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Exceptional values of meromorphic functions

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Abstract. If f is a transcendental meromorphic function in the plane, an upper bound is obtained for the number of exceptional values in the sense of Borel for f for distinct zeros of order $< k$, where k is a positive integer. Relations between the number of exceptional values of f and the deficiencies of f are also obtained. Earlier results of Valiron, Xiong Qing-Lai and others are improved and several new results are deduced.

I. Introduction. We denote by C the set of all (finite) complex numbers and by $\bar{C}$ the extended complex plane consisting of all (finite) complex numbers and ∞. By a meromorphic function we shall always mean a transcendental meromorphic function in the plane. If f is a meromorphic function, $a \in \bar{C}$ and $r > 0$, we use the following notations, of frequent use in the Nevanlinna theory, with their usual meaning; $m(r, a, f)$, $n(r, a, f)$, $\bar{n}(r, a, f)$, $N(r, a, f)$, $\bar{N}(r, a, f)$, $T(r, f)$, $\delta(a, f)$, $\Theta(a, f)$, $\Delta(a, f)$, etc. (see, e.g., [1] and [5]). If k is a positive integer, we also denote by $\bar{n}_k(r, a, f)$ the number of distinct zeros of order $\leqslant k$ of $f-a$ in $|z| \leqslant r$ (each zero is here counted only once irrespective of its multiplicity).

Thus, in the notation of [5], $\bar{n}_1(r, a, f) = n_s(r, a, f)$ and $\bar{n}_2(r, a, f) = \bar{n}_{12}(r, a, f)$. As usual, if $a = \infty$, then by a zero of $f-a$ we mean a pole of f. $\bar{N}_k(r, a, f)$ is defined in terms a $\bar{n}_k(r, a, f)$ in the obvious way. In what follows we write $\limsup g(r)$, $\liminf g(r)$, $g(r) = O(h(r))$ and $g(r) = o(h(r))$ for $\limsup\limits_{r\to\infty} g(r)$, $\liminf\limits_{r\to\infty} g(r)$, $g(r) = O(h(r))$ as $r \to \infty$ and $g(r) = o(h(r))$ as $r \to \infty$, respectively. If f is meromorphic, $a \in \bar{C}$ and k is a positive integer, we further define

$$\varrho_k(a, f) = \limsup \frac{\log^+ \bar{n}_k(r, a, f)}{\log r} = \limsup \frac{\log^+ \bar{N}_k(r, a, f)}{\log r},$$

$$\bar{\varrho}(a, f) = \limsup \frac{\log^+ \bar{n}(r, a, f)}{\log r} = \limsup \frac{\log^+ \bar{N}(r, a, f)}{\log r}$$

and

$$\varrho(a, f) = \limsup \frac{\log^+ n(r, a, f)}{\log r} = \limsup \frac{\log^+ N(r, a, f)}{\log r}.$$

If f is a meromorphic function of order ϱ, $0 \leqslant \varrho \leqslant \infty$, $a \in \bar{C}$ and k is a positive integer, then we say that a is

(i) an evB (*exceptional value in the sense of Borel*) for f for distinct zeros of order $\leqslant k$ if $\bar{\varrho}_k(a,f) < \varrho$,

(ii) an evB for f for distinct zeros if $\bar{\varrho}(a,f) < \varrho$,

(iii) an evB for f (for the whole aggregate of zeros) if $\varrho(a,f) < \varrho$.

Thus we call a an evB for f for simple zeros if $\bar{\varrho}_1(a,f) < \varrho$ and an evB for f for distinct simple and double zeros if $\bar{\varrho}_2(a,f) < \varrho$.

Valiron proved [6] the following generalizations of the classical theorem of Borel for entire functions of finite order.

THEOREM A. *Let f be an entire function of finite order ϱ. Then*

(i) *there exist at most two distinct elements of C which are evB for f for simple zeros,*

(ii) *if there exists $a \in C$ such that a is an evB for f for the joint sequence of simple and double zeros (double zeros being counted twice here), then $\bar{\varrho}_1(b,f) = \varrho$ for all $b \in C$ with $b \neq a$.*

In [5] Singh and Gopalakrishna obtained results stronger than the above for meromorphic functions of finite order. For entire functions of infinite order, Xiong [8] introduced a new definition of order and a corresponding new definition of an evB and showed that the above results of Valiron are valid for entire functions of infinite order with these new definitions. In this paper we obtain stronger results than those of Valiron for meromorphic functions of all orders (finite or infinite) with the usual definitions of order and evB even for functions of infinite order. We also obtain analogous results for other types of exceptional values.

Theorems and their proofs.

THEOREM 1. *Let f be a meromorphic function of order ϱ, $0 \leqslant \varrho \leqslant \infty$. If there exist distinct elements $a_1, a_2, \ldots, a_p$, $b_1, b_2, \ldots, b_q$, $c_1, c_2, \ldots, c_s$ in $\bar{C}$ such that $a_1, \ldots, a_p$ are evB for f for distinct zeros of order $\leqslant k$, $b_1, \ldots, b_q$ are evB for f for distinct zeros of order $\leqslant l$ and $c_1, \ldots, c_s$ are evB for f for distinct zeros of order $\leqslant m$, where k, l and m are positive integers, then*

$$\frac{pk}{k+1} + \frac{ql}{l+1} + \frac{sm}{m+1} \leqslant 2. \tag{1}$$

Proof. If $a \in \bar{C}$ and d is a positive integer, we clearly have

$$\bar{N}(r,a,f) \leqslant \frac{1}{d+1}\{d\bar{N}_d(r,a,f) + N(r,a,f)\}. \tag{2}$$

By Nevanlinna's second fundamental theorem, we have, for $r \geqslant r_0 > 0$,

$$(3) \quad (p+q+s-2)T(r,f) \leqslant \sum_{i=1}^{p} \bar{N}(r, a_i, f) + \sum_{j=1}^{q} \bar{N}(r, b_j, f) + \sum_{t=1}^{s} \bar{N}(r, c_t, f) + S(r,f),$$

where

$$(4) \quad \int_{r_0}^{r} \frac{S(x,f)}{x^{1+\lambda}}\, dx = O\left(\int_{r_0}^{r} \frac{\log T(x,f)}{x^{1+\lambda}}\, dx\right)$$

whenever $\lambda > 0$ (see [3], p. 69, and [1], Theorem 2.1). We have $\bar{\varrho}_k(a_i,f) < \varrho$ for $i = 1, \ldots, p$, $\bar{\varrho}_l(b_j,f) < \varrho$ for $j = 1, \ldots, q$ and $\bar{\varrho}_m(c_t,f) < \varrho$ for $t = 1, \ldots, s$. We choose a positive number $\lambda < \varrho$ such that $\bar{\varrho}_k(a_i,f) < \lambda$ for $i = 1, \ldots, p$, $\bar{\varrho}_l(b_j,f) < \lambda$ for $j = 1, \ldots, q$ and $\bar{\varrho}_m(c_t,f) < \lambda$ for $t = 1, \ldots, s$.

Then

$$(5) \quad \int_{r_0}^{\infty} \frac{\bar{N}_k(x, a_i, f)}{x^{1+\lambda}}\, dx < \infty, \quad \int_{r_0}^{\infty} \frac{\bar{N}_l(x, b_j, f)}{x^{1+\lambda}}\, dx < \infty \quad \text{and} \quad \int_{r_0}^{\infty} \frac{\bar{N}_m(x, c_t, f)}{x^{1+\lambda}}\, dx < \infty$$

for $i = 1, \ldots, p$, $j = 1, \ldots, q$ and $t = 1, \ldots, s$. Using (2), we obtain from (3), for $r \geqslant r_0$,

$$(p+q+s-2)T(r,f) \leqslant \frac{1}{k+1}\left\{k \sum_{i=1}^{p} \bar{N}_k(r, a_i, f) + \sum_{i=1}^{p} N(r, a_i, f)\right\} + \frac{1}{l+1}\left\{l \sum_{j=1}^{q} \bar{N}_l(r, b_j, f) + \sum_{j=1}^{q} N(r, b_j, f)\right\} + \frac{1}{m+1}\left\{m \sum_{t=1}^{s} \bar{N}_m(r, c_t, f) + \sum_{t=1}^{s} N(r, c_t, f)\right\} + S(r,f),$$

and so,

$$(6) \quad \left(\frac{pk}{k+1} + \frac{ql}{l+1} + \frac{sm}{m+1} - 2\right) T(r,f) \leqslant \frac{k}{k+1} \sum_{i=1}^{p} \bar{N}_k(r, a_i, f) + \frac{l}{l+1} \sum_{j=1}^{q} \bar{N}_l(r, b_j, f) + \frac{m}{m+1} \sum_{t=1}^{s} \bar{N}_m(r, c_t, f) + S(r,f),$$

since, for $a \in \bar{C}$, we have

$$N(r, a, f) \leqslant T\left(r, \frac{1}{f-a}\right) = T(r, f) + O(\log r).$$

From (4), it follows that

$$\int_{r_0}^{r} \frac{S(x,f)}{x^{1+\lambda}}\, dx = o\left(\int_{r_0}^{r} \frac{T(x,f)}{x^{1+\lambda}}\, dx\right).$$

Hence (6) yields

$$(7) \quad \left\{\frac{pk}{k+1} + \frac{ql}{l+1} + \frac{sm}{m+1} - 2 + o(1)\right\} \int_{r_0}^{r} \frac{T(x,f)}{x^{1+\lambda}}\, dx$$

$$\leqslant \frac{k}{k+1} \sum_{i=1}^{p} \int_{r_0}^{r} \frac{\bar{N}_k(x, a_i, f)}{x^{1+\lambda}}\, dx +$$

$$+ \frac{l}{l+1} \sum_{j=1}^{q} \int_{r_0}^{r} \frac{\bar{N}_l(x, b_j, f)}{x^{\;+\lambda}}\, dx + \frac{m}{m+1} \sum_{t=1}^{s} \int_{r_0}^{r} \frac{\bar{N}_m(x, c_t, f)}{x^{1+\lambda}}\, dx.$$

Thus, if $\frac{pk}{k+1} + \frac{ql}{l+1} + \frac{sm}{m+1} > 2$, then (7) would yield, by virtue of (5),

$$\int_{r_0}^{\infty} \frac{T(x,f)}{x^{1+\lambda}}\, dx < \infty,$$

which would imply $T(r,f) = o(r^{\lambda})$, so that we would have $\varrho =$ order of $f \leqslant \lambda$, which would be a contradiction. This completes the proof of the theorem.

Consequences of Theorem 1. Let f be a meromorphic function (of finite or infinite order).

(i) For $k \geqslant 3$, (1) yields $p \leqslant \frac{}{3}$ and hene it follows that there exist at most two elements in $\bar{C}$ which are evB for f for distinct zeros of order $\leqslant k$ if $k \geqslant 3$. This result cannot be improved, no matter what k we choose, for, 0 and ∞ are evB (for the whole aggregate of zeros) for e^{z}.

On the other hand, for $k = 3$, $p = 1$, $l = 4$ and $q = 1$, (1) yields $s \leqslant \frac{9}{20}(1+1/m)$, so that $s = 0$ for $m = 1$. Thus it follows that if there exist distinct elements a, b in $\bar{C}$ such that a is an evB for f for distinct zeros of order $\leqslant 3$ and b is an evB for f for distinct zeros of order $\leqslant 4$, then there exists no other element of $\bar{C}$ which is an evB for f for simple zeros.

(ii) For $k = 2$, (1) yields $2p/3 + ql/(l+1) \leqslant 2$, so that $p \leqslant 3$ and if $p = 3$, then $q = 0$ for any l. Thus there exist at most three elements in $\bar{C}$ which are evB for f for distinct simple and double zeros, and if there exist three such elements, then there is no other element of $\bar{C}$ which is an evB for f for simple zeros. This fact cannot be improved since there exists a meromorphic function g of positive order for which there exist three distinct elements a_1, a_2, a_3 in $\bar{C}$ such that $g-a_i$ has no zeros of order $\leqslant 2$ for $i=1, 2, 3$ [1], p. 45–46, so that a_1, a_2, a_3 are evB for g for zeros of order $\leqslant 2$.

Again, for $k=2$ and $p=1$, (1) yields $q \leqslant \frac{4}{3}(1+1/l)$, so that $q \leqslant 2$ for $l=1$ and $q \leqslant 1$ for $l=3$. Hence it follows that if there exists an element of $\bar{C}$ which is an evB for f for distinct simple and double zeros, then there exist at most two other elements of $\bar{C}$ which are evB for f for simple zeros and there exists at most one other element of $\bar{C}$ which is an evB for f for distinct zeros of order $\leqslant 3$.

Since ∞ is always an evB (for the whole aggregate of zeros) for an entire function, it follows that if f is an entire function, then there exist at most two elements of C which are evB for f for simple zeros, which is the result of Valiron (Theorem A, (i)) extended to entire functions of infinite order.

On the other hand, for $k=2$, $p=1$, $l=3$ and $q=1$, (1) yields $s \leqslant \frac{7}{12}(1+1/m)$, so that $s=0$ for $m=2$. Hence, if there exist distinct elements a, b in $\bar{C}$ such that a is an evB for f for distinct zeros of order $\leqslant 2$ and b is an evB for f for distinct zeros of order $\leqslant 3$, then there exists no other element of $\bar{C}$ which is an evB for f for distinct zeros of order $\leqslant 2$. It follows that if there exist three elements in $\bar{C}$ which are evB for f for distinct simple and double zeros, then there can be no element in $\bar{C}$ which is an evB for f for distinct zeros of order $\leqslant 3$.

For $k=2$, $p=1$, $l=6$ and $q=1$, (1) yields $s \leqslant \frac{10}{21}(1+1/m)$, so that $s=0$ when $m=1$. Hence, if there exist distinct elements a, b in $\bar{C}$ such that a is an evB for f for distinct simple and double zeros and b is an evB for f for distinct zeros of order $\leqslant 6$, then there exists no other element of $\bar{C}$ which is an evB for f for simple zeros. In particular, it follows that if f is an entire function and if there exists an element a in C which is an evB for f for distinct simple and double zeros, then there exists no other element of C which is an evB for f for simple zeros, which is the result of Valiron (Theorem A, (ii)) extended to entire functions of infinite order.

(iii) For $k=1$, (1) yields $p \leqslant 4$, whence it follows that there exist at most four elements in $\bar{C}$ which are evB f for simple zeros. This conclusion cannot be improved, for, as observed in [5], for Weierstrass's elliptic function $\wp(z)$, there exist four elements of $\bar{C}$ which are evB for simple zeros.

Also, from what was observed in (ii), it follows that if there exist four elements of $\bar{C}$ which are evB for f for simple zeros, then no element of $\bar{C}$ is an evB for f for distinct simple and double zeros.

We now consider meromorphic functions f of finite order and obtain relations between the number of elements of $\bar{C}$ which are evB for f for distinct zeros of order $\leqslant k$ and the deficiencies of f. We recall that if f is meromorphic and $a \in \bar{C}$, then we call a an evN (exceptional value in the sense of Nevanlinna) for f if $\delta(a,f) > 0$.

In what follows, f will always denote a meromorphic function of finite order ϱ and $\varrho(r)$ will denote a proximate order relative to $T(r,f)$, so that

(i) $\lim \varrho(r) = \varrho$, (ii) $\lim r\varrho'(r)\log r = 0$,

(iii) $T(r,f) \leqslant r^{\varrho(r)}$ for all $r \geqslant r_0$ and

(iv) $\limsup \dfrac{T(r,f)}{r^{\varrho(r)}} = 1$.

We first prove the following

LEMMA 1.

$$\limsup \frac{N(r,0,f')}{T(r,f)} \leqslant 2 - \Theta(\infty,f) - \sum_{a \in C} \delta(a,f).$$

Proof. It is well known that

$$\limsup \frac{T(r,f')}{T(r,f)} \leqslant 2 - \Theta(\infty,f). \tag{8}$$

Let $\langle a_i \rangle_{i=1}^{\infty}$ be an infinite sequence of distinct elements of C which includes every $a \in C$ for which $\delta(a,f) > 0$.

If q is a positive integer, then we have [5], Lemma 2,

$$\sum_{i=1}^{q} m(r,a_i,f) + N(r,0,f') \leqslant T(r,f') + o\big(T(r,f)\big).$$

Hence,

$$\sum_{i=1}^{q} \liminf \frac{m(r,a_i,f)}{T(r,f)} + \limsup \frac{N(r,0,f')}{T(r,f)} \leqslant \limsup \frac{T(r,f')}{T(r,f)} \leqslant 2 - \Theta(\infty,f).$$

Letting $q \to \infty$, we obtain (8).

THEOREM 2. *If there exists $a \in \bar{C}$ and positive integers k and q such that $(k+1)\Theta(a,f) + \sum_{b \in \bar{C}-\{a\}} \delta(b,f) > 2 - k(q-1)$, then there exist at most q elements of $\bar{C} - \{a\}$ which are evB for f for distinct zeros of order $\leqslant k$.*

Proof. Assume, without loss of generality, that $a = \infty$. Suppose, contrary to the theorem, that there exist $q+1$ elements $a_1, a_2, \ldots, a_{q+1}$ in C which are evB for f for distinct zeros of order $\leqslant k$.

Then

$$\limsup \frac{\log^+ \bar{N}_k(r, a_i, f)}{\log r} < \varrho \quad \text{for } i = 1, 2, \ldots, q+1,$$

whence it follows that

$$\bar{N}_k(r, a_i, f) = o(r^{\varrho(r)}) \quad \text{for } i = 1, 2, \ldots, q+1.$$

Also, since for each $b \in C$, a zero of $f-b$ of order $m > 1$ is a zero of f' of order $m-1$, we clearly have

$$\sum_{i=1}^{q+1} \bar{N}(r, a_i, f) \leqslant \sum_{i=1}^{q+1} \bar{N}_k(r, a_i, f) + \frac{1}{k} N(r, 0, f'). \tag{9}$$

By Nevanlinna's second fundamental theorem, we have

$$\begin{aligned} qT(r,f) &\leqslant \sum_{i=1}^{q+1} \bar{N}(r, a_i, f) + \bar{N}(r, \infty, f) + o\big(T(r,f)\big) \\ &\leqslant \sum_{i=1}^{q+1} \bar{N}_k(r, a_i, f) + \frac{1}{k} N(r, 0, f') + \bar{N}(r, \infty, f) + \\ &\qquad + o\big(T(r,f)\big) = o(r^{\varrho(r)}) + \frac{1}{k} N(r, 0, f') + \bar{N}(r, \infty, f). \end{aligned}$$

Hence

$$q \leqslant \frac{1}{k} \limsup \frac{N(r, 0, f')}{T(r,f)} + \limsup \frac{\bar{N}(r, \infty, f)}{T(r,f)}$$

since

$$\limsup \frac{T(r,f)}{r^{\varrho(r)}} = 1 \quad \text{and} \quad T(r,f) \leqslant r^{\varrho(r)} \quad \text{for all } r \geqslant r_0.$$

So, using (8), we obtain

$$q \leqslant \frac{1}{k} \Big\{ 2 - \Theta(\infty, f) - \sum_{b \in C} \delta(b, f) \Big\} + 1 - \Theta(\infty, f),$$

which yields

$$(k+1)\Theta(\infty, f) + \sum_{b \in C} \delta(b, f) \leqslant 2 - k(q-1),$$

which contradicts the hypothesis. This proves Theorem 2.

CONSEQUENCES OF THEOREM 2. (i) Let $k = 1$ in Theorem 2.

Taking $q = 3$, we see that if $\Theta(a, f) > 0$ for some $a \in \bar{C}$, then there exist at most there elements in $\bar{C} - \{a\}$ which are evB f or f for simple zeros. If follows that if there exist distinct elements a_1, a_2, a_3, a_4 in $\bar{C}$ which are evB for f for simple zeros, then $\Theta(a, f) = 0$ for all $a \in \bar{C} - \{a_1, a_2, a_3, a_4\}$.

Taking $q = 2$, it follows that if $2\Theta(a,f)+\sum_{b\in\bar{C}-\{a\}}\delta(b,f)>1$ for some $a\in\bar{C}$, then there exist at most two elements in $\bar{C}-\{a\}$ which are evB for f for simple zeros. In particular, this holds if there exists an $a\in\bar{C}$ such that $\Theta(a,f)>\frac{1}{2}$. If follows, therefore, that if there exist distinct elements a_1, a_2, a_3, a_4 in $\bar{C}$ which are evB for f for simple zeros, then $\Theta(a_i,f)\leqslant\frac{1}{2}$ for $i = 1, 2, 3, 4$.

If f is an entire function, then $\delta(\infty,f) = 1$ and ∞ is an evB for f. Hence it follows that if there exists an $a\in C$ such that $\Theta(a,f)>0$, then there exists at most one element in $C-\{a\}$ which is an evB for f for simple zeros. Therefore, if there exist distinct elements a, b in C which are evB for f for simple zeros, then $\Theta(c,f) = 0$ for all $c\in C-\{a,b\}$.

Taking $q = 1$, it follows that if

$$2\Theta(a,f)+\sum_{b\in\bar{C}-\{a\}}\delta(b,f)>2 \quad \text{for some } a\in\bar{C},$$

then there exists at most one element in $\bar{C}-\{a\}$ which is an evB for f for simple zeros. In particular, if f is an entire function, then $\Theta(\infty,f) = 1$, and so it follows that if there exists $b\in C$ such that $\delta(b,f)>0$, then there exists at most one element of C which is an evB for f for simple zeros. Thus, if f is entire and if there exist two elements of C which are evB for f for simple zeros, then $\delta(b,f) = 0$ for all $b\in C$, so that f has no finite evN. Again, if f is entire, then $\delta(\infty,f) = 1$ and ∞ is an evB for f, and so it follows that if there exists an $a\in C$ such that $\Theta(a,f)>\frac{1}{2}$, then there exists no element of $C-\{a\}$ which is an evB for f for simple zeros. Thus, if f is an entire function and if there exist distinct elements a, b in C which are evB for f for simple zeros, then $\Theta(a,f)\leqslant\frac{1}{2}$, $\Theta(b,f)\leqslant\frac{1}{2}$, $\delta(a,f) = 0$, $\delta(b,f) = 0$ and $\Theta(c,f) = 0$ for all $c\in C-\{a,b\}$.

(ii) If $\Theta(a,f)>0$ for some $a\in\bar{C}$, then, taking $q = 1$ in Theorem 2, we see that there exists at most one element in $\bar{C}-\{a\}$ which is an evB for f for distinct zeros of order $\leqslant k$, where $k = \left[\frac{2}{\Theta(a,f)}-1\right]+1$ (where, as usual, for any real number x, $[x]$ denotes the greatest integer $\leqslant x$). Thus, if there exist distinct elements a, b in $\bar{C}$ which are evB for f for distinct zeros, it follows that $\Theta(c,f) = 0$ for all $c\in\bar{C}-\{a,b\}$.

We now extend Theorems 1 and 2 to other types of exceptional values.

If $a\in\bar{C}$ and f is meromorphic, then Shah calls a an evE [4] for f if there exists a positive nondecreasing function $\varphi(x)$ satisfying $\int_A^\infty\frac{dx}{x\varphi(x)}<\infty$ (where $A>0$) such that

$$\limsup\frac{n(r,a,f)\varphi(r)}{T(r,f)}<\infty.$$

In this case, $\log r = o(\varphi(r))$ and since

$$\limsup \frac{n(r,a,f)\log r}{T(r,f)} \cdot \frac{\varphi(r)}{\log r} < \infty,$$

it follows that $n(r,a,f)\log r = o(T(r,f))$. Therefore $N(r,a,f) = o(T(r,f))$, since $N(r,a,f) \leqslant n(r,a,f)\log r$.

If $a \in \bar{C}$ and f is meromorphic, we call a an evE_2 for f for distinct zeros of order $\leqslant k$ if $\bar{N}_k(r,a,f) = o(T(r,f))$.

Then, under the hypothesis of Theorem 1 with 'evB' replaced by 'evE_2', we obtain, from (6),

$$\left\{\frac{pk}{k+1} + \frac{ql}{l+1} + \frac{sm}{m+1} - 2 + o(1)\right\} T(r,f) \leqslant S(r,f), \tag{10}$$

whence it follows that (1) holds, since otherwise, for $\lambda > 0$, (10) would imply that

$$\int_{r_0}^{r} \frac{T(x,f)}{x^{1+\lambda}}\,dx = O\left(\int_{r_0}^{r} \frac{S(x,f)}{x^{1+\lambda}}\,dx\right) = o\left(\int_{r_0}^{r} \frac{T(x,f)}{x^{1+\lambda}}\,dx\right),$$

which is impossible.

Thus, Theorem 1 and the consequences deduced therefrom hold if 'evB' is replaced everywhere by 'evE_2'.

If f is a meromorphic function of finite order and if a is an evE_2 for f for distinct zeros of order $\leqslant k$, where $a \in \bar{C}$, then

$$\bar{N}_k(r,a,f) = o(T(r,f)) = o(r^{\varrho(r)}),$$

where, as before, $\varrho(r)$ is a proximate order relative to $T(r,f)$, and so it follows that Theorem 2 and the consequences derived therefrom remain valid if 'evB' is replaced everywhere by 'evE_2'.

Again, if f is meromorphic and $a \in \bar{C}$, we call a an evE_1 for f for distinct zeros of order $\leqslant k$ if $\bar{n}_k(r,a,f) = o(T(r,f))$ (see also [2]).

Suppose that f is meromorphic of finite order $\varrho > 0$ and that $a \in \bar{C}$ is an evE_1 for f for distinct zeros of order $\leqslant k$. If $\varepsilon > 0$, then

$$\bar{n}_k(r,a,f) < \varepsilon T(r,f) \leqslant \varepsilon r^{\varrho(r)} \quad \text{for all } r \geqslant r_0,$$

where $\varrho(r)$ is, as before, a proximate order for $T(r,f)$.

So, for $r \geqslant r_0$,

$$\begin{aligned}\bar{N}_k(r,a,f) &= \int_0^{r_0} \frac{\bar{n}_k(x,a,f)}{x}\,dx + \int_{r_0}^{r} \frac{\bar{n}_k(x,a,f)}{x}\,dx \\ &\leqslant O(1) + \varepsilon \int_{r_0}^{r} x^{\varrho(x)-1}\,dx \\ &\sim \varepsilon \frac{r^{\varrho(r)}}{\varrho}.\end{aligned}$$

Thus $\bar{N}_k(r, a, f) = o(r^{\varrho(r)})$.

Now, under the hypothesis of Theorem 1 with $0 < \varrho < \infty$ and with 'evB' replaced everywhere by 'evE$_1$', we obtain, from (6),

$$\left(\frac{pk}{k+1} + \frac{ql}{l+1} + \frac{sm}{m+1} - 2\right) T(r, f) = o(r^{\varrho(r)}), \tag{11}$$

since $\varrho < \infty$, so that $S(r, f) = o(T(r, f))$. Hence (1) holds, since otherwise (11) would imply that $T(r, f) = o(r^{\varrho(r)})$, which would be a contradiction.

Thus Theorem 1 and the consequences deduced therefrom remain valid if 'evB' is replaced everywhere by 'evE$_1$' provided $0 < \varrho < \infty$. The same is true of Theorem 2 and its consequences.

Let $\mathscr{D}$ denote the class of all meromorphic functions f of finite order for which

$$\int_1^\infty \frac{T(r, f)}{r^{1+\varrho}}\, dr = \infty, \quad \text{where } \varrho \text{ is the order of } f.$$

If $f \in \mathscr{D}$ and $a \in \bar{C}$, Valiron calls [7] a an evG for f if

$$\int_1^\infty \frac{N(r, a, f)}{r^{1+\varrho}}\, dr < \infty,$$

where ϱ is the order of f.

We call a an evG for f for distinct zeros of order $\leqslant k$ if

$$\int_1^\infty \frac{\bar{N}_k(r, a, f)}{r^{1+\varrho}}\, dr < \infty.$$

If $f \in \mathscr{D}$, then, under the hypothesis of Theorem 1 with 'evB' replaced every where by 'evG', we obtain from

$$\left\{\frac{pk}{k+1} + \frac{ql}{l+1} + \frac{sm}{m+1} - 2\right\} \int_1^\infty \frac{T(r, f)}{r^{1+\varrho}}\, dr < \infty,$$

whence it follows that (1) holds, since otherwise we would have

$$\int_1^\infty \frac{T(r, f)}{r^{1+\varrho}}\, dr < \infty,$$

which would contradict the fact that $f \in \mathscr{D}$.

Thus Theorem 1 and its consequences hold if 'evB' is replaced everywhere by 'evG' provided $f \in \mathscr{D}$. A similar remark applies to Theorem 2 and its consequences.

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