

Continuity of total number of intersection

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Abstract. In the paper we present some theorems of the Rouché and Hurwitz type for intersection of continuous families of locally analytic sets. As applications we present a local version of Bezout's theorem and a very simple, geometrically clear proof of Bezout's theorem for intersection of algebraic sets.

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The organization of this paper is as follows. Sections 2–6 are of preparatory nature, where we collect together some known facts and derive their consequences for use in other sections. Sections 7–9 are principal for the paper. As applications of preceding sections a short proof of Bezout's theorem is presented in Section 10.

1. Introduction. It is a well-known result of complex algebraic geometry that a pure k -dimensional algebraic subset V of an n -dimensional complex projective space $P(M)$ intersects “almost all” complex $(n-k)$ -dimensional projective planes at the same number of points. This number is called the *degree* of V . If instead of the algebraic subset V we shall consider a pure k -dimensional locally analytic subset X of $P(M)$, then the preceding result can be regarded as a special case of the following theorem of this paper (Theorem 7.1).

THEOREM. *Let G be a subset of the Grassmann manifold $G_{n-k}(M)$ consisting of all $(n-k)$ -dimensional projective planes ξ such that $\xi \cap (\bar{X} \setminus X) = \emptyset$. Suppose that $G \neq \emptyset$. Then*

- (i) G is open in $G_{n-k}(M)$;
- (ii) let G' be a component of G ; if there exists $\xi \in G'$ such that $\xi \cap X = \emptyset$, then $\xi \cap X = \emptyset$ whenever $\xi \in G'$;
- (iii) the set

$$A' = \{\xi \in G' : \# \xi \cap X = \infty\}$$

is a properly included analytic subset of G' and the mapping

$$(*) \quad G' \setminus A' \ni \xi \rightarrow \sum_{a \in \xi \cap X} i(\xi \cdot X; a)$$

is continuous, where $i(\xi \cdot X; a)$ denotes the intersection multiplicity of ξ and X at a .

In this regard, one notices that the definition of the degree of a pure k -dimensional algebraic subset V of $P(M)$ can be understood in the following manner:

- (a) $G' = G = G_{n-k}(M)$, because $\bar{V} \setminus V = \emptyset$;
- (b) since A' is analytic in G' , we conclude that $G' \setminus A'$ is open and connected,
- (c) since $G' \setminus A'$ is connected, function $(*)$ takes only one value. This value is called the *degree* of V .

Of course, if X is a bounded, locally analytic subset of an affine part of $P(M)$, then $A' = \emptyset$ and (ii) is a trivial consequence of (iii). Moreover, the previous result generalizes the maximum principle for holomorphic mappings (see e.g. [11] and cf. Corollary 7.2 of this paper) and may also be used to formulate Hurwitz' type theorems.

Let X, Y be locally analytic subsets of a complex vector space N of complementary dimensions. Of course, X is contained in an open set $\Omega \subset N$. Then $\mathcal{B}^*(\Omega; X, Y)$ denotes the space of all biholomorphic mappings $f: \Omega \rightarrow f(\Omega) \subset N$ such that $f(\Omega)$ is bounded and $\overline{f(X)} \cap \bar{Y} = f(X) \cap Y$ with topology of local uniform convergence. Similarly for a connected open subset $\Omega \subset N$, $\mathcal{A}_\Omega(X, Y)$ denotes the spaces of all biholomorphic mappings $f: \Omega \rightarrow f(\Omega) \subset N$ such that $f(\Omega) \supset \bar{Y}$ and $(\bar{Y} \setminus Y) \cap f(X) = \emptyset$ with the topology of local uniform convergence.

In Section 8 we prove theorems on continuity of functions

$$\mathcal{B}^*(\Omega; X, Y) \ni f \rightarrow \#(f(X) \cdot Y) = \text{number of points of } f(X) \cap Y$$

counted with their multiplicities,

and

$$\mathcal{A}_\Omega(X, Y) \ni f \rightarrow \#(f(X) \cdot Y) \in \mathbb{Z}$$

(see Theorems 8.1, 8.2). Moreover, taking locally analytic sets X_1, \dots, X_k , $k \geq 2$, instead of X, Y and suitable space of biholomorphic mappings

$\sigma_1, \dots, \sigma_k$ we establish two results on continuity of the function

$$(\sigma_1, \dots, \sigma_k) \rightarrow \#(\sigma_1(X_1) \cdot \dots \cdot \sigma_k(X_k)) \in \mathbb{Z}.$$

The previous results state precisely the Poncelet vague idea of “principle of continuity” given in 1822, extend Rouché’s theorems (see Corollary 8.1) and make possible to obtain a local version of Bezout’s theorem (see Section 9) and to construct a very simple, geometrically clear proof of Bezout’s theorem (see Section 10).

2. Branched covering. Throughout the paper a special case of the Andreotti–Stoll theorems on a branched covering will be used. To make it clear for the reader we recall some definitions and some theorems that will be useful to us.

Let X, Y be analytic subsets of a complex manifold \mathcal{N} . Let $f: X \rightarrow Y$ be a holomorphic mapping. Then f is said to be an *s-sheeted branched covering* if the following conditions are satisfied:

- (1) the mapping f is surjective, open and proper,
- (2) for every $x \in X$ the fiber $f^{-1}(f(x))$ is finite,
- (3) there exists a thin analytic subset S of Y such that f is locally biholomorphic on $X \setminus f^{-1}(S)$ and such that

$$\begin{aligned} \# f^{-1}(y) &= s < \infty & \text{if } y \in Y \setminus S, \\ \# f^{-1}(y) &< s & \text{if } y \in S. \end{aligned}$$

Such set S is called a *critical set* of the branched covering f . The set $Y \setminus S$ is called a *regular set* of the branched covering f .

Now, suppose that $\varphi: X \rightarrow Y$ is holomorphic. If X', Y' are analytic subsets of X, Y , respectively, and $\varphi|_{X'}: X' \rightarrow Y'$ is a branched covering, then we write this branched covering (X', φ, Y') or shortly $\varphi|_{X'}$.

The following is a special case of the Andreotti–Stoll results (see [1], [15]).

THEOREM 2.1. *Suppose that X is a purely k -dimensional analytic subset of a complex manifold \mathcal{N} and Y is a connected, complex manifold of dimension k . If $f: X \rightarrow Y$ is a proper, holomorphic mapping with discrete fibres, then*

- (a) f is a branched covering,
- (b) the set S' consisting of all points of X where f is not locally biholomorphic is an analytic subset of X of codimension ≥ 1 ,
- (c) the set $f(S')$ is the critical set of the branched covering f .

A point $x \in S'$ is called a *critical point* of the branched covering f . A point $x \in X \setminus S'$ is called a *regular point* of the branched covering f .

3. Continuity of intersection. In this section we recall the definition of the topology of local uniform convergence and some results that will be useful to

us. The material for this section that is taken from [18] will be presented here without proofs.

Let X be a metric space. Let \mathcal{F}_X be a family of all closed subsets of X . We endow \mathcal{F}_X with the topology \mathcal{T}_X generated by the sets

$$U(S, K) = \{F \in \mathcal{F}_X: F \cap K = \emptyset \text{ and } F \cap U \neq \emptyset \text{ for } U \in S\}$$

corresponding to all compact subsets $K \subset X$, and all finite families S of open subsets of X . We call this topology the topology of local uniform convergence. It has been proved that the space \mathcal{F}_X is metrizable. We will write $F_v \rightarrow F$ if F is the limit set of the sequence $\{F_v\}$ in the topology of local uniform convergence.

Let us recall two simple lemmas.

LEMMA 3.1. *If $F, F_v \in \mathcal{F}_X$, $v = 1, 2, \dots$, then the following statements are equivalent:*

- (a) $F_v \rightarrow F$.
- (b) *Every point $x \in F$ is a limit point of a sequence $\{x_v\}$ such that $x_v \in F_v$ for $v = 1, 2, \dots$ and for every $x \in X \setminus F$ there exists a neighbourhood U of x such that $F_v \cap U = \emptyset$ for almost all indices v .*

LEMMA 3.2. *Let X, Y be metric spaces and let $f: X \rightarrow Y$ be a continuous mapping. Then*

- (a) *if f is a proper, then the mapping*
- (*)
$$\mathcal{F}_X \ni F \rightarrow f(F) \in \mathcal{F}_Y$$

is continuous,

- (b) *if f is a homeomorphism, then the mapping (*) is a homeomorphism too.*

Let N_1, N_2 be two finite dimensional, complex vector spaces. Let Ω be an open subset of N_1 . Let $f: \Omega \rightarrow N_2$ be a mapping. Throughout this paper we will identify f with its graph. Thus, if f is continuous, then f may be regarded as an element of $\mathcal{F}_{\Omega \times N_2}$.

LEMMA 3.3. *Let $f_v, f: N_1 \supset \Omega \rightarrow N_2$ be continuous mappings for $v = 1, 2, \dots$. Then the sequence $\{f_v\}$ converges to f uniformly on compact subsets of Ω if and only if $f_v \rightarrow f$ in the topology $\mathcal{F}_{\Omega \times N_2}$.*

Proof. If the sequence $\{f_v\}$ converges to f uniformly on compact subsets of Ω , then by virtue of Lemma 3.1 the sequence $\{f_v\}$ converges to f in the topology $\mathcal{F}_{\Omega \times N_2}$.

Conversely, let us suppose $f_v \rightarrow f$ in the topology $\mathcal{F}_{\Omega \times N_2}$. With $x_0 \in \Omega$ let us associate an open, connected neighbourhood U of x_0 such that \bar{U} is compact and $\bar{U} \subset \Omega$.

For every $\varepsilon > 0$ we define a neighbourhood \mathcal{U} of f by

$$\mathcal{U} = \mathcal{U}(\{(x, y) \in U \times N_2: |f(x) - y| < \varepsilon\}, \{(x, y) \in \bar{U} \times N_2: |f(x) - y| = \varepsilon\}).$$

Since U is connected, we conclude that if $f_v \in \mathcal{U}$ then $|f_v(x) - f(x)| < \varepsilon$. Thus convergence of $\{f_v\}$ in the topology $\mathcal{T}_{\Omega \times N_2}$ implies uniform convergence of $\{f_v\}$ on compact subsets of Ω .

Let Ω be an open subset of a complex vector space N . By $\mathcal{A}_p(\Omega)$ we will denote the subspace of \mathcal{F}_Ω consisting of all purely p -dimensional analytic subsets of Ω , the empty set included for $p = 0, \dots, n$, where $n = \dim N$.

THEOREM 3.1. (see [18], Theorem 3). *Let $V_0 \in \mathcal{A}_p(\Omega)$, $W_0 \in \mathcal{A}_q(\Omega)$ and $p+q \geq n$. If $V_0 \cap W_0 \in \mathcal{A}_{p+q-n}(\Omega)$, then the mapping*

$$\cap: \mathcal{A}_p(\Omega) \times \mathcal{A}_q(\Omega) \ni (V, W) \rightarrow V \cap W \in \mathcal{F}_\Omega$$

is continuous at the point (V_0, W_0) .

4. Projective space and projective closure. We give below (Sections 4, 5 and 6) some definitions and some known or easy to verify results that will be useful in the following sections.

Let M be a complex vector space of dimension $n+1$. For $0 \neq x \in M$ define $P(x) = Cx$. For $A \subset M$ define $P(A) = \{P(x): 0 \neq x \in A\}$. Then $P(M)$ with the standard complex structure is a connected, compact, complex manifold of dimension n called the *complex projective space* of M .

The mapping $P: M \setminus \{0\} \rightarrow P(M)$ is holomorphic and denoted by the same letter P for all vector spaces. Since the topology of $P(M)$ is generated by the sets $P(\Omega)$ corresponding to all open subsets Ω of M it follows that $P(M)$ is a topological subspace of \mathcal{F}_M .

Let N be a complex vector space of dimension n . Then N may be regarded as the affine part of $P(C \times N)$ relative to the hyperplane at infinity $N_\infty = P(\{0\} \times N)$.

If A is a subset of N , then by the natural embedding $A \rightarrow P(\{1\} \times A)$ it may be regarded as the subset $P(\{1\} \times A)$ of the projective space $P(C \times N)$ and \bar{A} denote the closure of A in the topology of $P(C \times N)$.

We call the set $A_\infty = \bar{A} \cap N_\infty$ the set of points of A at infinity.

LEMMA 4.1. *Let V be an algebraic subset of N of pure dimension k . Let V_0 be the algebraic cone in N such that $V_\infty = P(\{0\} \times V_0)$. If $0 \neq t_v$, $v = 1, 2, \dots$, is a sequence of complex numbers converging to $0 \in C$, then $t_v V = \{t_v x: x \in V\}$ converges (in the topology \mathcal{T}_N) to V_0 .*

Proof. Since $\dim V = \dim \bar{V} = \dim V_0 = k$ and $E(\bar{V}) \cap (\{0\} \times N) = \{0\} \times V_0$, the sets $\{0\} \times N$ and $E(\bar{V})$ intersects properly, where $E(\bar{V}) = \bigcup_{x \in \bar{V}} (P^{-1}(x) \cup \{0\})$ is the cone in $C \times N$ defining \bar{V} . Theorem on continuity of intersection (Theorem 3.1) implies that the sequence

$$\{\{t_v\} \times N\} \cap E(\bar{V}) = \{t_v\} \times t_v V, \quad v = 1, 2, \dots$$

converges to $\{0\} \times V_0$ (in the topology $\mathcal{T}_{C \times N}$). Let K be a compact subset of

C containing all complex numbers t_v , $v = 1, 2, \dots$. Since the mapping

$$K \times N \ni (t, x) \rightarrow x \in N$$

is continuous and proper, we conclude from Lemma 3.2 that $t_v V$ converges to V_0 .

LEMMA 4.2. *Let N_j be a complex vector space of finite dimension n_j and let X_j be an algebraic subset of N_j of pure dimension d_j , $j = 1, \dots, k$. Suppose that Y_1, \dots, Y_k are algebraic cones in N_1, \dots, N_k , respectively, such that $(X_j)_\infty = P(\{0\} \times Y_j)$ for $j = 1, \dots, k$. Then*

$$(X_1 \times \dots \times X_k)_\infty = P(\{0\} \times Y_1 \times \dots \times Y_k).$$

Proof. It is easy to verify (for example, applying Lemma 3.1) that the mapping

$$\mathcal{F}_{N_1} \times \dots \times \mathcal{F}_{N_k} \ni (F_1, \dots, F_k) \rightarrow F_1 \times \dots \times F_k \in \mathcal{F}_{N_1 \times \dots \times N_k}$$

is continuous. Now, our lemma follows at once from Lemma 4.1.

5. Grassmann Manifold. Let $\Lambda_p M$ denote the p -th exterior product of M . The Grassmann cone of order p is defined by

$$\tilde{G}_p(M) = \{v_0 \wedge \dots \wedge v_p \in \Lambda_{p+1} M : v_j \in M \text{ for } j = 0, \dots, p\}.$$

The Grassmann manifold $G_p(M) = P(\tilde{G}_p(M))$ of order p to M is a compact, connected, smooth, complex, algebraic subset of $P(\Lambda_{p+1} M)$ of dimension $(n-p)(p+1)$. In the natural manner $G_p(M)$ may be regarded as:

- 1° a submanifold of $P(\Lambda_{p+1} M)$,
- 2° a subspace of \mathcal{F}_M consisting of all $(p+1)$ -dimensional linear subspace of M ,
- 3° a subspace of $\mathcal{F}_{P(M)}$ consisting of all p -planes in $P(M)$.

Let N be a complex, n -dimensional vector space. The set of all k -dimensional affine subspaces of N is denoted by $\text{Aff}_k(N)$. We endow $\text{Aff}_k(N)$ with the topology of local uniform convergence. If we regard N as the affine part of $P(C \times N)$ relative to the hyperplane at infinity $N_\infty = P(\{0\} \times N)$, then the mapping

$$\text{Aff}_k(N) \ni L \rightarrow \bar{L} \in G_k(C \times N) \setminus G_k(\{0\} \times N)$$

is a homeomorphism.

For all $p \in \{1, 2, \dots, n\}$ the set $F_p(M)$ defined by

$$F_p(M) = \{(a, \xi) \in P(M) \times G_p(M) : a \in \xi\}$$

is a compact, connected, complex submanifold of $P(M) \times G_p(M)$ of dimension $(p+1)(n-p) + p$.

We shall use the two natural projection p_1, p_2 defined by

$$p_1: F_p(M) \ni (a, \xi) \rightarrow a \in P(M)$$

and

$$p_2: F_p(M) \ni (a, \xi) \rightarrow \xi \in G_p(M).$$

If $f: M \rightarrow M$ is a linear automorphism of M we clearly get an induced automorphism $F_p(f)$ of $F_p(M)$ defined by

$$F_p(f)(a, \xi) = (f(a), f(\xi)).$$

Notice that for every $(a, \xi), (b, \eta) \in F_p(M)$ there exists a linear automorphism f of M such that $F_p(f)(a, \xi) = (b, \eta)$. Therefore we clearly obtain that $\text{rank } dp_j$ is constant for $j = 1, 2$. Thus by Sard's theorem it is maximal. Then we have

LEMMA 5.1. *Projection p_1, p_2 are holomorphic, proper, surjective submersions.*

6. Intersection multiplicity. In this section we present a classical approach to the definition of the intersection multiplicity of $(k+1)$ -tuple of locally analytic subsets of a complex manifold. Our approach is like that of Draper [4].

Since the contents of this section is well-known, we present it here without proofs.

Let N be an n -dimensional, complex vector space. Let $V = (V_0, \dots, V_k)$ be a $(k+1)$ tuple of pure dimensional analytic subsets of an open neighbourhood of $0 \in N$ of the dimension d_0, \dots, d_k , respectively. Suppose that 0 is an isolated point of the intersection $V_0 \cap \dots \cap V_k$ and that the sets of V meet properly at 0 , i.e., $d_0 + \dots + d_k = kn$.

Let us consider in the product $N \times N^k = N \times \dots \times N$ the linear automorphism

$$f: N \times N^k \ni (x, y) \rightarrow (x, y - y_1, \dots, x - y_k) \in N \times N^k,$$

where $y = (y_1, \dots, y_k) \in N^k$.

Since 0 is an isolated point of the intersection $(N \times \{0\}) \cap f^{-1}(V_0 \times \dots \times V_k)$, there exist open connected neighbourhoods Ω, Ω_k of the point 0 in N and N^k , respectively, such that

- (i) $\bar{\Omega}, \bar{\Omega}_k$ are compact subsets of N, N^k , respectively,
- (ii) $(\bar{\Omega} \times \{0\}) \cap f^{-1}(V_0 \times \dots \times V_k) = \{0\}$,
- (iii) $(\partial\Omega \times \bar{\Omega}_k) \cap f^{-1}(V_0 \times \dots \times V_k) = \emptyset$,
- (iv) $f^{-1}(V_0 \times \dots \times V_k)$ is analytic in an open subset of N containing $\bar{\Omega} \times \bar{\Omega}_k$.

Let $\pi: N \times N^k \ni (x, y) \rightarrow y \in N^k$ be the natural projection. The set W

$= (\Omega \times \Omega_k) \cap f^{-1}(V_0 \times \dots \times V_k)$ is analytic in $\Omega \times \Omega_k$ and it follows from (ii) and (iii) that the restriction $\pi|V$ is proper. Since W is a purely nk -dimensional analytic subset of $\Omega \times \Omega_k$ it follows from Theorem 2.1 that $\pi|W: W \rightarrow \Omega_k$ is a μ -sheeted branched covering. The number μ is called the *intersection multiplicity* of V_0, \dots, V_k at 0, is written

$$\mu = i(V_0 \cdot \dots \cdot V_k; 0)$$

and it is not depend on Ω .

If V_0, \dots, V_k meet properly at an isolated point $P \neq 0$, then we define

$$i(V_0 \cdot \dots \cdot V_k; P) = i((V_0 - P) \cdot \dots \cdot (V_k - P); 0)$$

and we call $i(V_0 \cdot \dots \cdot V_k; P)$ the intersection multiplicity of V_0, \dots, V_k at P .

PROPERTY 6.1. *The intersection multiplicity is a local property, invariant with respect to translations.*

Let N_j be a complex vector space of dimension d_j , $j = 1, 2$. Let Ω_j be an open, connected subset of N_j . Let V be an analytic subset of $\Omega_1 \times \Omega_2$ of pure dimension d_1 such that the natural projection π_1 onto Ω_1 given by

$$\pi_1: V \ni (x, y) \rightarrow x \in \Omega_1$$

is proper. Then, by Theorem 2.1, π_1 is a μ -sheeted branched covering and we have

PROPERTY 6.2. *If a point $(a_1, a_2) \in \Omega_1 \times \Omega_2$ is such that*

$$V \cap (\{a_1\} \times \Omega_2) = \{(a_1, a_2)\}$$

then

$$i(V \cdot (\{a_1\} \times \Omega_2); (a_1, a_2)) = \mu.$$

Hence, $i(V \cdot (\{a_1\} \times \Omega_2); (a_1, a_2)) = 1$ if and only if V meets $\{a_1\} \times \Omega_2$ transversally at (a_1, a_2) .

Let $\Delta = \{(x_0, \dots, x_k) \in N^{k+1}: x_0 = x_1 = \dots = x_k\}$. Denote by δ the canonical embedding of N onto Δ , i.e., $\delta(x) = (x, \dots, x)$, and observe that $x \in V_0 \cap \dots \cap V_k$ if and only if $\delta(x) \in (V_0 \times \dots \times V_k) \cap \Delta$.

From Property 6.2 follows immediately next property of intersection multiplicity that will be useful in the sequel.

PROPERTY 6.3. *If local analytic sets V_0, \dots, V_k meet properly at an isolated point $a \in N$, then*

$$i(V_0 \cdot \dots \cdot V_k; a) = i((V_0 \times \dots \times V_k) \cdot \Delta; \delta(a)).$$

Hence $i(V_0 \cdot \dots \cdot V_k; a) = 1$ if and only if the sets V_0, \dots, V_k meet transversally at a .

LEMMA 6.1. *If $y = (y_1, \dots, y_k) \in \Omega_k$, then y is a point of the critical set of the branched covering $\pi|W$ if and only if there exists a point $x \in \Omega \cap V_0 \cap$*

$\cap(V_1 + y_1) \cap \dots \cap (V_k + y_k)$ such that $V_0, V_1 + y_1, \dots, V_k + y_k$ fail to meet transversally at x .

Remark 6.1. If P is an isolated point of proper intersection of V_0, \dots, V_k , then we may perturb them slightly to sets V'_0, \dots, V'_k having a finite number of transverse intersections near P . This number is just the intersection multiplicity at P .

LEMMA 6.2. *The intersection multiplicity of V is a biholomorphic invariant. More precisely, if U and U' are open subsets of N such that $P \in U$ and $\sigma: U \rightarrow U'$ is a biholomorphic mapping, then*

$$i(V_0 \cdot \dots \cdot V_k; P) = i(\sigma(V_0) \cdot \dots \cdot \sigma(V_k); \sigma(P)).$$

Let \mathcal{N} be a complex manifold of dimension n . Locally \mathcal{N} may be regarded as an open, connected subset of a complex vector space N of dimension n . Then Lemma 6.2 allows us to define (in a natural manner) the intersection multiplicity of locally analytic subsets of \mathcal{N} .

The total number of intersection $^*(V_0 \cdot \dots \cdot V_k)$ of locally analytic subsets of \mathcal{N} meeting properly in a finite set of points of \mathcal{N} is defined by

$$^*(V_0 \cdot \dots \cdot V_k) = \sum_{x \in V_0 \cap \dots \cap V_k} i(V_0 \cdot \dots \cdot V_k; x).$$

If V_0, \dots, V_k meet properly in a finite set of points on an open subset D of \mathcal{N} , then the number $^*(V_0 \cdot \dots \cdot V_k)|D$ defined by

$$^*(V_0 \cdot \dots \cdot V_k)|D = \sum_{x \in V_0 \cap \dots \cap V_k \cap D} i(V_0 \cdot \dots \cdot V_k; x)$$

is called the *total number of intersection of locally analytic subsets on D* .

Let $\Delta = \{(x_0, \dots, x_k) \in \mathcal{N}^{k+1} : x_0 = \dots = x_k\}$. Denote by $\delta: \mathcal{N} \ni x \rightarrow (x, \dots, x) \in \Delta$ the canonical embedding of \mathcal{N} onto Δ .

Since the intersection multiplicity is a biholomorphic invariant, Properties 6.2–6.3 imply the following

PROPERTY 6.4. *Let V_0, \dots, V_k be a locally analytic subsets of \mathcal{N} . Then*

(a) *if V_0, \dots, V_k meet properly at an isolated point $a \in \mathcal{N}$, then*

$$i(V_0 \cdot \dots \cdot V_k; a) = i((V_0 \times \dots \times V_k) \cdot \Delta; \delta(a)),$$

(b) *if V_0, \dots, V_k meet properly in a finite set of points, then*

$$^*(V_0 \cdot \dots \cdot V_k) = ^*((V_0 \times \dots \times V_k) \cdot \Delta),$$

(c) *$i(V_0 \cdot \dots \cdot V_k; a) = 1$ if and only if the sets V_0, \dots, V_k meet transversally at a .*

(d) *if $\sigma: \{0, \dots, k\} \rightarrow \{0, \dots, k\}$ is a permutation of the set $\{0, \dots, k\}$, then*

$$^*(V_0 \cdot \dots \cdot V_k) = ^*(V_{\sigma(0)} \cdot \dots \cdot V_{\sigma(k)}).$$

7. Continuity of intersection with linear subspaces. Let M be a complex vector space of dimension $n+1$. If X is a non-empty, purely k -dimensional, locally analytic subset of projective space $\mathbf{P}(M)$, define

$$G = G(X) = \{\xi \in G_{n-k}(M) : \xi \cap \partial X = \emptyset\},$$

where $\partial X = \bar{X} \setminus X$.

Now, we are in a position to state the main theorem.

THEOREM 7.1 (see [4], Th. 5.4). *Suppose that G is a non-empty subset of $G_{n-k}(M)$. Then the following statements hold:*

1. $G(X)$ is an open subset of $G_{n-k}(M)$.
2. (MAXIMUM PRINCIPLE.) *If G' is a connected component of G and there exists $\xi' \in G'$ such that $\xi' \cap X \neq \emptyset$, then $\xi \cap X = \emptyset$ for every $\xi \in G'$.*
3. *If there exists $\xi \in G'$ such that $\xi \cap X \neq \emptyset$, then*
 - (i) *the set A' defined by*

$$A' = \{\xi \in G' : \# \xi \cap X = \infty\}$$

is a proper analytic subset of G' and

- (ii) *the mapping*

$$G' \setminus A' \ni \xi \rightarrow \#(\xi \cdot X) \in \mathbf{Z}$$

is constant.

Proof. Let us take, for $p = n-k$ (see Section 5), the set

$$F_p(M) = \{(a, \xi) \in \mathbf{P}(M) \times G_p(M) : a \in \xi\}$$

and the projections $p_1: F_p(M) \ni (a, \xi) \rightarrow a \in \mathbf{P}(M)$ and $p_2: F_p(M) \ni (a, \xi) \rightarrow \xi \in G_p(M)$. The mapping p_2 is proper. Therefore, the set E defined by

$$E = \{\xi \in G_p(M) : \xi \cap \partial X \neq \emptyset = p_2(p_1^{-1}(\partial X))\}$$

is a compact subset of $G_p(M)$. Since $G = G_p(M) \setminus E$, the set G is open and by assumption $G \neq \emptyset$.

Let G' be a component of G . Denote

$$F'_p(M) = F_p(M) \cap (\mathbf{P}(M) \times G') = \{(a, \xi) \in F_p(M) : \xi \in G'\},$$

$$p'_1 = p_1|F'_p(M): F'_p(M) \rightarrow \mathbf{P}(M) \setminus \partial X,$$

$$Y = (p'_1)^{-1}(X).$$

Since X is a purely k -dimensional analytic subset of $\mathbf{P}(M) \setminus \partial X$, and since p'_1 is a holomorphic submersion (restriction of the submersion p_1), the set Y must be a purely $\dim G_p(M)$ analytic subset of $F'_p(M)$. Let us observe that $F'_p(M)$ is a closed submanifold of $\mathbf{P}(M) \times G'$. Hence Y is a purely $\dim G_p(M)$ -dimensional analytic subset of $\mathbf{P}(M) \times G'$. Since $p_2|Y: Y \rightarrow G'$ is proper, the set $p_2(Y)$ is an analytic subset of G' .

To prove 2, let us suppose that $\emptyset \neq p_2(Y) \subsetneq G'$. Let $\xi_0 \in G'$ be such that $\xi_0 \cap X \neq \emptyset$. Let us fix $a \in \xi_0 \cap X$.

If a is an isolated point of $\xi_0 \cap X$, then by the theorem on continuity of intersection (Theorem 3.1) ξ_0 is an interior point of $p_2(Y)$. Thus $p_2(Y) = G'$.

If a is not an isolated point of $\xi_0 \cap X$, then by the standard local analysis of analytic sets (see e.g. Chapter 7 of [20], especially Lemma 10C of Section 10), for every neighbourhood U of ξ_0 there exists $\xi \in U$ such that a is an isolated point of $\xi \cap X$. Since G' is open, the above shows that $p_2(Y) = G'$, and this contradicts the assumption.

To prove 3 let us observe that 2 implies $p_2(Y) = G'$. Let us fix $\xi \in G'$. By Chow's theorem $\xi \cap X$ is an algebraic subset of ξ and since ξ is algebraic in $P(M)$, so is $\xi \cap X$. Hence

$$^*(\xi \cap X) < \infty \quad \text{or} \quad \dim(\xi \cap X) \geq 1.$$

Moreover, since every $\xi \in G_{n-1}(M)$ must intersect every infinite algebraic subset of $P(M)$ (cf. [7], p. 182), we have

$$A' = \bigcap_{H \in G_{n-1}(M)} p_2(Y \cap (H \times G')).$$

Thus A' is an analytic subset of G' , and part (i) of 3 is proved.

Let us put $G'' = G' \setminus A'$, $Y'' = p_2^{-1}(G'') \cap Y$. We see that the triplet (Y'', p_2, G'') is an s -sheeted branched covering. Given $\xi \in G''$ such that ξ and X meet transversally at points $a_1, \dots, a_r \in P(M)$ we see that

$$r = \sum_{j=1}^r i(\xi \cdot X; a_j) = ^*(\xi \cdot X).$$

We show next ξ is a regular point for the branched covering (Y'', p_2, G'') , and hence $r = s$.

Given any $a = a_i \in \xi \cap X$, the point (a, ξ) is regular on Y'' because p_1 is a submersion and because a is regular on X . Since p_1 and p_2 are submersions, we calculate that

$$(*) \quad \text{Ker } d_{(a, \xi)}(p_2|Y'') = (T_a X \cap T_a \xi) \times \{0\}.$$

Hence $\text{Ker } d_{(a, \xi)}(p_2|Y'') = \{0\}$, because ξ and X meet transversally at a .

Thus, $d_{(a, \xi)} p_2|Y'': T_{(a, \xi)} Y'' \rightarrow T_\xi G_p(M)$ is an isomorphic mapping between tangent spaces. This implies that (a, ξ) is a regular point of the branched covering (Y'', p_2, G'') .

Let us fix an arbitrary $\xi \in G''$ and let $\{b^1, \dots, b^l\} = \xi \cap X$. Choose a complex, vector hyperplane H in M so that $b^j \notin P(H)$ for $j = 1, \dots, l$, and let L be a linear complement of H , i.e., $\dim L = 1$ and $L + H = M$. Let $0 \neq c \in L$ and let

$$\varphi: H \ni h \rightarrow P(c+h) \in P(M) \setminus P(H)$$

be a natural affine representation in $P(M)$. Then the sets $\xi_a = \varphi^{-1}(\xi)$, X_a

$= \varphi^{-1}(X)$, $b_a^j = \varphi^{-1}(b^j)$, $j = 1, \dots, l$, are affine parts of ξ , X , b^j , respectively.

Suppose that η is a complement of ξ_a in H . Then $H = \eta + \xi_a$, with $\dim \eta = k = \dim X_a$, and $\{b_a^1, \dots, b_a^l\} = X_a \cap \xi_a$. We may therefore find open, connected neighbourhoods D' of $0 \in \eta$ and D_1'', \dots, D_l'' of b_a^1, \dots, b_a^l in ξ_a , respectively, such that

1. $\overline{D'}, \overline{D_1''}, \dots, \overline{D_l''}$ are compact,
2. $\overline{D_i''} \cap \overline{D_j''} = \emptyset$ for $i \neq j$,
3. $(\overline{D'} + (\bigcup_{j=1}^l \partial D_j'')) \cap X_a = \emptyset$,
4. if $U_j = D' + D_j''$, then $\overline{U_j} \cap \partial X_a = \emptyset$ for $j = 1, \dots, l$.

It follows from the local analysis of analytic sets that there exists a sequence $\eta_v \in \eta$, $v = 1, 2, \dots$, such that

5. $\eta_v \in D'$ for $v = 1, 2, \dots$ and $\eta_v \rightarrow 0$ when $v \rightarrow \infty$,
6. $\eta_v + \xi_a$ meet transversally X in $U = \bigcup_{j=1}^l U_j$ exactly at $\sum_{a \in \xi \cap X} i(X \cdot \xi; a)$

points.

Setting $\xi_v = \varphi(\eta_v + \xi_a) \subset P(M)$ we obtain a sequence $\{\xi_v\}$ with limit set ξ in the topology $\mathcal{T}_{P(M)}$. It suffices to show that $\xi_v \cap X \subset \bigcup_{j=1}^l (U_j)$ for almost all $v \in N$. Suppose on the contrary that for every $v \in N$ there exists $x_v \in (\xi_v \cap X) \setminus \bigcup_{j=1}^l \varphi(U_j)$. Having found sequence $\{x_v\}$, we may assume that x_v tends to a limit point $x_0 \in \bar{X} \setminus \bigcup_{j=1}^l \varphi(U_j)$. Hence it would follow that $x_0 \in \partial X \cap \xi$, which contradicts the assumption.

Now, we shall establish an affine version of Theorem 7.1.

THEOREM 7.2. *Let N be a complex vector space of dimension n . Let X be a bounded, purely k -dimensional, locally analytic subset of N . If*

$$D = \{\xi \in \text{Aff}_{n-k}(N): \xi \cap \partial X = \emptyset\},$$

then D is a non-empty, open subset of $\text{Aff}_{n-k}(N)$ and if D' is a component of D , then the function

$$D' \ni \xi \rightarrow \#(\xi \cdot X)$$

is constant in D' .

Proof. Let $M = \mathbb{C} \times N$. Then N may be regarded as an affine part of $P(M) = \bar{N}$. The mapping

$$\psi: \text{Aff}_{n-k}(N) \ni \xi \rightarrow \bar{\xi} \in G_{n-k}(M) \setminus G_{n-k}(\{0\} \times N)$$

is a homeomorphic mapping. Hence $\psi(D') \subset G'$ is an open, connected subset

of G . Then there exists a component G' of G such that $\psi(D') \subset G'$. Moreover, in this case the set A' is empty, and our result follows from Theorem 7.1.

COROLLARY 7.1. *Let X be an irreducible, locally analytic subset of $\mathbf{P}(M)$ of the dimension one. Let the set Ω_0 defined by*

$$\Omega_0 = \{\xi \in G_{n-1}(M) : \xi \cap \bar{X} = \emptyset\}$$

be a non-empty subset of $G_{n-1}(M)$. If $\xi_0 \in \bar{\Omega}_0$ and $\xi \cap X \neq \emptyset$, then $X \subset \xi_0$.

Proof. Suppose that a point $a \in \xi_0 \cap X$ is an isolated point of $\xi_0 \cap X$. From Theorem 7.1 there would exist a neighbourhood U of ξ_0 such that all subspaces ξ belonging to U intersect X and this contradicts $\xi_0 \in \partial\Omega_0$. Therefore, every point $a \in \xi_0 \cap X$ is not isolated. Hence $\dim \xi_0 \cap X \geq 1$ at every point and since the set X is irreducible, we have $X \subset \xi_0$.

COROLLARY 7.2 (MAXIMUM PRINCIPLE, cf. [19], p. 55–56). *Let U be an open, connected subset of \mathbf{C}^n . Let $f: U \rightarrow \mathbf{C}^m$ be a holomorphic mapping. If there exists a hyperplane H in \mathbf{C}^m such that $f(U) \cap H \neq \emptyset$ and for every neighbourhood $\mathcal{U} \subset \text{Aff}_{m-1}(\mathbf{C}^m)$ of H there exists a hyperplane $H' \in \mathcal{U}$ such that $f(U) \cap H' = \emptyset$, then $f(U) \subset H$.*

Proof. Let a be a point of U such that $f(a) \in H$. If for every affine complex line l through a we have $f(l \cap U) \subset H$, then $f(U) \subset H$. Thus, it is easy to see that we may assume U to be a bounded, connected, open subset of \mathbf{C} . Then f is a local, irreducible analytic subset of $\mathbf{C} \times \mathbf{C}^m$ of dimension one such that $(a, f(a)) = f \cap (\mathbf{C} \times H)$. Hence $(a, f(a))$ is an isolated point of $f \cap (\mathbf{C} \times H)$ or $f \cap (\mathbf{C} \times H) = f$. If $(a, f(a))$ is an isolated point of $f \cap (\mathbf{C} \times H)$, then in virtue of Theorem 7.2 there exists a neighbourhood \mathcal{U} of H such that $f \cap (\mathbf{C} \times H') \neq \emptyset$ for $H' \in \mathcal{U}$ and this contradicts our assumption.

8. Principle of continuity. We usually encounter Rouché's theorem as theorem on equality of number of zero points of f and $f+g$, where f, g are holomorphic in a bounded domain $D \subset \mathbf{C}^n$ satisfying a special boundary condition. There exists a few proofs of the Rouché theorem. One of them, the most popular, dates from the beginning of our century (see e.g. [3] and [2]). The idea of this proof consists in observation that the family $(f + \lambda g)_{\lambda \in [0,1]}$ has constant number of zero points (counted with its multiplicities). In this section this idea will be extended to the case of intersection of families of locally analytic subsets of a finite dimensional complex vector space. Theorems that will be obtained will contain "almost all" versions of Rouché's theorems and can be used to prove of Hurvitz' and Bezout's type theorems.

We begin with two lemmas.

LEMMA 8.1. *Let N_1, N_2 be complex vector spaces of dimensions n_1, n_2 respectively. Let X be a locally analytic subset of $N_1 \times N_2$ of pure dimension n_2 such that $(\partial X) \cap (N_1 \times \{0\}) = \emptyset$ and $\pi_1(X)$ is bounded, where $\pi_1: N_1 \times N_2 \ni (x, y) \rightarrow x \in N_1$ is the natural projection.*

If Ω is an open subset of N_1 such that $\Omega \supset X \cap (N_1 \times \{0\})$, then there exists a neighbourhood U of 0 in N_2 such that if $f: \Omega \rightarrow N_2$ is holomorphic in Ω and $f(\Omega) \subset U$, then

$$*(X \cdot f) = *(X \cdot (N_1 \times \{0\})).$$

Proof. Since $X \cap (N_1 \times \{0\}) = \bar{X} \cap (N_1 \times \{0\}) \subset \overline{\pi_1(X)}$, the set $X \cap (N_1 \times \{0\})$ is a compact analytic subset of N_1 . Hence $X \cap (N_1 \times \{0\})$ is finite.

Let $\pi_2: N_1 \times N_2 \rightarrow N_2$ be the natural projection onto N_2 . Since ∂X is a closed subset of $\pi_1(X) \times N_2$ and $\pi_1(X)$ is compact, it follows that $\pi_2(\partial X)$ is a closed subset of N_2 . Since $0 \notin \pi_2(\partial X)$, this implies that there exists an open neighbourhood U' of $0 \in N_2$ such that $(N_1 \times U') \cap \partial X = \emptyset$.

Let $X \cap (N_1 \times \{0\}) = \{x_1, \dots, x_k\}$. Let $B_1(x_j, r_j) \subset \Omega$ be an open ball in N_1 of radius $r_j > 0$ about x_j such that $\overline{B_1(x_i, r_i)} \cap \overline{B_1(x_j, r_j)} = \emptyset$ for $i \neq j$ and $i, j = 1, \dots, k$. Then there exists a ball $B_2(0, r)$ in N_2 such that $B_2(0, r) \subset U'$ and $(\partial B_1(x_i, r_i) \times \overline{B_2(0, r)}) \cap X = \emptyset$ for $i = 1, \dots, k$ and obviously $X \cap (N_1 \times B_2(0, r)) = X \cap (\bigcup_{i=1}^k B_1(x_i, r_i) \times B_2(0, r))$.

Let $U = B_2(0, r)$. We can assume that $\Omega = B_1(x_1, r_1)$. Since the intersection number is invariant with respect to translation, we can also assume that $x_1 = 0$. Let Δ' be the linear subspace of $(N_1 \times N_2)^2$ defined by

$$\Delta' = \{(x_1, tx_2), (tx_1, x_2) : x_i \in N_i, i = 1, 2\}.$$

Of course, $\Delta^1 = \Delta$ is the diagonal in $(N_1 \times N_2)^2$ and

$$*(X \cdot f) = *((X \times f) \cdot \Delta^1).$$

On the other hand,

$$*((X \times f) \cdot \Delta^0) = *(X \cdot (N_1 \times \{0\})),$$

because f meet transversally $\{0\} \times N_2$, exactly at one point set and it is easy to see that

$$\partial(X \times f) \cap \Delta' = \emptyset \quad \text{for } t \in [0, 1].$$

Now, Lemma 8.1 is a simple consequence of Theorem 7.2.

LEMMA 8.2. Let X be a bounded, purely k -dimensional analytic subset of an open set Ω of an n -dimensional, complex vector space N . Let Y be a purely $(n-k)$ -dimensional, locally analytic subset of N such that $\bar{X} \cap \bar{Y} = X \cap Y$. Let $f: \Omega \rightarrow f(\Omega) \subset N$ be a biholomorphic mapping. Then there exists $\varepsilon > 0$ such that if $|f(x) - x| < \varepsilon$ for $x \in \Omega$, then

$$(a) \quad \overline{f(X)} \cap \bar{Y} = f(X) \cap Y$$

and

$$(b) \quad *(f(X) \cdot Y) = *(X \cdot Y).$$

Proof. Let us fix $\varepsilon > 0$. Given a biholomorphic mapping f such that $|f(x) - x| < \varepsilon$ for $x \in \Omega$, we obtain

$$\partial f(X) \subset \bigcup_{x \in X} \overline{B(x, \varepsilon)} \quad \text{and} \quad \overline{f(X)} \subset \bigcup_{x \in X} \overline{B(x, \varepsilon)},$$

where $B(x, \varepsilon)$ is a ball of radius ε about x . Since ∂X and \bar{X} are compact subsets of N and ∂Y is closed, it is clear that there exists $\varepsilon > 0$ such that if $|f(x) - x| < \varepsilon$ for $x \in \Omega$, then $\overline{f(X)} \cap \bar{Y} = f(X) \cap Y$.

Let us consider the linear biholomorphism Φ of $N \times N$ given by

$$\Phi(x, y) = (x, y - x).$$

Writing $Z = \Phi(X \times Y)$, we have

$$\partial Z \cap (N \times \{0\}) = \Phi(\partial(X \times Y) \cap \Delta) = \emptyset,$$

where $\Delta = \{(x, x) : x \in N\}$ is the diagonal in $N \times N$. By Property 6.4, since the total number of intersection is a biholomorphic invariant,

$$\#(X \cdot Y) = \#((X \times Y) \cdot \Delta) = \#(Z \cdot (N \times \{0\})).$$

Also $\pi_1(Z) = X$ is bounded and by Lemma 8.1 there exists $\varepsilon > 0$ such that if $f: \Omega \rightarrow f(\Omega) \subset N$ is a biholomorphism with $|f(x) - x| < \varepsilon$ for $x \in \Omega$, then

$$(*) \quad (Z \cdot (N \times \{0\})) = \#(Z \cdot (f - \text{id})) = \#(X \cdot Y),$$

where id is the identity mapping on Ω .

Next, let $\Phi_1: \Omega \times N \rightarrow f(\Omega) \subset N$ be the biholomorphism defined by

$$\Phi_1(x, y) = (f(x), x + y).$$

We see that

$$\Phi_1(f - \text{id}) = \{(f(x), f(x)) : x \in \Omega\} = \Delta \cap (f(\Omega) \times f(\Omega)) = \Delta_{f(\Omega)}$$

and

$$\Phi_1(Z) = f(X) \times Y.$$

Thus

$$\#((f - \text{id}) \cdot Z) = \#(\Delta_{f(\Omega)} \cdot (f(X) \times Y)) = \#(\Delta(f(X) \times Y)) = \#(f(X) \cdot Y)$$

and by (*) we end the proof of Lemma 8.2.

Let X be a purely k -dimensional analytic subset of an open set $\Omega \subset N$. Let Y be a purely $(n - k)$ -dimensional, locally analytic subset of N . Let $\mathcal{B} = \mathcal{B}(\Omega)$ be the set of all biholomorphic mapping $f: \Omega \rightarrow f(\Omega) \subset N$ such that $f(\Omega)$ is bounded. We endow $\mathcal{B}(\Omega)$ with the topology of uniform convergence. By $\mathcal{B}^* = \mathcal{B}^*(\Omega; X, Y)$ we denote the subspace of \mathcal{B} consisting of all $f \in \mathcal{B}$ such that

$$\overline{f(X)} \cap \bar{Y} = f(X) \cap Y.$$

Now, the Rouché theorem can be proved.

THEOREM 8.1. *The set \mathcal{B}^* is open in \mathcal{B} and the function*

$$\mathcal{B}^* \ni f \rightarrow {}^*(f(X) \cdot Y) \in \mathbb{Z}$$

is continuous.

Proof. $\mathcal{B}(\Omega) = \emptyset$ is trivial. If $\mathcal{B} \neq \emptyset$ then $\mathcal{B}^* \neq \emptyset$. Suppose that $f_0 \in \mathcal{B}^*$ and consider the sets $X_1 = f_0(X)$, $\Omega_1 = f_0(\Omega)$ and Y . By Lemma 8.2 there exists $\varepsilon > 0$ such that if $f_1: \Omega_1 \rightarrow f(\Omega_1) \subset N$ is a biholomorphic mapping with $|f_1(x) - x| < \varepsilon$ for $x \in \Omega_1$, then $\overline{f_1(X_1)} \cap \bar{Y} = f_1(X_1) \cap Y$ and

$${}^*(f_1(X_1) \cdot Y) = {}^*(X_1 \cdot Y).$$

Clearly, $|f(x) - f_0(x)| < \varepsilon$ for $x \in \Omega$ if and only if $|f \circ f_0^{-1}(x) - x| < \varepsilon$ for $x \in \Omega_1$. Summarizing, if $f \in \mathcal{B}$ satisfies the assumption

$$|f(x) - f_0(x)| < \varepsilon \quad \text{for } x \in \Omega$$

then setting $f_1 = f \circ f_0^{-1}$ we have

$$\overline{f(X)} \cap \bar{Y} = \overline{f_1(X_1)} \cap \bar{Y} = f_1(X_1) \cap Y = f(X) \cap Y$$

and

$${}^*(\overline{f(X)} \cdot \bar{Y}) = {}^*(X_1 \cdot Y) = {}^*(f_0(X) \cdot Y)$$

and so the result is proved.

We will now construct an example showing that all assumptions of the preceding theorem are indispensable.

EXAMPLE 8.1. Let $\Omega = \{z \in \mathbb{C} : |z| < 1\}$. Let us consider the sequence $\{f_v\}$ of biholomorphic mappings

$$f_v: \Omega \times \mathbb{C} \ni (z, t) \rightarrow (z, z^v + t) \in \Omega \times \mathbb{C}$$

which is locally uniformly convergent to the identity mapping

$$f: \Omega \times \mathbb{C} \ni (z, t) \rightarrow (z, t) \in \Omega \times \mathbb{C}.$$

Setting $X = \Omega \times \{0\}$ and $Y = \Omega \times \{\frac{1}{2}\}$, we have

$${}^*(X \cdot Y) = 0 \quad \text{and} \quad {}^*(f_v(X) \cdot Y) = v \rightarrow \infty \quad \text{when } v \rightarrow \infty.$$

PROPOSITION 8.1. *Let Ω_j be an open subset of N and let X_j be an analytic subset of Ω_j of pure dimension d_j , $j = 1, \dots, k$. Suppose that $d_1 + \dots + d_k = (k-1)n$. Let T be a connected topological space and let*

$$T \ni t \rightarrow \sigma_i^t \in \mathcal{B}(\Omega_i), \quad i = 1, \dots, k,$$

be continuous mappings. If for every $t \in T$

$$\bigcap_{j=1}^k \overline{\sigma_j^t(X_j)} = \bigcap_{j=1}^k \sigma_j^t(X_j),$$

then the function

$$T \ni t \rightarrow {}^{\#}(\sigma_1^t(X_1) \cdot \dots \cdot \sigma_k^t(X_k))$$

is constant.

Proof. Let us set $X = X_1 \times \dots \times X_k$, $\Omega = \Omega_1 \times \dots \times \Omega_k$, $f_t = \sigma_1^t \times \dots \times \sigma_k^t$ and $Y = \Delta_k = \{(x, \dots, x) \in N^k: x \in N\}$. Under this notation the assumptions of Proposition 8.1 become

$$\overline{f_t(X)} \cap \bar{Y} = f_t(X) \cap Y$$

and for every $t \in T$, $f_t \in \mathcal{B}^*(\Omega; X, Y)$. Now, by Theorem 8.1, the function

$$T \ni t \rightarrow {}^{\#}(f_t(X) \cdot Y) \in \mathbb{Z}$$

is continuous and since T is connected, it is constant. Also, by Property 6.4,

$${}^{\#}(f_t(X) \cdot Y) = {}^{\#}(\sigma_1^t(X_1) \cdot \dots \cdot \sigma_k^t(X_k))$$

and so the result is proved.

Let Ω be an open subset of N . Let $\mathcal{A}(\Omega)$ be the set of all biholomorphic mappings $f: \Omega \rightarrow f(\Omega) \subset N$. We endow $\mathcal{A}(\Omega)$ with the topology of local uniform convergence.

LEMMA 8.3. Suppose that $f_0 \in \mathcal{A}(\Omega)$. If U is an open, connected subset of $f_0(\Omega)$ such that \bar{U} is compact and $\bar{U} \subset f_0(\Omega)$, then there exists a neighbourhood \mathcal{U} of f_0 in $\mathcal{A}(\Omega)$ such that

(a) if $f \in \mathcal{U}$ then $f(\Omega) \supset \bar{U}$

and

(b) the mapping

$$\varphi: \mathcal{U} \ni f \rightarrow f^{-1}|_U \in \mathcal{A}(U)$$

is continuous.

Proof. The set $D = f^{-1}(U)$ is open and connected. Since $\bar{D} \subset \Omega$ and \bar{D} is compact, we can find an open subset $D' \subset \Omega$ such that $\bar{D} \subset D' \subset \bar{D}' \subset \Omega$ and \bar{D}' is compact.

The set \mathcal{U} defined by

$$\mathcal{U} = \{f \in \mathcal{A}(\Omega): f(D) \cap U \neq \emptyset \text{ and } f(\partial D') \cap \bar{U} = \emptyset\}$$

is a neighbourhood of f_0 in $\mathcal{A}(\Omega)$. Making use of the facts that $f \cap (D' \times U)$ is a pure n -dimensional analytic subset of $D' \times U$ and for any $f \in \mathcal{U}$ the projection

$$f \cap (D' \times U) \ni (x, y) \rightarrow y \in U$$

is proper, we now conclude (e.g. by virtue of Theorem 2.1) that $f(D') \supset U$. Hence $f(\Omega) \supset \bar{U}$ whenever $f \in \mathcal{U}$.

To prove the continuity let us observe that $\varphi = \varphi_2 \circ \varphi_1$, where φ_1, φ_2

are defined by

$$\begin{aligned}\varphi_1: \mathcal{U} \ni f &\rightarrow f \cap (D' \times U) \in \mathcal{F}_{D' \times U}, \\ \varphi_2: \varphi_1(\mathcal{U}) \ni f \cap (D' \times U) &\rightarrow f^{-1}|U \in \mathcal{A}(\mathcal{U}) \subset \mathcal{F}_{U \times N}\end{aligned}$$

and that by virtue of Lemma 3.3 the mappings φ_1, φ_2 are continuous.

Let X be a purely k -dimensional analytic subset of an open, connected subset $\Omega \subset N$. Let Y be a bounded, purely $(n-k)$ -dimensional analytic subset of N . Let $\mathcal{A}_\Omega(X, Y)$ be the subspace of $\mathcal{A}(\Omega)$ defined by

$$\mathcal{A}_\Omega(X, Y) = \{f \in \mathcal{A}(\Omega): f(\Omega) \supset \bar{Y} \text{ and } \partial Y \cap f(X) = \emptyset\}.$$

THEOREM 8.2. *The function*

$$\mathcal{A}_\Omega(X, Y) \ni f \rightarrow * (f(X) \cdot Y) \in Z$$

is continuous (locally constant).

PROOF. Suppose $f_0 \in \mathcal{A}_\Omega(X, Y)$. First we observe that there exists an open, connected subset U of $f_0(\Omega)$ such that $\bar{Y} \subset U$ and \bar{U} is compact. We can also choose an open subset G so that $Y \subset G \subset \bar{G} \subset U$ and Y is analytic in G .

To prove the result we will readopt the conventions of Theorem 8.1 to $\Omega_1 = G, f_1 = f_0^{-1}|G, X_1 = Y$ and $Y_1 = X$. Since $f_1 \in \mathcal{B}(\Omega_1)$ and

$$\begin{aligned}\overline{f_1(X_1)} \cap \bar{Y}_1 &= \overline{f_0^{-1}(Y)} \cap \bar{X} = f_0^{-1}(\bar{Y}) \cap \bar{X} = f_0^{-1}(\bar{Y} \cap \overline{f_0(X)}) \\ &= f_0^{-1}(Y \cap f_0(X)) = f_0^{-1}(Y) \cap X = f_1(X_1) \cap Y_1,\end{aligned}$$

it follows that $f_1 \in \mathcal{B}^*(\Omega_1; X_1, Y_1)$. In view of Theorem 8.1 we can choose a neighbourhood V of f_1 (in the topology of uniform convergence) such that

$$\overline{g(Y)} \cap \bar{X} = g(Y) \cap X$$

and

$$*(g(Y) \cdot X) = *(f_1(Y) \cdot X) = *(Y \cdot f_0(X))$$

whenever $g \in V$.

Since $f_0 \in \mathcal{A}_\Omega(X, Y)$ and \bar{G} is a compact subset of U , it follows from Lemma 8.3 that there exists a neighbourhood V' of f_0 in $\mathcal{A}_\Omega(X, Y)$ such that $f(\Omega) \supset U$ and $(f^{-1}|U)|G \in V$ whenever $f \in V'$. Hence

$$*(f^{-1}(Y) \cdot X) = *(Y \cdot f_0(X))$$

which implies that

$$*(Y \cdot f(X)) = *(Y \cdot f_0(X))$$

whenever $f \in V'$, because f is a biholomorphic mapping.

COROLLARY 8.1 (Rouché, cf. [15] and [8]). *Let T be a connected topological space. Let Ω be an open subset of N . Let*

$$f: T \times \Omega \rightarrow N$$

be a continuous mapping. For every $t \in T$, define $f_t: \Omega \rightarrow N$ by

$$f_t(z) = f(t, z)$$

and suppose that f_{\pm} is holomorphic for every $t \in T$. Let G be an open subset of Ω such that \bar{G} is compact and $\bar{G} \subset \Omega$. Suppose that

$$f_t^{-1}(0) \cap (\bar{G} \cap G) = \emptyset \quad \text{for } t \in T.$$

Then the function

$$T \ni t \rightarrow \#(f_t \cdot (G \times \{0\})) \in \mathbb{Z}$$

is constant.

Proof. Letting $\Omega' = \Omega \times N$, $X = \Omega \times \{0\}$, $Y = G \times \{0\}$ we will prove Corollary 8.1 by using the preceding result. Defining

$$F_t: \Omega' \ni (x, y) \rightarrow (x, y + f_t(x)) \in \Omega' \quad \text{for } t \in T$$

we see that F_t is a biholomorphic mapping and the mapping $T \ni t \rightarrow F_t$ is continuous in the topology of local uniform convergence and our assumptions may be restated as follows:

$$\bar{Y} \subset \Omega', \quad F_t(\Omega') = \Omega' \supset \bar{Y}, \quad F_t(X) = \{(x, f_t(X)): x \in \Omega\} = f_t,$$

$$\partial Y \cap F_t(X) = (\partial G \times \{0\}) \cap f_t = f_t^{-1}(0) \cap (\partial G \times \{0\}) = \emptyset$$

and the mapping

$$T \ni t \rightarrow F_t \in \mathcal{A}_{\Omega}(X, Y)$$

is continuous. Now, by making use of Theorem 8.2, we complete the proof.

PROPOSITION 8.2. Let $\Omega_1, \dots, \Omega_k$ be open and connected subsets of N . Let X_j be an analytic subset of Ω_j of pure dimension d_j , $j = 1, \dots, k$. Suppose that $d_1 + \dots + d_k = (k-1)n$. Let D be an open subset of N such that \bar{D} is compact. Let T be a connected topological space and let for every $t \in T$, $i = 1, \dots, k$, a biholomorphic mapping

$$\sigma_i^t: \Omega_i \rightarrow \sigma_i^t(\Omega) \subset N$$

be given. If all the mappings

$$T \ni t \rightarrow \sigma_i^t, \quad i = 1, \dots, k,$$

are continuous in the topology of local uniform convergence, $\sigma_i^t(\Omega_i) \supset \bar{D}$ if $t \in T$, $i = 1, \dots, k$, and

$$\bigcap_{i=1}^k \sigma_i^t(X_i) \cap \partial D = \emptyset \quad \text{for } t \in T,$$

then the function

$$T \ni t \rightarrow \#(\sigma_1^t(X_1) \cdot \dots \cdot \sigma_k^t(X_k))$$

is constant.

Proof. Setting $\Omega = \Omega_1 \times \dots \times \Omega_k$, $f_t = \sigma_1^t \times \dots \times \sigma_k^t$ for $t \in T$ and $Y = \{(x, \dots, x) \in N^k: x \in D\}$, we see that all the assumptions of Theorem 8.2 are satisfied. Hence, since T is connected, it follows that the function

$$T \ni t \rightarrow {}^*(\sigma_1^t(X_1) \cdot \dots \cdot \sigma_k^t(X_k)) = {}^*(f_t(X) \cdot Y)$$

is constant.

9. Existence and number of points of intersection. Let D be an open subset of N . We say that D is a *starlike domain* if there exists a bounded, open neighbourhood U of the point 0 in N such that

- (a) $\{tx: t \geq 0\} \cap \partial U = \{x\}$ for all $x \in \partial U$ and
- (b) there exists a biholomorphic mapping $\psi: D \rightarrow U$.

THEOREM 9.1. Let N_1, \dots, N_k be complex vector spaces of dimensions n_1, \dots, n_k , respectively. Let D_j be a starlike domain in N_j for $j = 1, \dots, k$. Let V_j be a pure $d_j = \sum_{i \neq j} n_i$ dimensional analytic subset of $D_1 \times \dots \times D_k$ for $j = 1, \dots, k$. If there exist compact subsets K_1, \dots, K_k of D_1, \dots, D_k , respectively such that

$$V_j \subset D_1 \times \dots \times D_{j-1} \times K_j \times D_{j+1} \times \dots \times D_k,$$

then

- (i) natural projection p_j of V_j onto $D_1 \times \dots \times D_{j-1} \times D_{j+1} \times \dots \times D_k$ defined by

$$p_j(x_1, \dots, x_k) = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_k)$$

is s_j — sheeted branched covering for $j = 1, \dots, k$, and

- (ii) ${}^*(V_1 \cdot \dots \cdot V_k) = s_1 \cdot \dots \cdot s_k$.

Proof. Taking, if necessary, suitable chosen biholomorphic mappings $\psi_j: D_j \rightarrow \psi(D_j)$, $j = 1, \dots, k$ and replacing D_j by $\psi(D_j)$; V_j by $(\psi_1, \dots, \psi_k)(V_j)$; K_j by $\psi_j(K_j)$ for $j = 1, \dots, k$, we can assume that

- (a) D_1, \dots, D_k are open, bounded subsets of N_1, \dots, N_k , respectively,
- (b) $0 \in D_j$ for $j = 1, \dots, k$,
- (c) $\{tx_j: t \geq 0\} \cap \partial D_j = \{x_j\}$ for all $x_j \in \partial D_j$, $j = 1, \dots, k$.

Let $0 \neq x_j \in N_j$. Then there exists exactly one point $\tilde{x}_j \in \partial D_j$ such that $\{tx_j: t \geq 0\} \cap \partial D_j = \{\tilde{x}_j\}$, $j = 1, \dots, k$. The functions

$$\varphi_j: N_j \ni x_j \rightarrow \begin{cases} 0 & \text{if } x_j = 0, \\ \frac{|x_j|}{|\tilde{x}_j|} & \text{if } x_j \neq 0, \end{cases} \quad j = 1, \dots, k,$$

are well defined and

$$D_j = \{x_j \in N_j: \varphi_j(x_j) < 1\} \quad \text{and} \quad \partial D_j = \{x_j \in N_j: \varphi_j(x_j) = 1\}$$

for $j = 1, \dots, k$.

Let $V = V_1 \times \dots \times V_k$. Then

$$\partial V = \bigcup_{i=1}^k \bar{V}_1 \times \dots \times \bar{V}_{i-1} \times \partial V_i \times \bar{V}_{i+1} \times \dots \times \bar{V}_k.$$

Let us take $t \in [0, 1]$. The linear subspace Δ^t of $\mathcal{N} = (N_1 \times \dots \times N_k)^k$ defined by

$$\Delta^t = \{((x_1, tx_2, \dots, tx_k), \dots, (tx_1, \dots, tx_{k-1}, x_k)) : x_i \in N_i, i = 1, \dots, k\}$$

is $q = \sum_{i=1}^k n_i$ dimensional subspace of \mathcal{N} and we see that $\Delta = \Delta^1$ is the diagonal in \mathcal{N} .

An easy calculation show that the mapping

$$[0, 1] \ni t \rightarrow \Delta^t \in \text{Aff}_q(\mathcal{N}) \subset \mathcal{T}_{\mathcal{N}}$$

is continuous and

$$\#(V \cdot \Delta^0) = \prod_{j=1}^k \#(V_j \cdot (\{0\} \times \dots \times \{0\} \times N_j \times \{0\} \times \dots \times \{0\})) = s_1 \cdot \dots \cdot s_k.$$

Thus, according to Theorem 7.2, it suffices to show that

$$\partial V \cap \Delta^t = \emptyset \quad \text{for } t \in [0, 1].$$

Let us take $t = 1$ and suppose that there exists a point $(x, \dots, x) \in \partial V$. Following the assumption there would exist an index i such that $x \in \partial V_i$ and $x \in \bar{V}_j$ for $j \neq i$. Since $x \in \partial V_i$ there would exist $r \neq i$ such that $\varphi_r(x_r) = 1$ and $x \in \bar{V}_r$. Hence it would follow that $\varphi_r(x_r) < 1$ and this contradicts $\varphi_r(x_r) = 1$.

Now, suppose that $t \in [0, 1)$ and $\Delta^t \cap \partial V \neq \emptyset$. Then there would exist a point $x \in \mathcal{N}$ and $i \in \{1, \dots, k\}$ such that $(tx_1, \dots, tx_{i-1}, x_i, x_{i+1}, \dots, tx_k) \in \partial V_i$ and $(tx_1, \dots, tx_{j-1}, x_j, tx_{j+1}, \dots, tx_k) \in \bar{V}_j$ for $j \neq i$. Hence there would exist $j \neq i$ such that $\varphi_j(tx_j) = 1$ and this follow $\varphi_j(x_j) > 1$ and this contradicts $(tx_1, \dots, tx_{j-1}, x_j, tx_{j+1}, \dots, tx_k) \in \bar{V}_j$.

10. Bézout's theorem. To formulate Bézout's theorem we need definitions of multiplicity of intersection along component and of degree of algebraic subset of $\mathbf{P}(M)$. Since these definitions are well known, we present they here without proof of correctness. However, our proof of Bezout's theorem will realize and generalize an old Poncelet's and Severi's idea (see [13]) of proving the Bezout theorem for intersection of two plane curves.

Let Ω be an open subset of N and let X_j be an analytic subset of Ω of pure dimension d_j , $j = 1, \dots, k$. Let us suppose that B is a component of the analytic set $X_1 \cap \dots \cap X_k$ of pure dimension $t = \sum_{j=1}^k d_j - (k-1)n > 0$. Let B^* be the subset of B consisting of regular points of $X_1 \cap \dots \cap X_k$.

THEOREM 10.1 (Draper [4]). *Let H be a submanifold of Ω of dimension $n-t$ meeting B^* transversally at a point $P \in B^*$. Then the intersection multiplicity $i(X_1 \cdot \dots \cdot X_k, H; P)$ does not depend on H and P .*

The constant number in Theorem 10.1 is called the *multiplicity of intersection* of X_1, \dots, X_k along B , and is denoted by $i(X_1 \cdot \dots \cdot X_k; B)$.

Let X be an algebraic subset of $\mathbf{P}(M)$ of pure dimension k . Since $\partial X = \emptyset$, the set $G = G(X)$ defined in Section 7 by

$$G(X) = \{\xi \in G_{n-k}(M) : \xi \cap \partial X = \emptyset\}$$

is equal to $G_{n-k}(M)$. Therefore, by Theorem 7.1 and Chow's theorem the set A defined by

$$A = \{\xi \in G_{n-k}(M) : \#(\xi \cap X) = \infty\}$$

is a proper algebraic subset of $G_{n-k}(M)$ and there exists a positive integer s such that

$$\#(\xi \cdot X) = s \quad \text{for all } \xi \in G_{n-k}(M) \setminus A.$$

This number s is called the *degree of X* and is denoted by $\deg X$.

THEOREM 10.2 (Bezout, see e.g. [9], Theorem 5.16). *Let V_j be an algebraic subset of $\mathbf{P}(M)$ ($\dim M = n+1$) of pure dimension d_j , $j = 1, \dots, k$. Suppose that V_1, \dots, V_k intersect properly, i.e., the set $W = V_1 \cap \dots \cap V_k$ has the pure dimension $t = \sum_{j=1}^k d_j - (k-1)n \geq 0$. If W_1, \dots, W_l are all components of W , then*

$$\sum_{j=1}^l i(V_1 \cdot \dots \cdot V_k; W_j) \deg W_j = \prod_{j=1}^k \deg V_j.$$

Proof. We can find a linear, projective subspace V_0 of $\mathbf{P}(M)$ of dimension $n-t$ such that V_0 meets W transversally. The preceding results show that

$$\#(V_0 \cdot \dots \cdot V_k) = \sum_{j=1}^l i(V_1 \cdot \dots \cdot V_k; W_j) \deg W_j.$$

Since the set $V_0 \cap \dots \cap V_k$ is finite, we can find a projective hyperplane $H \subset \mathbf{P}(M)$ such that $V_0 \cap \dots \cap V_k \cap H = \emptyset$. We shall regard H as hyperplane at infinity and without loss of generality we can assume that $M = \mathbf{C} \times N$ and $H = \mathbf{P}(\{0\} \times N) = N_\infty$. For the sake of consistency, we shall denote affine parts of V_0, \dots, V_k by the same letters. Then all points of $V_0 \cap \dots \cap V_k$ lie in N with unchanging multiplicity of intersection. Furthermore

$$(V_0)_\infty \cap \dots \cap (V_k)_\infty = \emptyset \quad \text{and} \quad V_j = (V_j)_\infty \quad \text{for } j = 1, \dots, k.$$

Let us consider the product $V = V_0 \times \dots \times V_k \subset N^{k+1}$. Let $\Delta = \{(x_0, \dots, x_k) \in N^{k+1} : x_0 = x_1 = \dots = x_k\}$ be the diagonal in N^{k+1} . Then

$$\#(V_0 \cdot \dots \cdot V_k) = \#(V \cdot \Delta).$$

Let \tilde{V}_j be an algebraic cone in N such that $(V_j)_\infty = P(\{0\} \times \tilde{V}_j)$ for $j = 0, \dots, k$. Since V_0, \dots, V_k have no point in common with N_∞ ; therefore $\tilde{V}_0 \cap \dots \cap \tilde{V}_k = \{0\}$. This shows that

$$P(\{0\} \times \Delta) \cap P(\{0\} \times \tilde{V}_0 \times \dots \times \tilde{V}_k) = \emptyset.$$

Therefore, by Lemma 4.2 we have

$$\bar{\Delta} \cap \partial V = \emptyset.$$

Let ξ_0, \dots, ξ_k be affine planes in N of codimensions $\dim L$, $\dim V_1, \dots, \dim V_k$, respectively, such that $\#(\xi_i \cap V_i) = \deg V_i$ for $i = 0, \dots, k$. Clearly ξ_j meets V_j transversally for $j = 0, \dots, k$ and all points of intersection lie in N . The subspace $\xi = \xi_0 \times \dots \times \xi_k$ meets V transversally at $\deg V_1 \cdot \dots \cdot \deg V_k$ points and it follows from Lemma 4.2 that

$$\bar{\xi} \cap \partial V = \emptyset.$$

Let $p = \text{codim } V = kn$. Keeping notation of Section 5, we see that

$$\{\eta \in G_p(C \times N): \eta \cap \partial V = \emptyset\} = p_2(p_1^{-1}(V))$$

and since p_2 is proper, it is analytic (by Remmert's theorem). Hence, the set $G = G(V)$ which is defined in Section 7 is connected. Furthermore, $\Delta \in G(V)$ and $\bar{\xi} \in G(V)$, and Theorem 7.1 shows that

$$\#(V \cdot \Delta) = \#(V \cdot \bar{\xi})$$

and in view of Property 6.4(b), this completes our proof.

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