delta.$$

Let $\bar{u}, \bar{v} \in V$ be such that

$$\bar{u}(x) = \begin{cases} u_0(x), & |x| \leqslant n_0, \\ 0, & |x| \geqslant n_0+1, \end{cases}$$

$$\bar{v}(x) = \begin{cases} v_0(x), & |x| \leqslant n_0, \\ 2, & |x| \geqslant n_0+1. \end{cases}$$

(One can choose any $C > 2\alpha(1-2\alpha)^{-1}$ instead of 2.)

We may show, as in (iv), that $\varrho(u_0, \bar{u}) < \delta$ and $\varrho(v_0, \bar{v}) < \delta$. Let $t \geqslant \max\{T, n_0+1\}$. Then $(S_t\bar{u})(0) = 0$, $(S_t\bar{v})(0) = 2$ and

$$\varrho(S_t\bar{u}, S_t\bar{v}) = \sum_{n=1}^{\infty} 2^{-n}(h \circ \varrho_n)(S_t\bar{u}, S_t\bar{v}) \geqslant \frac{1}{2} \cdot \frac{2}{2+1} > \alpha,$$

since $\alpha < \frac{1}{4}$. Theorem 1 is proved.

4. Chaos in the space $C^r(\boldsymbol{R}, \boldsymbol{R})$. Let us fix an integer $r \geqslant 1$ and let $V = C^r(\boldsymbol{R}, \boldsymbol{R})$ be the set of all r times continuously differentiable functions from $\boldsymbol{R}$ into $\boldsymbol{R}$. For $u, v \in V$ let us put

$$\varrho_n^r(u, v) = \sum_{i=0}^{r} \max |u^{(i)}(x) - v^{(i)}(x)|, \qquad n = 1, 2, \ldots,$$

$$\varrho^r(u, v) = \sum_{n=1}^{\infty} 2^{-n} (h \circ \varrho_n^r)(u, v).$$

Then (V, ϱ^r) is a complete metric space and the convergence in ϱ^r is the uniform convergence with derivatives up to the r-th order on compact subsets of $\boldsymbol{R}$.

For $u \in V$ and $t \in \boldsymbol{R}$ let $(S_t u)(x) = u(x-t)$ for $x \in \boldsymbol{R}$. $\{S_t\}_{t \in \boldsymbol{R}}$ is a group of transformations of V. As in Section 3 we will prove that the dynamical system $\{S_t\}$ is generically chaotic. We may fix $\alpha \in (0, \frac{1}{4})$ as above. One can check that the arguments of (i)–(v) in Section 3 remain valid for the new dynamical system. There is only a little change in (iv) and (v). Namely, we have to choose the functions u, v, $\bar{u}$, $\bar{v}$ from $C^r(\boldsymbol{R}, \boldsymbol{R})$, and this is possible since there exists a decomposition of unity for the set $\{x\colon |x| \leqslant n_0\}$ and its open covering set $\{x\colon |x| < n_0 + \frac{1}{2}\}$.

5. Chaos in the space $C^1(\boldsymbol{R}, \boldsymbol{R}^m)$. Let $V = C^1(\boldsymbol{R}, \boldsymbol{R}^m)$ be the set of all functions from $\boldsymbol{R}$ into $\boldsymbol{R}^m$ with a continuous derivative. For $u, v \in V$ let us put

$$\varrho_n(u, v) = \max_{|x| \leqslant n} \sum_{j=1}^{m} \big(|u_j(x) - v_j(x)| + |u_j'(x) - v_j'(x)|\big), \qquad n = 1, 2, \ldots,$$

$$\varrho(u, v) = \sum_{n=1}^{\infty} 2^{-n} (h \circ \varrho_n)(u, v),$$

where $u = (u_1, \ldots, u_m)$, $v = (v_1, \ldots, v_m)$. Then (V, ϱ) is a complete metric space. Again, for $u \in V$ and $t \in \boldsymbol{R}$ let $(S_t u)(x) = u(x-t)$ for $x \in \boldsymbol{R}$. $\{S_t\}_{t \in \boldsymbol{R}}$ is a dynamical system on V and we will prove that it is generically chaotic.

The arguments of (i)–(iii) in Section 3 may be repeated without change. In (iv) and (v) we have to define the functions u, v, $\bar{u}$, $\bar{v}$ in a new way.

So, in (iv), for T, ε, u_0, v_0, δ, n_0 fixed as in 3, (iv), let $u, v \in C^1(\boldsymbol{R}, \boldsymbol{R}^m)$ be such that

$$u(x) = \begin{cases} u_0(x), & |x| \leqslant n_0, \\ 0, & |x| \geqslant n_0 + 1, \end{cases}$$

$$v(x) = \begin{cases} v_0(x), & |x| \leqslant n_0, \\ 0, & |x| \geqslant n_0 + 1, \end{cases}$$

where $0 = (0, \ldots, 0) \in \boldsymbol{R}^m$.

In (v), for T, u_0, v_0, δ, n_0 fixed as in 3, (v), let $\bar{u}, \bar{v} \in C^1(\boldsymbol{R}, \boldsymbol{R}^m)$ be such that

$$\bar{u}(x) = \begin{cases} u_0(x), & |x| \leqslant n_0, \\ 0, & |x| \geqslant n_0+1, \end{cases}$$

$$\bar{v}(x) = \begin{cases} v_0(x), & |x| \leqslant n_0, \\ (2, 0, \ldots, 0), & |x| \geqslant n_0+1. \end{cases}$$

(In both the above-mentioned constructions we use a decomposition of unity in each coordinate separately.)

6. Chaos in the space $C^2(\boldsymbol{R}, \boldsymbol{R}) \times C^1(\boldsymbol{R}, \boldsymbol{R})$. We put

$$V = \{(\varphi, \psi) |\ \varphi \in C^2(\boldsymbol{R}, \boldsymbol{R}),\ \psi \in C^1(\boldsymbol{R}, \boldsymbol{R})\},$$

$$\varrho(u, v) = \max\{\varrho^2(\varphi_u, \varphi_v), \varrho^1(\psi_u, \psi_v)\},$$

where $u = (\varphi_u, \psi_u)$, $v = (\varphi_v, \psi_v)$, $(u, v) \in V$. (See Section 4 for the definitions of ϱ^2 and ϱ^1.) Again (V, ϱ) is a complete metric space. For $u = (\varphi, \psi) \in V$ and $t \in \boldsymbol{R}$ let

$$(S_t u)(x) = (\varphi(x-t), \psi(x-t)) \quad \text{for } x \in \boldsymbol{R}.$$

Then $\{S_t\}_{t \in \boldsymbol{R}}$ is a dynamical system on V and we will prove that it is generically chaotic. It is obvious that $\{S_t\}$ is a product flow built of flows in C^2 and C^1, respectively, as in Section 4 with $m = 1$. Thus it suffices to repeat twice, for each coordinate of the product, the constructions of the functions u, v, $\bar{u}$, $\bar{v}$ in (iv) and (v).

We may summarize the results of Sections 3–6 in the following theorem.

THEOREM 1. *The shift groups in the spaces of continuously differentiable functions from* $\boldsymbol{R}$ *into* $\boldsymbol{R}^m$ *are generically chaotic in the metric of uniform convergence on compact sets.*

7. Chaos of Bernoulli shifts. It is well known that the so-called *Baker's Transformation* is a mixing ([2]). So we can say that this transformation is *chaotic* in the sense of ergodic theory. One can also easily prove that Baker's Transformation is isomorphic with the shift in the set of two-sided sequences of 0's and 1's. We shall generalize this example and prove that the so-called Bernoulli shifts are generically chaotic. We shall treat Bernoulli shifts in the topological context and not in the measure-theoretical one. (See [5], e.g.) Let us notice that Bernoulli shifts as K-systems are also mixings (see [3], [5], [7]).

Let us fix a positive integer $N \geq 2$ and real numbers p_i such that $0 < p_i < 1$ $(i = 1, \ldots, N)$, $\sum_{i=1}^{N} p_i = 1$. Let $a_0 = 0$, $a_k = \sum_{l=1}^{k} p_l$ for $k = 1, \ldots, N$. Further, let us define recursively a family of sets $A_n = \{a_0^{(n)}, \ldots, a_{N^n}^{(n)}\}$. Namely, let

$$A_1 = \{a_0, \ldots, a_N\}, \quad \text{i.e.,} \quad a_i^{(1)} = a_i \quad \text{for } i = 0, \ldots, N,$$

$$A_{n+1} = \{a_{i-1}^{(n)} + a_j(a_i^{(n)} - a_{i-1}^{(n)}) |\ i = 1, \ldots, N^n,\ j = 0, \ldots, N\}, \quad n = 1, 2, \ldots,$$

and the points of A_{n+1} are ordered increasingly from $a_0^{(n+1)} = 0$ to $a_{N^{n+1}}^{(n+1)} = 1$.

Let us observe that $\# A_n = N^n + 1$, $n = 1, 2, \ldots$ Let us put $K = [0, 1)^2$ and $d(P, Q) = |x_1 - x_2| + |y_1 - y_2|$ for $P(x_1, y_1)$, $Q(x_2, y_2)$ in K. (K, d) is a metric space.

We define a transformation $\tau\colon K \to K$ in the following way:

$$\tau(x, y) = (x', y'), \quad \text{where } x' = (x - a_{i-1})p_i^{-1},\ y' = p_i y + a_{i-1}$$
$$\text{for } x \in [a_{i-1}, a_i),\ i = 1, \ldots, N \text{ and } y \in [0, 1).$$

The function τ is one-to-one and maps K onto K. It is easy to see that, for $n = 1, 2, \ldots$, $\tau^n\colon K \setminus A_n \to K$ is one-to-one and continuous. It is also easily seen that $K \setminus A$ is a dense subset of K, where $A = \bigcup_{n=1}^{\infty} A_n$. Let us put $K_1 = K \setminus A$.

Now we are going to construct an isometric model of the system (K, d, τ) which we shall call the Bernoulli shift.

Let B^* be the family of all two-sided sequences with values in the set $\{0, 1, \ldots, N-1\}$, i.e.,

$$B^* = \{(\alpha_i)_{i \in Z} |\ \alpha_i \in \{0, 1, \ldots, N-1\},\ i \in Z\}.$$

Let B be the set of those sequences in B^* which have neither left nor right period built of the sign $N-1$. We define a map $\sigma\colon B \to B$ such that for $\alpha = (\alpha_i)_{i \in Z} \in B$ and $\beta = (\beta_i)_{i \in Z} \in B$ we put

$$\sigma(\alpha) = \beta \quad \text{iff} \quad \beta_i = \alpha_{i+1} \quad \text{for } i \in Z.$$

Thus σ is the shift to the left on B.

Now, let $(x, y) \in K_1$. We put $x = 0{,}\alpha_0\alpha_1\alpha_2\ldots$ and $y = 0{,}\alpha_{-1}\alpha_{-2}\alpha_{-3}\ldots$, where the right-hand sides stand for the expansions of the numbers x, y, respectively, in the "weighted" digital system with the base N and with the weights $p_1, \ldots, p_N$. (That is to say, $\alpha_0 = i$ if $x \in [a_i, a_{i+1})$, then $\alpha_1 = j$ if $x \in [a_i + (a_{i+1} - a_i)a_j,\ a_i + (a_{i+1} - a_i)a_{j+1})$, and so on. The expansion of y is defined similarly.)

Further, let us define a map $\varphi\colon K_1 \to B$ such that

$$\varphi(x, y) = (\alpha_i)_{i \in Z} \quad \text{for any } (x, y) \in K_1,$$

where α_i $(i \in Z)$ are the digits of the expansions of x and y obtained in the way described above.

It is easy to see that φ is well-defined, one-to-one and maps K_1 onto B and that $\varphi \circ \tau = \sigma \circ \varphi$, i.e., that the following diagram commutes:

$$\begin{array}{ccc} K_1 & \xrightarrow{\tau} & K_1 \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ B & \xrightarrow[\sigma]{} & B. \end{array}$$

More generally, we have: $\varphi \circ \tau^n = \sigma^n \circ \varphi$ for $n = 1, 2, \ldots$

By means of the map φ we transport the metric from K_1 to B setting $\varrho(\alpha, \beta) = d(\varphi^{-1}(\alpha), \varphi^{-1}(\beta))$ for $\alpha, \beta \in B$. One can see that (B, ϱ) is a metric space, σ: $B \to B$ is a bijective continuous map and φ: $K_1 \to B$ is an isometry. Thus (K, τ) and (B, σ) are isometrically isomorphic dynamical systems.

We call the system (B, σ) we have just constructed the "Bernoulli shift".

Now we shall prove

THEOREM 2. *The Bernoulli shift σ is generically chaotic on B* (in other words: *the map τ is generically chaotic on K*).

Proof. We may compute either in K or in B and we are going to choose the context so as to make computations simpler. Let us fix $\lambda \in (0, \min_{1 \leqslant i \leqslant N} p_i)$. For $n = 1, 2, \ldots$ and $\varepsilon > 0$ we put

$$\begin{aligned} L_{n,\varepsilon} &= \{(\alpha, \beta) \in B^2 \mid \inf_{k \geqslant n} \varrho(\sigma^k(\alpha), \sigma^k(\beta)) < \varepsilon\} \\ &= \{(P, Q) \in K_1^2 \mid \inf_{k \geqslant n} d(\tau^k(P), \tau^k(Q)) < \varepsilon\}, \\ U_n &= \{(\alpha, \beta) \in B^2 \mid \sup_{k \geqslant n} \varrho(\sigma^k(\alpha), \sigma^k(\beta)) > \lambda\} \\ &= \{(P, Q) \in K_1^2 \mid \sup_{k \geqslant n} d(\tau^k(P), \tau^k(Q)) > \lambda\}, \end{aligned}$$

where $\alpha = \varphi(P)$, $\beta = \varphi(Q)$ and the set equalities are to be understood in the sense of the isomorphism φ.

By the Intersection Proposition it suffices to prove that each of the sets $L_{n,\varepsilon}$, U_n is open and dense.

(i) First we shall prove that the sets $L_{n,\varepsilon}$ are open. Indeed, let us fix $n \geqslant 1$, $\varepsilon > 0$ and $(P_0, Q_0) \in L_{n,\varepsilon}$. There exist $\eta \in (0, \varepsilon)$ and an integer $k \geqslant n$ such that $d(\tau^k(P_0), \tau^k(Q_0)) < \varepsilon - \eta$. Since τ^k is continuous, there exist neighbourhoods E, F of P_0, Q_0, respectively, which are disjoint from A_k and such that

$$d(\tau^k(P), \tau^k(P_0)) < \tfrac{1}{2}\eta \quad \text{for } P \in E \quad \text{and} \quad d(\tau^k(Q), \tau^k(Q_0)) < \tfrac{1}{2}\eta \quad \text{for } Q \in F.$$

Then, for $(P, Q) \in E \times F$, we have

$$d(\tau^k(P), \tau^k(Q)) \leqslant d(\tau^k(P), \tau^k(P_0)) + d(\tau^k(P_0), \tau^k(Q_0)) + d(\tau^k(Q_0), \tau^k(Q)) < \varepsilon$$

and, consequently, $E \times F \subset L_{n,\varepsilon}$.

(ii) In order to prove that the sets U_n are open let us fix $n \geqslant 1$ and $(P_0, Q_0) \in U_n$. There exist $\eta > 0$ and an integer $k \geqslant n$ such that

$d(\tau^k(P_0), \tau^k(Q_0)) > \lambda+\eta$. Choosing the neighbourhoods E, F as in (i) we obtain $E\times F \subset U_n$.

(iii) To prove that the sets $L_{n,\varepsilon}$ are dense, let us fix $n \geqslant 1$, $\varepsilon > 0$, $(\alpha^0, \beta^0)\in B^2$ and $\delta > 0$. There exists an integer $k_0 \geqslant 1$ such that $(\max_{1\leqslant i\leqslant N} p_i)^{k_0} < \frac{1}{2}\min\{\varepsilon, \delta\}$. Let us put, for $i\in \boldsymbol{Z}$,

$$\alpha_i = \begin{cases}\alpha_i^0, & |i| \leqslant k_0,\\ i(\operatorname{mod} N), & |i| > k_0,\end{cases} \qquad \beta_i = \begin{cases}\beta_i^0, & |i| \leqslant k_0,\\ i(\operatorname{mod} N), & |i| > k_0,\end{cases}$$

and let $\alpha = (\alpha_i)_{i\in\boldsymbol{Z}}$, $\beta = (\beta_i)_{i\in\boldsymbol{Z}}$. Then $(\alpha, \beta)\in B^2$ and $\varrho(\alpha^0, \alpha) < \delta$, $\varrho(\beta^0, \beta) < \delta$. Further, let $k \geqslant \max\{n, 2k_0+1\}$. Then $\alpha_i^k = \beta_i^k$ for $|i| \leqslant k_0$, where $(\alpha_i^k)_i = \sigma^k(\alpha)$, $(\beta_i^k)_i = \sigma^k(\beta)$. Thus $\varrho(\sigma^k(\alpha), \sigma^k(\beta)) < \varepsilon$.

(iv) In order to prove that the sets U_n are dense, let us fix $n \geqslant 1$, $(\alpha^0, \beta^0)\in B^2$ and $\delta > 0$. There exists an integer $k \geqslant n$ such that $(\max_{1\leqslant i\leqslant N} p_i)^k < \frac{1}{2}\delta$. For $i\in\boldsymbol{Z}$, let us put

$$\bar{\alpha}_i = \begin{cases}\alpha_i^0, & |i| \leqslant k,\\ i(\operatorname{mod} N), & |i| > k,\end{cases} \qquad \bar{\beta}_i = \begin{cases}\beta_i^0, & |i| \leqslant k,\\ k(\operatorname{mod} N), & i = k+1,\\ i(\operatorname{mod} N), & |i| > k,\ i \neq k+1,\end{cases}$$

and let $\bar{\alpha} = (\bar{\alpha}_i)_{i\in\boldsymbol{Z}}$, $\bar{\beta} = (\bar{\beta}_i)_{i\in\boldsymbol{Z}}$. Then $(\bar{\alpha}, \bar{\beta})\in B^2$ and $\varrho(\alpha^0, \bar{\alpha}) < \delta$, $\varrho(\beta^0, \bar{\beta}) < \delta$. Further,

$$\varrho(\sigma^k(\bar{\alpha}), \sigma^k(\bar{\beta})) \geqslant \min_{1\leqslant i\leqslant N} p_i > \lambda.$$

Theorem 2 is proved.

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Reçu par la Rédaction le 26.02.1987