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On infinite systems of Volterra integral equations in Banach spaces

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Abstract. In this paper we consider the infinite system of integral equations

$$(1) \qquad x_i(t) = p_i(t) + \int_0^t f_i(t, s, x_1(s), x_2(s), \ldots)\, ds \qquad (i = 1, 2, \ldots),$$

where t belongs to a compact interval $[0, a]$ and p_i, x_i, f_i are functions with values in a Banach space E_i. We prove that under suitable assumptions the set of all continuous solutions of (1) is a compact R_δ in the Fréchet space $C([0, a], E_1 \times E_2 \times \ldots)$

Let $J = [0, a]$ be a compact interval in R and let E_i be a Banach space with the norm $\|\cdot\|_i$ $(i = 1, 2, \ldots)$. We introduce the following notation:

$E = E_1 \times E_2 \times \ldots$ — the Fréchet space of all infinite sequences $x = (x_i)$, $x_i \in E_i$ for $i = 1, 2, \ldots$, with the paranorm

$$|x| = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{\|x_i\|_i}{1 + \|x_i\|_i};$$

$C_i = C(J, E_i)$ — the Banach space of all continuous functions $u: J \to E_i$ with the norm $\|u\|_{ic} = \sup\{\|u(t)\|_i : t \in J\}$;

$C = C(J, E)$ — the Fréchet space of all continuous functions $u: J \to E$ with the paranorm $|u|_c = \sup\{|u(t)| : t \in J\}$;

$\alpha_i, \alpha, \alpha_c$ — the measures of non-compactness in E_i, E, C, respectively (Kuratowski [6], p. 318).

Assume that for each positive integer i:

I. $p_i: J \to E_i$ is a continuous function.

II. $(t, s, x) \to f_i(t, s, x)$ is a mapping of the set $\{0 \leqslant s \leqslant t \leqslant a, x \in E\}$ into E_i, which satisfies the following conditions:

1° for any fixed $x \in E$ and $t \in J$ the function $s \to f_i(t, s, x)$ is strongly L-measurable on $[0, t]$;

2° for any fixed $t, s, 0 \leqslant s \leqslant t \leqslant a$, the function $x \to f_i(t, s, x)$ is continuous in E;

3° there exist real-valued functions $(t, s) \to m_i(t, s)$ and $(\tau, t, s) \to r_i(\tau, t, s)$ $(0 \leqslant s \leqslant t \leqslant \tau \leqslant a)$ such that

(i) for any fixed t, τ the functions $s \to r_i(\tau, t, s)$ and $s \to m_i(t, s)$ are L-integrable on $[0, t]$;

(ii) $\sup\{\|f_i(\tau, s, x) - f_i(t, s, x)\|_i : x \in E\} \leqslant r_i(\tau, t, s)$ and $\sup\{\|f_i(t, s, x)\|_i : x \in E\} \leqslant m_i(t, s)$;

(iii) $\lim\limits_{\tau - t \to 0+} \Big[\int\limits_t^\tau m_i(\tau, s)ds + \int\limits_0^t r_i(\tau, t, s)ds\Big] = 0$ for fixed t or τ.

III. $a_i\big(f_i(t, [0, t] \times X)\big) \leqslant h\big(t, a(X)\big)$ for each $t \in J$ and for every subset X of E, where $h: J \times R_+ \to R_+$ is a Kamke function, i.e., h satisfies the assumptions:

1° for any fixed $t \in J$ the function $z \to h(t, z)$ is continuous and non-decreasing on R_+;

2° for any $b > 0$ there exist real-valued functions $t \to m(t)$ and $(\tau, t) \to r(\tau, t)$ $(0 \leqslant t \leqslant \tau \leqslant a)$ such that

$$\sup\{|h(t, z) - h(\tau, z)| : 0 \leqslant z \leqslant b\} \leqslant r(\tau, t),$$

$$\sup\{h(t, z) : 0 \leqslant z \leqslant b\} \leqslant m(t),$$

and

$$\lim_{\tau - t \to 0+} [(\tau - t)m(\tau) + tr(\tau, t)] = 0 \quad \text{for fixed } t \text{ or } \tau;$$

3° for any $q, 0 < q \leqslant a$, the function identically equal to zero is the unique continuous solution of the integral equation $z(t) = \int\limits_0^t h\big(t, z(s)\big)ds$ defined on $[0, q]$.

We consider the infinite system of integral equations

$$(1) \qquad x_i(t) = p_i(t) + \int\limits_0^t f_i\big(t, s, x_1(s), x_2(s), \ldots\big)ds \quad \text{for } t \in J,\ i = 1, 2, \ldots,$$

where $\int$ denotes the Bochner integral.

THEOREM. *The set S of all continuous solutions of* (1) *is a compact R_δ in C, i.e., S is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts.*

Proof. Let us fix a positive integer i. Put

$$F_i(x)(t) = p_i(t) + \int\limits_0^t f_i\big(t, s, x(s)\big)ds \quad \text{for } x \in C \text{ and } t \in J.$$

Fix $t_0 \in J$. For any $x \in C$ and $t \in J$ we have

$$\|F_i(x)(t) - F_i(x)(t_0)\|_i \leqslant \|p_i(t) - p_i(t_0)\|_i + \int_{t_0}^{t} \big\|f_i(t, s, x(s))\big\|_i ds + \\ + \int_0^{t_0} \big\|f_i(t, s, x(s)) - f_i(t_0, s, x(s))\big\|_i ds$$

$$\leqslant \|p_i(t) - p_i(t_0)\|_i + \int_{t_0}^{t} m_i(t, s)\,ds + \int_0^{t_0} r_i(t, t_0, s)\,ds \qquad \text{when } t \geqslant t_0,$$

and

$$\|F_i(x)(t) - F_i(x)(t_0)\|_i \leqslant \|p_i(t) - p_i(t_0)\|_i + \int_t^{t_0} m_i(t_0, s)\,ds + \int_0^{t} r_i(t_0, t, s)\,ds$$

when $t \leqslant t_0$. By II. 3° and by the continuity of p_i, it hence follows that $\lim\limits_{t\to t_0} F_i(x)(t) = F_i(x)(t_0)$ uniformly in $x \in C$, which proves that $F_i(C)$ is an equicontinuous subset of C_i. Since J is compact, this implies that the set $F_i(C)$ is equiuniformly continuous, and therefore the numbers

$$w_i(d) = \sup\{\|u(t) - u(s)\|_i \colon\ u \in F_i(C),\ t, s \in J,\ |t-s| \leqslant d\}$$

tend to zero as $d \to 0+$.

Assume that $x^n, x \in C$ and $\lim\limits_{n\to\infty} |x^n - x|_c = 0$. Since

$$\lim_{n\to\infty} f_i\big(t, s, x^n(s)\big) = f_i\big(t, s, x(s)\big) \qquad \text{and}$$

$$\big\|f_i\big(t, s, x^n(s)\big) - f_i\big(t, s, x(s)\big)\big\|_i \leqslant 2m_i(t, s) \qquad \text{for each } 0 \leqslant s \leqslant t \leqslant a,$$

the Lebesgue theorem shows that

$$\lim_{n\to\infty} \int_0^t \big\|f_i\big(t, s, x^n(s)\big) - f_i\big(t, s, x(s)\big)\big\|_i ds = 0,$$

i.e., $\lim\limits_{n\to\infty} F_i(x^n)(t) = F_i(x)(t)$ for each $t \in J$, and hence, by the equicontinuity of $F_i(C)$, $\lim\limits_{n\to\infty} \|F_i(x^n) - F_i(x)\|_{ic} = 0$. This shows that F_i is a continuous mapping of C into C_i.

Consequently, the mapping $F\colon C \to C$, defined by

$$F(x) = \big(F_1(x), F_2(x), \ldots\big) \qquad \text{for } x \in C,$$

is continuous. Let

$$w(d) = \sup\{|u(t) - u(s)| \colon\ u \in F(C),\ t, s \in J,\ |t-s| \leqslant d\}.$$

Since

$$w(d) \leqslant \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{w_i(d)}{1 + w_i(d)} \qquad \text{and} \qquad \lim_{d\to 0} w_i(d) = 0 \qquad \text{for each } i,$$

we see that $w(d) \to 0$ as $d \to 0$.

We shall prove that

(2) If $v^n \in C$, $n = 1, 2, \ldots$, and $\lim_{n\to\infty} |v^n - F(v^n)|_c = 0$, then (v^n) has a convergent subsequence.

Suppose that $v^n \in C$, $n = 1, 2, \ldots$, and

$$\lim_{n\to\infty} |v^n - F(v^n)|_c = 0. \tag{3}$$

Put $V = \{v^n: n = 1, 2, \ldots\}$ and $V(t) = \{u(t): u \in V\}$ for $t \in J$. Denote by I the identity mapping on C. From (3) it follows that $(I-F)(V)$ is an equiuniformly continuous subset of C. Since

$$V \subset (I-F)(V) + F(V), \tag{4}$$

and the set $F(V)$ is equiuniformly continuous, the set V is also equiuniformly continuous, so that the numbers

$$w(V, d) = \sup\{|u(s)-u(t)|: u \in V, t, s \in J, |t-s| \leqslant d\}$$

tend to zero as $d \to 0$. Because

$$|u(t)-z(t)| \leqslant |u(s)-z(s)| + 2w(V, |t-s|) \quad \text{for } u, z \in V \text{ and } t, s \in J,$$

we get

$$\big|a\big(V(t)\big) - a\big(V(s)\big)\big| \leqslant 2w(V, |t-s|) \quad \text{for } t, s \in J,$$

and therefore the function $t \to v(t) = a\big(V(t)\big)$ is continuous on J.

For fixed $t \in J$ we divide the interval $[0, t]$ into n parts $0 = t_0 < t_1 < \ldots < t_n = t$ in such a way that $\Delta t_k = t_k - t_{k-1} = t/n$ for $k = 1, \ldots, n$. Let $V(t_{k-1}, t_k) = \{u(s): u \in V, t_{k-1} \leqslant s \leqslant t_k\}$. Using the same argument as in the proof of Lemma 2.2 in [2], we can prove that

$$a\big(V(t_{k-1}, t_k)\big) = \sup\big\{a\big(V(s)\big): t_{k-1} \leqslant s \leqslant t_k\big\}.$$

Hence, by the continuity of v, there exists $s_k \in [t_{k-1}, t_k]$ such that

$$a\big(V(t_{k-1}, t_k)\big) = v(s_k). \tag{5}$$

Further, by the mean value theorem ([1]; Theorem V. 10.4), we obtain

$$F_i(x)(t) = p_i(t) + \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f_i\big(t, s, x(s)\big)\,ds \in p_i(t) + \sum_{k=1}^{n} \Delta t_k \,\overline{\operatorname{conv}}\, f_i\big(t, [t_{k-1}, t_k] \times V(t_{k-1}, t_k)\big) \quad \text{for each } x \in V,$$

and therefore

$$F_i(V)(t) \subset p_i(t) + \sum_{k=1}^{n} \Delta t_k \,\overline{\operatorname{conv}}\, f_i\big(t, [0, t] \times V(t_{k-1}, t_k)\big),$$

where $F_i(V)(t) = \{F_i(x)(t): x \in V\}$. By III, (5), and the corresponding properties of a_i (cf. [2]), it hence follows that

$$\begin{aligned} a_i\big(F_i(V)(t)\big) &\leqslant \sum_{k=1}^{n} \Delta t_k\, a_i\big(\overline{\operatorname{conv}} f_i(t,\, [0,t]\times V(t_{k-1},t_k))\big) \\ &= \sum_{k=1}^{n} \Delta t_k\, a_i\big(f_i(t,\, [0,t]\times V(t_{k-1},t_k))\big) \leqslant \sum_{k=1}^{n} \Delta t_k\, h\big(t,\, a(V(t_{k-1},t_k))\big) \\ &= \sum_{k=1}^{n} \Delta t_k\, h\big(t,\, v(s_k)\big). \end{aligned}$$

But if $n\to\infty$, then $\sum\limits_{k=1}^{n} \Delta t_k h\big(t, v(s_k)\big) \to \int\limits_0^t h\big(t, v(s)\big)\,ds$. Thus

$$a_i\big(F_i(V)(t)\big) \leqslant \int_0^t h\big(t, v(s)\big)\,ds \quad \text{for } i=1,2,\ldots \tag{6}$$

Now we shall show that

$$a(A_1\times A_2\times\ldots) \leqslant \sum_{i=1}^{\infty} \frac{1}{2^i}\cdot \frac{a_i(A_i)}{1+a_i(A_i)} \tag{7}$$

for any sequence of bounded sets $A_i \subset E_i$ $(i=1,2,\ldots)$.

Let $A = A_1\times A_2\times\ldots$ and $\varepsilon>0$. For any i there exists a finite family of sets $(B_{ij})_{j=1,\ldots,n_i}$ such that

$$A_i \subset \bigcup_{j=1}^{n_i} B_{ij} \quad \text{and} \quad \delta_i(B_{ij}) \leqslant a_i(A_i)+\varepsilon.$$

Choose a positive integer k such that $1/2^k \leqslant \varepsilon$. The family of sets $B_{j_1,\ldots,j_k} = B_{1j_1}\times B_{2j_2}\times\ldots\times B_{kj_k}\times A_{k+1}\times A_{k+2}\times\ldots$ $(1\leqslant j_1\leqslant n_1,\ldots,1\leqslant j_k\leqslant n_k)$ is a finite cover of A. Moreover,

$$\begin{aligned} \delta(B_{j_1,\ldots,j_k}) &\leqslant \sum_{i=1}^{k} \frac{1}{2^i}\cdot\frac{\delta_i(B_{ij_i})}{1+\delta_i(B_{ij_i})} + \sum_{i=k+1}^{\infty} \frac{1}{2^i}\cdot\frac{\delta_i(A_i)}{1+\delta_i(A_i)} \\ &\leqslant \sum_{i=1}^{k} \frac{1}{2^i}\cdot\frac{a_i(A_i)+\varepsilon}{1+a_i(A_i)+\varepsilon} + \sum_{i=k+1}^{\infty}\frac{1}{2^i} \leqslant \sum_{i=1}^{k}\frac{1}{2^i}\cdot\frac{a_i(A_i)}{1+a_i(A_i)} + \\ &\qquad + \sum_{i=1}^{k}\frac{1}{2^i}\cdot\frac{\varepsilon}{1+\varepsilon} + \frac{1}{2^k} \leqslant \sum_{i=1}^{\infty}\frac{1}{2^i}\cdot\frac{a_i(A_i)}{1+a_i(A_i)} + 2\varepsilon. \end{aligned}$$

Hence

$$a(A) \leqslant \sum_{i=1}^{\infty}\frac{1}{2^i}\cdot\frac{a_i(A_i)}{1+a_i(A_i)} + 2\varepsilon.$$

Since the last inequality is satisfied for every $\varepsilon > 0$, we get

$$a(A) \leqslant \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{a_i(A_i)}{1+a_i(A_i)}.$$

From (6) and (7) it follows that

$$a\big(F(V)(t)\big) \leqslant \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{a_i\big(F_i(V)(t)\big)}{1+a_i\big(F_i(V)(t)\big)} \leqslant \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{\int_0^t h\big(t, v(s)\big)\,ds}{1+\int_0^t h\big(t, v(s)\big)\,ds}$$
$$\leqslant \int_0^t h\big(t, v(s)\big)\,ds.$$

On the other hand, from (3) it follows that $\lim_{n\to\infty} |v^n(t) - F(v^n)(t)| = 0$, which implies $a\big((I-F)(V)(t)\big) = 0$. Consequently, by (4), we obtain

$$a\big(V(t)\big) \leqslant a\big((I-F)(V)(t)\big) + a\big(F(V)(t)\big) = a\big(F(V)(t)\big),$$

so that

$$v(t) \leqslant \int_0^t h\big(t, v(s)\big)\,ds \quad \text{for } t \in J.$$

Applying now Theorem 2 of [3], we deduce that $v(t) = 0$ for each $t \in J$. In the same way as in the proof of Theorem 2.3 in [2], we can show that

$$a_c(V) = \sup\{a\big(V(t)\big):\ t \in J\} = 0.$$

From this we conclude that the sequence (v^n) has a convergent subsequence, which settles (2).

For any positive integer n write $a_n = a/n$ and

$$F^n(x)(t) = \begin{cases} p(0) & \text{for } 0 \leqslant t \leqslant a_n, \\ F(x)(t-a_n) & \text{for } a_n \leqslant t \leqslant a, \end{cases}$$

for each $x \in C$, where $p(0) = \big(p_1(0), p_2(0), \ldots\big)$. Obviously, F^n is a continuous mapping $C \to C$ and

$$|F^n(x) - F(x)|_c \leqslant w(a_n) \quad \text{for every } x \in C.$$

Let $T = I - F$ and $T^n = I - F^n$. Then T, T^n are continuous mappings $C \to C$ and

$$\lim_{n\to\infty} |T^n(x) - T(x)|_c = 0 \quad \text{uniformly on } C.$$

Fix n. Assume that $y \in C$. We define a finite sequence (x^k), $k = 1, 2, \ldots, n$, of continuous functions by the formulas

$$x^1(t) = y(t) + p(0) \qquad \text{for } t \in J,$$

$$x^{k+1}(t) = \begin{cases} x^k(t) & \text{for } 0 \leqslant t \leqslant ka_n, \\ y(t) + F(x^k)(t - a_n) & \text{for } ka_n \leqslant t \leqslant a. \end{cases}$$

It can easily be verified that

$$x^k(t) = y(t) + F^n(x^k)(t) \qquad \text{for } 0 \leqslant t \leqslant ka_n,\ k = 1, \ldots, n,$$

and consequently $T^n(x^n) = y$. Conversely, if $T^n(x) = y$ and $x \in C$, then $x(t) = x^k(t)$ for $0 \leqslant t \leqslant ka_n$, $k = 1, \ldots, n$, and therefore $x = x^n$. This proves that T^n is a bijection $C \to C$.

Now assume that $\lim\limits_{j\to\infty} |T^n(u^j) - T^n(u)|_c = 0$, where u^n, $u \in C$. Because $u^j(t) = T^n(u^j)(t) + p(0)$ and $u(t) = T^n(u)(t) + p(0)$ for $0 \leqslant t \leqslant a_n$, $\lim\limits_{j\to\infty} u^j(t) = u(t)$ uniformly on $[0, a_n]$. Further,

$$u^j(t) = T^n(u^j)(t) + F(u^j)(t - a_n) \quad \text{and} \quad u(t) = T^n(u)(t) + F(u)(t - a_n)$$
$$\text{for } a_n \leqslant t \leqslant 2a_n,$$

and

$$\lim_{j\to\infty} F(u^j)(t - a_n) = F(u)(t - a_n) \qquad \text{uniformly on } [a_n, 2a_n].$$

This implies

$$\lim_{j\to\infty} u^j(t) = u(t) \qquad \text{uniformly on } [0, 2a_n].$$

By repeating this argument, we infer that $\lim\limits_{j\to\infty} u^j(t) = u(t)$ uniformly on $[0, ka_n]$ for $k = 1, \ldots, n$, i.e., $\lim\limits_{j\to\infty} u^j = u$ in C. This proves the continuity of $(T^n)^{-1}$. Therefore T^n is a homeomorphism $C \to C$.

Hence, applying Theorem 4 of [9], we conclude that the set $T^{-1}(0)$ is a compact R_δ. It is clear that $S = T^{-1}(0)$, and this ends the proof.

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