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Remark on a partial differential inequality of the first order

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In known theorems on a partial differential inequality of the first order

$$\frac{\partial z}{\partial x} \leqslant f\left(x, y, z, \frac{\partial z}{\partial y}\right), \tag{1}$$

the function f is supposed to be of class C^2 or to satisfy assumptions implying the existence of solutions for the equation

$$\frac{\partial z}{\partial x} = f\left(x, y, z, \frac{\partial z}{\partial y}\right) \tag{2}$$

with arbitrary smooth initial data.

In this note we give a theorem on the inequality (1) under weaker assumptions which do not imply the existence of solutions for (2) even for smooth initial data. We shall deal with the case of two independent variables but the result can be easily extended on the general case.

We shall consider functions satisfying differential inequalities on the set S: $a \leqslant x \leqslant b$, $g(x) \leqslant y \leqslant h(x)$, $a < b$, $g(x) < h(x)$ on $\langle a, b\rangle$, where $g(x), h(x)$ are functions of class C^1 on $\langle a, b\rangle$. On the boundary of S we shall consider sets A: $x = a$, $g(a) \leqslant y \leqslant h(a)$; G: $a \leqslant x \leqslant b$, $y = g(x)$, H: $a \leqslant x \leqslant b$, $y = h(x)$.

Now we shall prove a variant of a well-known theorem on a (strong) partial differential inequality

$$\frac{\partial z}{\partial x} < f\left(x, y, z, \frac{\partial z}{\partial y}\right). \tag{3}$$

We start with a lemma.

Lemma. *Suppose that*

(4) *function u is continuous and possesses the derivatives u_x, u_y (not necessarily continuous) on S and total differential on $G \cup H$,*

(5) *if for certain point $(\xi, \eta) \in S$ function u satisfies equalities $u(\xi, \eta) = u_y(\xi, \eta) = 0$ then $u_x(\xi, \eta) < 0$,*

(6) *if* $(\xi,\eta)\in G$, $u(\xi,\eta)=0$ *and* $u_y(\xi,\eta)\leqslant 0$ *then* $u_x(\xi,\eta)+g'(\xi)u_y(\xi,\eta)<0$,

(7) *if* $(\xi,\eta)\in H$, $u(\xi,\eta)=0$, $u_y(\xi,\eta)\geqslant 0$ *then* $u_x(\xi,\eta)+h'(\xi)u_y(\xi,\eta)<0$,

(8) $u(x,y)<0$ *on* A.

Under above assumptions

$$u(x,y)<0 \quad \text{on } S. \tag{9}$$

Remark 1. The existence of derivatives u_x, u_y or total differential at such points (x,y) that $u(x,y)\neq 0$ is not necessary. It was assumed to make announcement of the Lemma simpler.

Proof of the Lemma. Suppose relation (9) fails to be satisfied. In virtue of (4), (8) there exists a point $(\xi,\eta)\in S$ that

$$u(x,y)<0 \quad \text{for} \quad x<\xi,\ (x,y)\in S, \tag{10}$$

and

$$u(\xi,\eta)=0. \tag{11}$$

Obviously $u(\xi,y)\leqslant 0$ for $(\xi,y)\in S$. Therefore either (i) $(\xi,\eta)\in G$ and $u_y(\xi,\eta)\leqslant 0$ or (ii) $(\xi,\eta)\in H$ and $u_y(\xi,\eta)\geqslant 0$ or (iii) $(\xi,\eta)\notin G\cup H$ and $u_y(\xi,\eta)=0$. In virtue of (8) $\xi>a$. In the case (i) we have

$$\begin{aligned}\frac{d}{dx}u\big(x,g(x)\big)_{|x=\xi}&=u_x\big(\xi,g(\xi)\big)+g'(\xi)u_y\big(\xi,g(\xi)\big)\\&=u_x(\xi,\eta)+g'(\xi)u_y(\xi,\eta)<0\end{aligned}$$

because of (6) which gives contradiction with (10), (11). Similarly in the case (ii)

$$\frac{d}{dx}u\big(x,h(x)\big)_{|x=\xi}=u_x(\xi,\eta)+h'(\xi)u_y(\xi,\eta)<0$$

because of (7) which gives contradiction with (10), (11). In case (iii) the contradiction follows from (5).

THEOREM 1. *Let us assume that functions* $u=z(x,y)$, $u=s(x,y)$ *satisfy assumption* (4) *and the differential inequalities*

$$z_x<f(x,y,z,z_y) \quad \text{on } S, \tag{12}$$

$$s_x\geqslant f(x,y,s,s_y) \quad \text{on } S, \tag{13}$$

where function $f(x,y,z,q)$ *is defined for* $(x,y)\in S$, $z=z(x,y)$, $q=z_y(x,y)$ *as well as for* $(x,y)\in S$, $z=s(x,y)$, $q=s_y(x,y)$ (f *is not necessarily continuous on its domain* $D(f)$) *and satisfies the inequalities*

$$f(x,y,z,p)-f(x,y,z,q)\geqslant -g'(x)(p-q) \quad \text{for} \quad p>q,\ (x,y)\in G, \tag{14}$$

$$f(x,y,z,p)-f(x,y,z,q)\leqslant -h'(x)(p-q) \quad \text{for} \quad p>q,\ (x,y)\in H. \tag{15}$$

Moreover we assume that

(16) $$z(x, y) < s(x, y) \quad \text{on } A.$$

Then the inequality

(17) $$z(x, y) < s(x, y) \quad \text{on } S$$

is satisfied.

For the proof of Theorem 1 we put $u(x, y) = z(x, y) - s(x, y)$ and apply Lemma. We shall show (7), conditions (5), (6) can be shown similarly, (8) results from (16) and u satisfies (4) because z, s do. To prove (7) assume that $(\xi, \eta) \in H$, $u(\xi, \eta) = 0$ and $u_y(\xi, \eta) \geqslant 0$. We have $z(\xi, \eta) = s(\xi, \eta)$, $z_y(\xi, \eta) \geqslant s_y(\xi, \eta)$ and therefore by (15)

$$f\big(\xi, \eta, z(\xi, \eta), z_y(\xi, \eta)\big) - f\big(\xi, \eta, s(\xi, \eta), s_y(\xi, \eta)\big) \leqslant -h'(\xi)\big(z_y(\xi, \eta) - s_y(\xi, \eta)\big).$$

By (12), (13) it follows that

$$z_x(\xi, \eta) - s_x(\xi, \eta) < -h'(\xi)\big(z_y(\xi, \eta) - s_y(\xi, \eta)\big) \quad \text{or} \quad u_x(\xi, \eta) + h'(\xi) u_y(\xi, \eta) < 0.$$

Assumptions of Lemma being satisfied, we have (9) and inequality (17) results.

Now we formulate the main result.

THEOREM 2. *Let us assume that functions $u = z(x, y)$, $u = s(x, y)$ satisfy property* (4) *and the differential inequalities*

(18) $$z_x \leqslant f(x, y, z, z_y) \quad \text{on } S,$$

(19) $$s_x \geqslant f(x, y, s, s_y) \quad \text{on } S.$$

Function $f(x, y, z, q)$ satisfies (14), (15) *and the inequality*

(20) $$f(x, y, s, q) - f(x, y, z, q) \leqslant l(x, s - z)$$

for $s > z$, $(x, y, s, q) \in D(f)$, $(x, y, z, q) \in D(f)$ where $l(x, u)$ is such a continuous function defined for $a \leqslant x \leqslant b$, $u \geqslant 0$ that

(21) *$u = 0$ is the unique solution of the ordinary differential equation $u' = l(x, u)$ with the initial condition $u(a) = 0$.*

Moreover we assume that

(22) $$z(x, y) \leqslant s(x, y) \quad \text{on } A.$$

Then the inequality

(23) $$z(x, y) \leqslant s(x, y) \quad \text{on } S$$

is satisfied (1).

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(1) I have presented this theorem in a course on differential equations given in 1956/57 at the Mathematical Institute of the Polish Academy of Sciences.

Proof. We have

$$l(x, 0) = 0 \quad \text{for} \quad a \leqslant x \leqslant b \tag{24}$$

because of (21). Denote by $u_n(x)$ a solution of the equation

$$u' = l(x, u) + \frac{1}{n} \tag{25}$$

with the initial condition

$$u(a) = \frac{1}{n} . \tag{26}$$

From (24), (25) and (26) we can deduce that

$$u_n(x) > 0 \quad \text{on } \langle a, b \rangle \tag{27}$$

or on a largest subinterval of $\langle a, b \rangle$ on which $u_n(x)$ exists. In virtue of (21) by a well-known theorem on continuous dependence of integrals on right side of an ordinary differential equation and initial data we obtain

$$u_n(x) \to 0 \quad \text{on } \langle a, b \rangle \text{ as } n \to \infty \tag{28}$$

($u_n(x)$ are defined on $\langle a, b \rangle$ for $n \geqslant N$, where N is sufficiently large integer). In virtue of (25) we have

$$u_n' > l(x, u_n) . \tag{29}$$

Consider functions

$$z(n, x, y) = z(x, y) - u_n(x) \quad \text{for} \quad n \geqslant N \text{ (}^2\text{)} . \tag{30}$$

We shall show that $z = z(n, x, y)$ satisfies inequality (12).

Indeed,

$$z_x(n, x, y) = z_x(x, y) - u_n'(x) < z_x(x, y) - l\big(x, u_n(x)\big)$$

because of (29). By (18)

$$z_x(n, x, y) < f\big(x, y, z(x, y), z_y(x, y)\big) - l\big(x, u_n(x)\big) .$$

In virtue of (20), (30), (27) we have the inequality

$$f\big(x, y, z(x, y), z_y(x, y)\big) - f\big(x, y, z(n, x, y), z_y(n, x, y)\big) - l\big(x, u_n(x)\big) \leqslant 0 .$$

Subtracting the left side of the latter inequality from the right side of the former one we obtain inequality (12) for $z = z(n, x, y)$. The inequality $z(n, x, y) < s(x, y)$ on A results from (22), (30), (27). Therefore, by Theorem 1,

(²) A substitution of that kind was independently applied to partial differential inequalities by P. Besala.

we have the inequality $z(n, x, y) < s(x, y)$ on S $(n \geqslant N)$. In virtue of (30), (28) we obtain (23). The proof of Theorem 2 is thus complete.

Remark 2. Instead of inequality (20) we can assume Lipschitz condition

$$|f(x, y, s, q) - f(x, y, z, q)| \leqslant M|s - z| \quad \text{on } D(f),$$

because as is well known function $u = 0$ is the unique solution of the equation $u' = Mu$ with the initial condition $u(a) = 0$.

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