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Special solutions of a functional equation

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The equation in question is

(1) $$\varphi(x) = h(x, \varphi[f(x)]),$$

where φ is the unknown function, x is a real variable, and all functions appearing in (1) are real. We assume that Ω is a plane set with the properties:

(2) for every $x \in \Omega$ there exists a closed rectangle P such that $x \in P \subset \Omega$,

and

(3) $$(0, 0) \in \Omega,$$

and let I be an interval (open, closed or half-open, finite or not) such that

(4) $$0 \in I.$$

Following Choczewski [1] we shall call a function g on I *regular* iff it is continuous in I and has a derivative at $x = 0$ (in I).

A value z_0 is said to be *admissible at a point* $(x_0, y_0) \in \Omega$ iff there exist two closed intervals, I_1 and I_2, each of them having zero as one end, and such that $xx_0 \geqslant 0$ for $x \in I_1$, $yz_0 \geqslant 0$ for $y \in I_2$ and $I_1 \times I_2 \subset \Omega$. Note that every value is admissible if $(0, 0)$ is an inner point of Ω.

We assume the following conditions regarding the functions f and h.

(i) f is regular in an interval I fulfilling (4) and $0 < f(x)/x < 1$ for $x \in I$, $x \neq 0$.

(ii) h is continuous in a set Ω fulfilling (2) and (3), moreover,

(5) $$h(x, y) = Ax + By + o(|x| + |y|), \quad (x, y) \to (0, 0).$$

Further, h fulfils a Lipschitz condition

(6) $$|h(x, y_1) - h(x, y_2)| \leqslant L|y_1 - y_2|$$

in a neighbourhood of $(0, 0)$ ([1]).

([1]) A neighbourhood of a plane point will always be meant relatively to Ω, and a neighbourhood of a point on the real line — relatively to I.

(iii) The function h takes only admissible values in a neighbourhood of $(0, 0)$.

(iv) We have

$$Lf'(0) < 1. \tag{7}$$

THEOREM 1. *Under conditions* (i)-(iv) *there exists exactly one regular solution* φ *of equation* (1) *in a neighbourhood of* $x = 0$ *such that*

$$\varphi(0) = 0. \tag{8}$$

This solution φ *is given as the limit of successive approximations:*

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x) \tag{9}$$

with

$$\varphi_{n+1}(x) = h\big(x, \varphi_n[f(x)]\big), \tag{10}$$

where φ_0 *is an arbitrary function regular in a neighbourhood of* $x = 0$ *with*

$$\varphi_0'(0) = A/\big(1-Bf'(0)\big). \tag{11}$$

Proof. If φ is a regular solution of (1) fulfilling (8) in an interval $J \subset I$, where $0 \in J$, then the function $\psi(x) = \varphi(x)/x$, defined as $\varphi'(0)$ for $x = 0$, is a continuous solution of the equation

$$x\psi(x) = h\big(x, f(x)\psi[f(x)]\big) \tag{12}$$

in J; and conversely, if ψ is a continuous solution of equation (12) in J, then $\varphi(x) = x\psi(x)$ is a regular solution of equation (1) in J. Thus we need only prove that equation (12) has a unique continuous solution in a neighbourhood of zero.

Write

$$H(x, z) = x^{-1}h\big(x, f(x)z\big) \tag{13}$$

with

$$H(0, z) = A + Bf'(0)z. \tag{14}$$

By (5) H is continuous in a neighbourhood of $(0, 0)$. On the other hand, (6) implies that

$$|H(x, z_1) - H(x, z_2)| \leqslant \frac{f(x)}{x} L|z_1 - z_2| \tag{15}$$

in a neighbourhood of $(0, \eta)$, for arbitrary η (the neighbourhood depends on η). It follows from (15), (14) and (7) that

$$|Bf'(0)| < 1. \tag{16}$$

If a function $\psi(x)$, defined and continuous in a neighbourhood of zero satisfies equation (12), then the value

$$\eta = \psi(0) \tag{17}$$

fulfils the equation

$$\eta = H(0, \eta). \tag{18}$$

In view of (14) and (16) equation (18) has the unique solution

$$\eta = \frac{A}{1 - Bf'(0)}. \tag{19}$$

Now, we may find closed intervals I_1, I_2 such that $0 \in I_1$, $\eta \in I_2$, and the function H is defined in $I_1 \times I_2$ (if possible, we choose I_1 and I_2 in such a manner that 0 resp. η belong to the interior of I_1 resp. I_2). Further, we may fix I_1 and I_2 in such a manner that (15) holds in $I_1 \times I_2$ with

$$L\frac{f(x)}{x} < \vartheta < 1 \tag{20}$$

(cf. (7)) and

$$H(x, z) \in I_2 \quad \text{for } (x, z) \in I_1 \times I_2. \tag{21}$$

The possibility of a realization of condition (21) follows from conditions (15) with (20) and, in the case $\eta = 0$, from assumption (iii).

The existence and uniqueness of ψ for $x \in I_1$, as well as the formula

$$\psi(x) = \lim_{n\to\infty} \psi_n(x) \tag{22}$$

with

$$\psi_{n+1}(x) = H\big(x, \psi_n[f(x)]\big), \tag{23}$$

where ψ_0 is an arbitrary continuous function in I_1 fulfilling (17) and such that $\psi_0(x) \in I_2$ for $x \in I_1$, result from Banach's fixed point principle for contraction maps. Relations (9) and (10) are a consequence of (22) and (23), whereas (11) follows from (17) and (19).

THEOREM 2. *Under conditions* (i)-(iv) ([2]), *if, moreover, the functions f and h are convex and increasing* (*h with respect to either variable*) *in a neighbourhood of* 0 *resp.* (0, 0), *then equation* (1) *has a unique convex solution in a neighbourhood of zero, fulfilling* (8). *This solution is given by* (9) *with* (10), *where φ_0 is an arbitrary convex and increasing function in a neighbourhood of zero, fulfilling* (11).

Proof. Assume first that 0 is an end-point of I. Then every convex function in a neighbourhood of $x = 0$ is regular, which implies the uniqueness of the solution. The existence results from formula (9), since under our conditions all functions φ_n are convex.

([2]) The regularity of f need not be assumed. In this case formula (11) must be postulated for left-hand side and right-hand side derivatives (if 0 is an inner point of I).

If 0 is an inner point of I, we need only split I into two subintervals, each with zero as an end-point.

In order to obtain global results we must make further assumptions. Put

$$\Omega_x = \{y\colon (x, y) \in \Omega\},$$

and suppose that

(v) For every $x \in I$ the set Ω_x is an interval.

(vi) For every $x \in I$ we have $h(f(x), \Omega_{f(x)}) \subset \Omega_x$.

THEOREM 3. *Under conditions* (i)-(vi) *there exists exactly one regular solution* φ *of equation* (1) *in* I *fulfilling condition* (8). *This solution is given in* I *by* (9) *with* (10), *where* φ_0 *is an arbitrary regular function in* I *fulfilling* (11).

THEOREM 4. *Under conditions* (i)-(vi) (3), *if, moreover, the functions* f *and* h *are convex and increasing* (*h with respect to either variable*) *in* I *resp.* Ω, *then equation* (1) *has a unique convex solution* φ *in* I *fulfilling condition* (8). *This solution is given by* (9) *with* (10), *where* φ_0 *is an arbitrary convex function in* I *fulfilling* (11).

Theorems 3 and 4 result from Theorems 1 and 2 and from the extension theorem ([3], Theorem 3.3; the assumption, appearing in [3], that Ω is open may be released without any change in the argument). The convexity of φ in Theorem 4 follows from the fact that all the functions φ_n are convex in I.

Remark. Convex solutions of equation (1) have recently been studied (under different conditions) by J. Matkowski and Z. Kominek [4], [2].

(3) Footnote (2) applies also in this case.

References

[1] B. Choczewski, *Regular solutions of a linear functional equation of the first order*, A Nehézipari Müszaki Egyetem Közleményei (Miskolc) 30 (1970), p. 255 - 262.

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[3] M. Kuczma, *Functional equations in a single variable*, Monografie Mat. 46, Warszawa 1968.

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