ANNALES
POLONICI MATHEMATICI
XXXIX (1981)

Some remarks on a new pseudo-differential metric

by KYONG T. HAHN (University Park, Penn.)

Stefan Bergman in memoriam

Abstract. Let $S(\Delta, M)$ be the family of injective holomorphic mappings of the unit disc $\Delta \subset C$ into a complex manifold M. Following the method of the Kobayashi–Royden peudodifferential metric K_M and using the family $S(\Delta, M)$, the author introduces a new pseudodifferential metric S_M on the complex manifold M and studies some basic properties of this metric. The metric S_M has a distance-decreasing property under injective holomorphic mappings and coincides with the Carathéodory–Reiffen and Kobayashi–Royden differential metrics on any bounded symmetric domain. One of the interesting features of S_M is that it is larger than or equal to K_M, but differs from K_M on some spaces. For example, S_M defines a complete proper metric on $M = C-\{0\}$, while K_M is trivial. More generally, the n-dimensional complex projective space $P_n(C)$ minus $(n+1)$ hyperplanes in general position is hyperbolic with respect to S-metric, but the space $P_n(C)$ minus n hyperplanes in general position is not. An immediate consequence of this fact is that every injective holomorphic function in C^m must take every complex number. Other peculiarities of the metric S_M are also given.

1. Introduction. Let $\mathcal{S}(\Delta, M)$ be the family of (1-1) holomorphic mappings of the unit disc $\Delta \subset C$ into complex manifold M. In this note we introduce, using the family $\mathcal{S}$, a new pseudo-differential metric S_M analogous to the Kobayashi–Royden pseudo-differential metric K_M and study some basic properties of this metric. The metric S_M has a distance-decreasing property under (1-1) holomorphic mappings and coincides with the Carathéodory–Reiffen and Kobayashi–Royden differential metrics on any bounded symmetric domain. One of the interesting features of S_M lies in the fact that it is larger than or equal to K_M, but differs from K_M on some spaces. For example, S_M defines a complete proper metric on $M = C - \{0\}$, while K_M is trivial. More generally, the n-dimensional complex projective space $P_n(C)$ minus $(n+1)$ hyperplanes in general position is hyperbolic with respect to S-metric, but the space $P_n(C)$ minus n hyperplanes in general position is not. An immediate consequence of this fact is that there is no (1-1) holomorphic mapping of C^m into $P_n(C)$ minus $(n+1)$ hyperplanes in general position.

2. Preliminaries. Let M be a complex manifold of dimension n and $T(M)$ the complex tangent bundle on M. Following Grauert and Reckziegel ([1]), we define a *differential metric* on M by an upper-semicontinuous function

$$F_M\colon T(M) \to R^+ \cup \{0\}$$

such that for each $(z, \xi) \in T(M)$

$$(1) \qquad F_M(z, \lambda\xi) = |\lambda| F_M(z, \xi), \quad \lambda \in C,$$

and

$$(2a) \qquad F_M(z, \xi) > 0 \quad \text{for } \xi \neq 0.$$

We say that F_M is a *pseudo-differential metric* if, instead of condition (2a), it satisfies

$$(2b) \qquad F_M(z, \xi) \geqslant 0 \quad \text{for } (z, \xi) \in T(M).$$

The Carathéodory–Reiffen and Kobayashi–Royden pseudo-differential metrics are two well-known examples of pseudo-differential metrics.

Let $H(M, \Delta)$ be the family of holomorphic mappings of M into Δ. The Carathéodory–Reiffen metric (CR-metric) is defined by

$$(3) \qquad C_M(z, \xi) = \sup\{|df(z)\xi|\colon f \in H(M, \Delta) \ni f(z) = 0\}$$

([6]), while the Kobayashi–Royden metric (KR-metric) is

$$(4a) \qquad K_M(z, \xi) = \inf\{|v|\colon \exists f \in H(\Delta, M) \ni f(0) = z, f'(0)v = \xi\}$$

([7]), where $H(\Delta, M)$ is the family of holomorphic mappings of Δ into M.

In terms of a differential metric F_M, the KR-metric may be written in the form:

$$(4b) \qquad K_M(z, \xi) = \inf\left\{\frac{F_M(z, \xi)}{F_M(f(0), f'(0))}\colon \exists f \in H(\Delta, M) \ni f(0) = z\right\}.$$

If, in (4b), F_M is replaced by C_M, then

$$(5) \qquad K_M(z, \xi) \geqslant C_M(z, \xi), \quad (z, \xi) \ni T(M).$$

For,

$$(6) \qquad C_M(f(0), f'(0)) = \sup\{|dg(f(0))f'(0)|\colon g \in H(M, \Delta) \ni g(z) = 0\}$$
$$= \sup\{|d(g \circ f)(0)|\colon g \circ f \in H(\Delta, \Delta) \ni (g \circ f)(0) = 0\} \leqslant 1,$$

by the classical Schwarz lemma on $g \circ f$.

3. Basic properties of S-metric. Let $\mathscr{S}(\Delta, M)$ be the family of (1-1) holomorphic mappings of Δ into M. Analogous to the KR-metric, we define

$$(7) \qquad S_M(z, \xi) = \inf\{|v|\colon \exists f \in \mathscr{S}(\Delta, M) \ni f(0) = z, f'(0)v = \xi\}$$
$$= \inf\left\{\frac{F_M(z, \xi)}{F_M(f(0), f'(0))}\colon \exists f \in \mathscr{S}(\Delta, M) \ni f(0) = z\right\}$$

for $(z, \xi) \in T(M)$ and for any differential metric F_M on M. Using the argument of H. Royden ([7]) we see that S_M is a pseudo-differential metric. Furthermore, S_M satisfies the following:

PROPOSITION 1. *Let $f\colon M \to N$ be a* (1-1) *holomorphic mapping of M into another complex manifold N. Then*

$$S_N(f(z), df(z)\xi) \leqslant S_M(z, \xi), \quad (z, \xi) \in T(M).$$

If, in particular, $f\colon M \to N$ is onto, then

$$S_N(f(z), df(z)\xi) = S_M(z, \xi).$$

PROPOSITION 2. *For any complex manifold M,*

$$K_M(z, \xi) \leqslant S_M(z, \xi), \quad (z, \xi) \in T(M).$$

Let z and w be any two points in M. As usual, we define the integrated metric by

$$\mathscr{S}_M(z, w) = \inf \int_\gamma S_M(z, dz), \tag{8}$$

where the inf runs over all piecewise regular curves γ joining z and w in M.

Following S. Kobayashi ([3]), we define a pseudo-distance τ_M on M as follows: For any two points z and w in M, choose a chain of points $z = z_0, z_1, \ldots, z_k = w$ of M, points $a_1, \ldots, a_k$, $b_1, \ldots, b_k$ of Δ, and functions $f_1, \ldots, f_k \in \mathscr{S}(\Delta, M)$ such that

$$f_i(a_i) = z_{i-1} \quad \text{and} \quad f_i(b_i) = z_i, \quad i = 1, 2, \ldots, k.$$

Then

$$\tau_M(z, w) = \inf \sum_{i=1}^{k} \varrho_\Delta(a_i, b_i), \tag{9}$$

where the inf runs over all possible chains of the points connecting z and w in the manner described above and ϱ_Δ is the Poincaré metric on Δ. Then τ_M is an inner metric in the sense of Rinow, i.e., $\tau_M = \mathscr{S}_M$, as it easily follows from the method of Royden ([7]).

We note that the notion of pseudo-differential metric may as well be defined on a domain D in any infinite dimensional normed linear space X exactly in the same way as before.

We state the following theorem in a general normed linear space setting.

THEOREM 1. *Let B be the unit ball in any normed linear space X which is homogeneous. Then*

$$S_B(z, \xi) = K_B(z, \xi) = C_B(z, \xi), \quad z \in B, \ \xi \in X. \tag{10}$$

In particular, (10) *holds for any bounded symmetric domain of finite dimension.*

Proof. By the Schwarz lemma, for all $f \in \mathscr{S}(\Delta, B)$ with $f(0) = 0$,

$|f'(0)| \leqslant 1$, where $|\ |$ denotes the norm in X. See [2] for example. Hence,

$$|v| = \frac{|\xi|}{|f'(0)|} \geqslant |\xi|$$

for all $f \in \mathcal{S}(\Delta, B)$ with $f(0) = 0$ and $f'(0)v = \xi$. Therefore,

$$S_B(0, \xi) \geqslant |\xi|.$$

The mapping $h(z) = (x/|x|)z$, $x \in B$, $x \neq 0$, belongs to $\mathcal{S}(\Delta, B)$ and satisfies: $h(0) = 0$, $h'(0) = x/|x|$. Since $|h'(0)v| = |\xi|$ implies $|v| = |\xi|$, we have

$$S_B(0, \xi) = |\xi|. \tag{11}$$

Here $h(z)$ serves as an extremal map. So,

$$S_B(0, \xi) = K_B(0, \xi) = C_B(0, \xi) = |\xi|. \tag{12}$$

Since B is homogeneous, any point of B can be mapped by a (1-1) holomorphic mapping to the origin. By the invariant property of S_B, K_B and C_B, we have

$$S_B(z, \xi) = K_B(z, \xi) = C_B(z, \xi), \quad z \in B,\ \xi \in X. \tag{13}$$

Equalities (13) hold for any bounded symmetric domains of finite dimension when we observe that the homogeneous unit ball of any normed linear space includes all bounded symmetric domains of finite dimension including two exceptional cases.

In spite of this similarity of S_M and K_M, these two metrics behave differently.

Let $M = C - \{0\}$. Since C is a covering surface of M, $K_M(z, \xi) = 0$ for $(z, \xi) \in T(M)$. On the other hand, S_M is not identically zero. In fact, we have

THEOREM 2. *Let M be any domain in C with $M \neq C$. Then*

$$S_M(z, \xi) \geqslant |\xi|/4\delta(z) \tag{14}$$

and the integrated metric $\mathcal{S}_M$ satisfies:

$$\mathcal{S}_M(z_1, z_2) \geqslant \tfrac{1}{4}\left|\log\frac{\delta(z_1)}{\delta(z_2)}\right|, \quad z_1, z_2 \in M, \tag{15}$$

where $\delta(z)$ denotes the distance from $z \in M$ to $C \backslash M$. Furthermore, $\mathcal{S}_M$ is a complete metric.

Proof. Since M is a proper subset of C, $\delta(z)$ is finite for every $z \in M$. It is well known ([5]) that if f is a (1-1) holomorphic mapping of Δ into C, then for $z \in \Delta$,

$$\tfrac{1}{4}(1 - |z|^2)|f'(z)| \leqslant \operatorname{dist}(f(z), \partial f(\Delta)). \tag{16}$$

In particular, for $z = 0$,

$$\tfrac{1}{4}|f'(0)| \leqslant \operatorname{dist}(f(0), \partial f(\Delta)). \tag{17}$$

If $f(0) = z$, then

$$\frac{1}{|f'(0)|} \geqslant \frac{1}{4\delta(z)} \tag{18}$$

and, hence,

$$S_M(z, \xi) = \inf\left\{\frac{|\xi|}{|f'(0)|} : f \in \mathscr{S}(\Delta, M) \ni f(0) = z\right\} \geqslant |\xi|/4\delta(z). \tag{19}$$

Therefore, the integrated metric satisfies

$$\mathscr{S}_M(z_1, z_2) = \inf \int_\gamma S_M(z, dz) \geqslant \inf \int_\gamma \frac{|dz|}{4\delta(z)}, \tag{20}$$

where the inf is taken over all piecewise regular curves γ connecting z_1 and z_2 in M. Since

$$|z_1 - z_2| \geqslant |\delta(z_1) - \delta(z_2)|$$

for any z_1 and z_2 in M, we have

$$\left|\frac{d\delta(z)}{dz}\right| \leqslant 1. \tag{21}$$

Therefore,

$$\mathscr{S}_M(z_1, z_2) \geqslant \tfrac{1}{4} \inf \int_\gamma \left|\frac{d\delta(z)}{\delta(z)}\right| \geqslant \tfrac{1}{4} \inf \left|\int_\gamma \frac{d\delta(z)}{\delta(z)}\right| \geqslant \tfrac{1}{4}\left|\log \frac{\delta(z_1)}{\delta(z_2)}\right|, \tag{22}$$

which completes the proof.

COROLLARY 1. *There is no* (1-1) *holomorphic mapping of* C *into itself which omits at least one point.*

Let M and N be two complex manifolds. Applying the distance decreasing property of CR- and KR-metrics to the projections $p\colon M \times N \to M$ and $q\colon M \times N \to N$, we have

$$C_{M\times N} \geqslant \max(C_M, C_N) \tag{23}$$

and

$$K_{M\times N} \geqslant \max(K_M, K_N). \tag{24}$$

The opposite inequality also holds for the KR-metric. In fact, we prove:

LEMMA 1. *Let* M *and* N *be two complex manifolds. Then*

$$K_{M\times N} \leqslant \max(K_M, K_N) \tag{25}$$

and

$$S_{M\times N} \leqslant \max(S_M, S_N). \tag{26}$$

Proof. Let F_M and F_N be differential metrics of M and N, respectively. Then $\max(F_M, F_N)$ defines a differential metric on $M \times N$. Therefore, for each $(z, w) \in M \times N$.

$$(27) \quad K_{M\times N}((z,w),(\xi,\eta)) = \inf\left\{\frac{\max\{F_M(z,\xi), F(w,\eta)\}}{\max\{F_M(f(0), f'(0)), F_N(g(0), g'(0))\}}: \exists\, h = (f,g) \in H(\Delta, M\times N) \ni h(0) = (z,w)\right\}$$

for $(\xi, \eta) \in T_z(M) \times T_w(N)$. Let $\varepsilon > 0$. By the definition of $K_M(z, \xi)$ there exists an $f_1 \in H(\Delta, M)$ with $f_1(0) = z$ such that

$$(28) \qquad \frac{F_M(z,\xi)}{F_M(f_1(0), f_1'(0))} < K_M(z,\xi)+\varepsilon.$$

Similarly, there exists a $g_1 \in H(\Delta, N)$ with $g_1(0) = w$ such that

$$(29) \qquad \frac{F_N(w,\eta)}{F_N(g_1(0), g_1'(0))} < K_N(w,\eta)+\varepsilon.$$

Therefore, there exists an $h_1 = (f_1, g_1) \in H(\Delta, M \times N)$ with $h_1(0) = (z, w)$ such that

$$\begin{aligned}(30) \quad K_{M\times N}((z,w),(\xi,\eta)) &\leqslant \frac{\max\{F_M(z,\xi), F_N(w,\eta)\}}{\max\{F_M(f_1(0), f_1'(0)), F_N(g_1(0), g_1'(0))\}}, \\ &\leqslant \max\left\{\frac{F_M(z,\xi)}{F_M(f_1(0), f_1'(0))}, \frac{F_N(w,\eta)}{F_N(g_1(0), g_1'(0))}\right\} \\ &< \max\{K_M(z,\xi)+\varepsilon, K_N(w,\eta)+\varepsilon\}\end{aligned}$$

for every $\varepsilon > 0$. This proves (25). It is clear that the same argument works for $S_{M\times N}$ when the family $H(\Delta, M\times N)$ is replaced by $\mathscr{S}(\Delta, M\times N)$.

It is not likely that the same inequality in Lemma 1 holds for the CR-metric. Combining (24) and (25), we have

COROLLARY 2. *Let M and N be two complex manifolds. Then*

$$K_{M\times N} = \max\{K_M, K_N\}.$$

The same result is, however, not true for the S-metric. It can be shown by the following example: Let $M = C-\{0\}$ and $N = C$. Then $K_M = K_N = S_N = 0$ and $S_M(z,\xi) \geqslant |\xi|/4\delta(z)$. Therefore,

$$(31) \qquad \max(S_M, S_N)((z,w),(\xi,\eta)) \geqslant |\xi|/4\delta(z).$$

But, $S_{M\times N} = 0$. To see this, let $(z,w) \in M\times N$ and $(\xi,\eta) \in C^2$. Let $h_n \in \mathscr{S}(\Delta, M\times N)$ be given by

$$(32) \qquad h_n(\lambda) = (f_n, g_n)(\lambda) = (ze^{2\pi i\lambda},\ n\lambda + w), \qquad \lambda \in \Delta.$$

Then $h_n(0) = (z, w)$ and $h'_n(0) = (2\pi iz, n)$. From definition,

$$(33)\qquad S_{M\times N}((z, w), (\xi, \eta)) \leqslant \frac{\sqrt{|\xi|^2+|\eta|^2}}{\sqrt{|f'_n(0)|^2+|g'_n(0)|^2}} = \frac{\sqrt{|\xi|^2+|\eta|^2}}{\sqrt{4\pi^2|z|^2+n^2}},$$

which shows: $S_{M\times N} = 0$. But, we can prove:

THEOREM 3. *Let M and N be two complex manifolds. Then*

$$(34)\qquad \min[\max\{K_M, S_N\}, \max\{S_M, K_N\}] \leqslant S_{M\times N} \leqslant \max\{S_M, S_N\}.$$

Proof. The second inequality of (34) was proved in Lemma 1. Therefore, we only need to prove the first inequality. By definition,

$$(35)\quad S_{M\times N}((z, w), (\xi, \eta)) = \frac{\max\{F_M(z, \xi), F_N(w, \eta)\}}{\sup\max\{F_M(f(0), f'(0)), F_N(g(0), g'(0))\}},$$

where the sup is taken for $(f, g) \in \mathscr{S}(\Delta, M\times N)$ such that $(f, g)(0) = (z, w)$. For $(f, g) \in \mathscr{S}(\Delta, M\times N)$, the following three cases are possible.

1° $f \in \mathscr{S}(\Delta, M)$ and $g \in \mathscr{S}(\Delta, N)$;

2° $f \in \mathscr{S}(\Delta, M)$ and $g \in H(\Delta, N)$;

3° $f \in H(\Delta, M)$ and $g \in \mathscr{S}(\Delta, N)$.

Case 1°. Replacing F_M by S_M and F_N by S_N in (35), we have

$$S_{M\times N}((z, w), (\xi, \eta)) \geqslant \max\{S_M(z, \xi), S_N(w, \eta)\}.$$

Case 2°. Replacing F_M by S_M and F_N by K_N in (35), we have

$$S_{M\times N}((z, w), (\xi, \eta)) \geqslant \max\{S_M(z, \xi), K_N(z, \eta)\}.$$

Case 3°. Replacing F_M by K_M and F_N by S_N in (35), we have

$$S_{M\times N}((z, w), (\xi, \eta)) \geqslant \max\{K_M(z, \xi), S_N(w, \eta)\}.$$

In any case we have the first inequality of (35), since $S_M \geqslant K_M$ and $S_N \geqslant K_N$.

4. Hyperbolicity in S-metric.

DEFINITION. Let M be a complex manifold furnished with a pseudo-differential metric F_M. M is said to be *hyperbolic* with respect to F_M if for each $z_0 \in M$ there exists a neighbourhood $U(z_0)$ and a constant $c > 0$ such that

$$(36)\qquad F_M(z, \xi) \geqslant c|\xi| \quad \text{for } z \in U(z_0) \text{ and } \xi \in T_z(M).$$

From Theorem 2, $M = C-\{0\}$ is hyperbolic with respect to S_M-metric. Moreover, we have

THEOREM 4. *Let $\tilde{M} = C-\{0\}\times \ldots \times C-\{0\}$ be the Cartesian product of n copies of $C-\{0\}$. Then $\tilde{M}$ is hyperbolic with respect to $S_{\tilde{M}}$.*

Proof. Repeated use of the first inequality of Theorem 3 and inequality (14) imply

$$S_{\tilde{M}}(z, \xi) \geqslant \tfrac{1}{4} \min \left\{ \frac{|\xi_j|}{\delta(z_j)} : j = 1, 2, \ldots, n \right\}, \tag{37}$$

where $z = (z_1, \ldots, z_n) \in \tilde{M}$, $\xi = (\xi_1, \ldots, \xi_n) \in C^n$ and $\delta(z_j)$ denotes the distance from z_j to the boundary of $C - \{0\}$. The hyperbolicity of $\tilde{M}$ is now clear from (37).

THEOREM 5. *Let M be the n-dimensional complex projective space which omits $(n+1)$ hyperplanes in general position. Then M is hyperbolic with respect to S_M.*

Proof. Without loss of any generality we may assume that

$$M = \pi\{(z_0, \ldots, z_n) \in C^{n+1} - \{0\} : z_j \neq 0,\ j = 0, 1, \ldots, n\}, \tag{38}$$

where $\pi \colon C^{n+1} - \{0\} \to P_n(C)$ is the canonical projection. That is,

$$M = P_n(C) - H_0 \cup H_1 \cup \ldots \cup H_n, \tag{39}$$

where $H_j = \pi\{(z_0, \ldots, z_n) \in C^{n+1} - \{0\} : z_j = 0\}$, $j = 0, 1, \ldots, n$. Since M is biholomorphically equivalent to $\tilde{M} = C - \{0\} \times \ldots \times C - \{0\}$ which is hyperbolic with respect to S_M by Theorem 4, M is also hyperbolic with respect to S_M.

In view of $S_{C^m} = 0$ for all $m \in N$, we have the generalization of Corollary 1.

COROLLARY 3. *There can not be any (1-1) holomorphic mapping from C^m into $P_n(C)$ minus $(n+1)$ hyperplanes in general position. Equivalently, there is no (1-1) holomorphic mapping of C^m into C^n minus n hyperplanes in general position.*

THEOREM 6. *Let $M = C \times C - \{0\} \times \ldots \times C - \{0\}$ be the Cartesian product of C with $(n-1)$ copies of $C - \{0\}$. Then M is not hyperbolic with respect to S_M.*

Proof. Let z^0 be any point in M. We want to know if there exists a neighbourhood $U(z^0)$ and $c > 0$ such that

$$S_M(z, \xi) \geqslant c \max_{1 \leqslant j \leqslant n} |\xi_j|$$

or

$$\max_{2 \leqslant j \leqslant n} \frac{|\xi_j|}{\delta(z_j)} \geqslant 4c \max_{1 \leqslant j \leqslant n} |\xi_j|. \tag{40}$$

For definiteness, we assume that

$$\frac{|\xi_2|}{\delta(z_2)} = \max_{2 \leqslant j \leqslant n} \frac{|\xi_j|}{\delta(z_j)}. \tag{41}$$

If $\max_{1 \leqslant j \leqslant n} |\xi_j| = |\xi_2|$, then

$$\frac{|\xi_2|}{\delta(z_2)} \geqslant 4c\,|\xi_2|.$$

or

(42) $$\frac{1}{\delta(z_2)} \geqslant 4c.$$

Since $\delta(z_2) < \infty$, there exists a neighbourhood $U(z^0)$ of z^0 and $c > 0$ so that (42) holds for all $z \in U(z^0)$. If $\max|\xi_j| = |\xi_k|$ for $k \neq 2$, then

$$\frac{|\xi_2|}{\delta(z_2)} \geqslant 4c\,|\xi_k|$$

or

(43) $$\frac{|\xi_2|}{|\xi_k|} > 4c\delta(z_2).$$

Keeping ξ_2 finite and taking $|\xi_k|$ sufficiently large we may let $\varepsilon > |\xi_2|/|\xi_k|$ for any given $\varepsilon > 0$. From (43), $c < \varepsilon/4\delta(z_2)$ and there is no $c > 0$ which satisfies (40).

COROLLARY 4. *The n dimensional complex projective space $P_n(C)$ minus n hyperplanes in general position is not hyperbolic with respect to S-metric.*

In this connection, we show by an example that the hyperbolicity in the sense of Wong ([8]) differs from that of Royden ([7])

EXAMPLE. Let $M = C - \{0, 1\}$ and $N = C$. Since $K_C = 0$, from Corollary 2,

(44) $$K_{M\times N}((z, w), (\xi, \eta)) = K_M(z, \xi) > 0$$

for $(z, w) \in M \times N$ and $(\xi, \eta) \in C^2$. Therefore, $M \times N$ is hyperbolic with respect to KR-metric in the sense of Wong, but it fails to be hyperbolic in the sense of Royden.

To show this, we may assume that $M = \Delta$ and $N = C$. Then

(45) $$K_{\Delta\times C}((z, w), (\xi, \eta)) = K_\Delta(z, \xi) = \frac{|\xi|}{1-|z|^2}$$

for $(z, w) \in \Delta \times C$ and $(\xi, \eta) \in C^2$. Let (z_0, w_0) be any point in $\Delta \times C$. We need to check if there exists a neighbourhood U of (z_0, w_0) and a constant $c > 0$ such that

(46) $$\frac{|\xi|}{1-|z|^2} \geqslant n \max(|\xi|, |\eta|) \quad \text{for } (\xi, \eta) \in C^2 \quad \text{and } (z, w) \in U.$$

If $|\xi| \geqslant |\eta|$, it clearly holds, while if $|\eta| > |\xi|$,

(47) $$\frac{|\xi|}{|\eta|} \geqslant c(1-|z_0|^2).$$

Since $\eta \in C$ may be chosen arbitrarily for each finite ξ, $\varepsilon = |\xi|/|\eta|$ can be made arbitrarily small. Since $z_0 \in \Delta$, there exists an r such that $|z_0| \leqslant r < 1$. From (47), $c \leqslant \frac{\varepsilon}{1-|z_0|^2} < \frac{\varepsilon}{1-r^2}$. This shows that there is no positive constant c which satisfies (46).

Concerning the continuity of S-metric we can only prove:

THEOREM 7. *Let M be a Riemann surface. Then $S_M(z, \xi)$ is continuous on $T(M)$.*

Proof. If M is of genus 0, then $S_M = 0$ and, hence, it is continuous. If M is of genus greater than 0, then S_M defines a proper metric in which case

$$S_M(z, \xi) = \inf\left\{\frac{|\xi|}{|f'(0)|}: \exists f \in \mathscr{S}(\Delta, M) \ni f(0) = z\right\}.$$

Let $g(\zeta) = \frac{f(\zeta)-f(0)}{f'(0)}$. Then g is univalent in Δ and satisfies: $g(0) = 0$, $g'(0) = 1$. Since the class of univalent functions g, normalized as above, forms a normal family, so does the class $\mathscr{S}(\Delta, M)$. Let $(z, \xi) \in T(M)$ and let $(z_k, \xi_k) \to (z, \xi)$. Let $\varepsilon > 0$. Suppose that $f_k \in \mathscr{S}(\Delta, M)$ is a sequence such that $f_k(0) = z_k$, $f_k'(0) = \xi_k$ and

$$\frac{|\xi_k|}{|f_k'(0)|} < S_M(z_k, \xi_k)+\varepsilon \tag{48}$$

for sufficiently large k. Since $f_k \in \mathscr{S}(\Delta, M)$ is a normal family, there exists a subsequence $\{f_{k_j}\}$ such that $f_{k_j}(z) \to f(z)$ uniformly in Δ. The limit function f is either univalent or constant in Δ. We may assume that $\xi \neq 0$. Then f can not be constant. Therefore, f is in $\mathscr{S}(\Delta, M)$ and satisfies: $f(0) = z$ and $f'(0)v = \xi$. Thus,

$$S_M(z, \xi) = \frac{|\xi|}{|f'(0)|} < S_M(z_k, \xi_k)+\varepsilon.$$

This proves the continuity of $S_M(z, \xi)$, since S_M is always upper-semi-continuous.

Acknowledgement. The author would like to thank the SNU-AID Project for its generous support in his research activity at Seoul National University.

References

[1] H. Grauert and H. Reckziegel, *Hermitesche Metriken und normale Familien holomorphic Abbildungen*, Math. Z. 89 (1965), p. 108–125.

[2] L. Harris, *Bounded symmetric homogeneous domains in infinite dimensional spaces*, Proc. on Infinite Dimensional Holomorphy, Lecture Notes in Math. 364, Springer-Verlag, 1974.

[3] S. Kobayashi, *Invariant distances on complex manifolds and holomorphic mappings*, J. Math. Soc. Japan 19 (1967), p. 460–480.

[4] — *Intrinsic distances, measures and geometric function theory*, Bull. Amer. Math. Soc. 82 (1976), p. 357–416.

[5] Chr. Pommerenke, *Univalent functions*, Vandenboeck and Ruprecht Göttingen, Germany, 1975.

[6] H. J. Reiffen, *Die differential geometrischen Eigenschaften der invarianten Distanzfunktion von Carathéodory*, Schr. Math. Inst. Univ. Münster No. 26 (1963).

[7] H. L. Royden, *Remarks on the Kobayashi metric*, Several Complex Variables, II (Proc. International Conf., Univ. of Maryland (1970), Lecture Notes in Math., vol. 185, Springer-Verlag, Berlin 1971, p. 125–137.

[8] B. Wong, *On the holomorphic curvature of some intrinsic metrics*, Proc. Amer. Math. Soc. 65 (1977), p. 57–61.

DEPARTMENT OF MATHEMATICS, THE PENNSYLVANIA STATE UNIVERSITY
UNIVERSITY PARK, PENNSYLVANIA
DEPARTMENT OF MATHEMATICS, SEOUL NATIONAL UNIVERSITY, SEOUL, KOREA

Reçu par la Rédaction le 21. 12. 1978