ANNALES
POLONICI MATHEMATICI
XV (1964)

Applications of Denjoy analogue, I*
(Sufficient conditions for a function to be monotone)

by K. M. GARG ** (Lucknow)

In an earlier paper [1] the author has proved a theorem (1) called as "Denjoy analogue". In this note we investigate some applications of this theorem.

In 1951 T. Ważewski proved (2) that

(I) "If $f(x)$ is a continuous function in an interval I and Q denotes the set of points $x \in I$ for which

$$(*) \qquad D^+ f(x) < 0 ,$$

a necessary and sufficient condition for $f(x)$ to be monotone non-decreasing in I is that

$$(**) \qquad mf(Q) = 0 ."$$

It is also known (see Saks [4], p. 204, Theorem (7.2)) that

(II) "If $f(x)$ is a finite function of one variable such that (i) $\limsup_{h\to 0+} f(x-h) \leqslant f(x) \leqslant \limsup_{h\to 0+} f(x+h)$ at every point x, and (ii) $D^+ f(x) \geqslant 0$ at every point x except at most at those of an enumerable set, then the function $f(x)$ is monotone non-decreasing. "

---

* This research project was formerly supported by the Ministry of Scientific Research and Cultural Affairs, Govt. of India, and at present by the University Grants Commission of India.

** The author wishes to express his gratitude to Dr. U. K. Shukla for help and suggestions.

(1) For the sake of convenience this theorem is reproduced as follows:

"Given an arbitrary finite real function $f(x)$ defined in a linear interval I, the Dini derivates of $f(x)$ at each point x, except possibly at a set $A \subset I$ for which $mf(A) = 0$, satisfy one of the following four relations:

$$(\text{a.1}) \qquad D^+ f(x) = D_+ f(x) = D^- f(x) = D_- f(x) \neq 0 ,$$

$$(\text{a.2}) \qquad D^+ f(x) = +\infty , \quad D_- f(x) = -\infty , \quad D_+ f(x) = D^- f(x) \neq 0 ,$$

$$(\text{a.3}) \qquad D_+ f(x) = -\infty , \quad D^- f(x) = +\infty , \quad D^+ f(x) = D_- f(x) \neq 0 ,$$

$$(\text{a.4}) \qquad D^+ f(x) = D^- f(x) = +\infty , \quad D_+ f(x) = D_- f(x) = -\infty ."$$

(2) See Ważewski [5], p. 117, Theorem 1. The present enunciation follows from the theorem just on considering the function $-f(x)$.

By an application of Denjoy analogue we prove (in § 1, Theorem 1) that the sufficiency part [3] of (I) remains valid even if (i) the continuity of $f(x)$ is replaced by the condition (i) of (II), (ii) the measure in (**) is replaced by interior measure, and (iii) the set Q is replaced by one of its subsets. We also prove (in § 1, Theorem 2) that (II) remains still valid if the points where $D^+f(x) < 0$ form a set of measure zero and if the points where $D^+f(x) = -\infty$ form a set whose power is less than that of the continuum.

The monotony of continuous functions which fulfil the Banach's conditions [4] (T_1) and (T_2) has been investigated in §§ 2 and 3.

1. Arbitrary real functions. Let $f(x)$ be a finite real function of a real variable such that

$$\limsup_{h\to 0+} f(x-h) \leqslant f(x) \leqslant \limsup_{h\to 0+} f(x+h) \tag{1}$$

at every point x, and let

$$E = \{x;\ D^+f(x) \leqslant 0\}\,. \tag{2}$$

In case $f(E)$ contains a non-degenerate interval, we evidently have

$$m_i f(E) > 0\,. \tag{3}$$

But, according to the Denjoy analogue (see footnote [1]), we have

$$m_i f(E) = m_i f(E_1 + E_2)\,, \tag{4}$$

where

$$E_1 = \{x;\ f'(x) \text{ exists } ^{(5)} \text{ and is } < 0\}\,, \tag{5}$$

and

$$E_2 = \{x;\ D^+f(x) = D_-f(x) < 0\,,\ D_+f(x) = -\infty\,,\ D^-f(x) = +\infty\}\,. \tag{6}$$

Thus, if $m_i f(E_1+E_2) = 0$, the set $f(E)$ contains no non-degenerate interval, and so, according to a well known theorem [6] of A. Zygmund, $f(x)$ is then monotone non-decreasing.

Hence we have the following

---

(3) The necessity part remains trivially valid. In fact, if a function $f(x)$, continuous or not, is monotone non-decreasing in I, then it is known that all the four derivates of $f(x)$ are > 0 throughout I. (See Natanson [2], p. 208, Lemma 1.)

(4) For the definitions of Banach's conditions (T_1) and (T_2) see Saks [4], p. 277.

(5) The derivate $f'(x)$ exists at a point x whenever the four derivates of $f(x)$ are equal at x, whether finite or infinite.

(6) Viz. "If $f(x)$ is a finite function of a real variable such that (i) $\limsup_{h\to 0+} f(x-h) \leqslant f(x) \leqslant \limsup_{h\to 0+} f(x+h)$ at every point x, and (ii) the set of the values assumed by $f(x)$ at the points x where $D^+f(x) \leqslant 0$ contains no non-degenerate interval, then the function $f(x)$ is monotone non-decreasing." (See Saks [4], p. 203, Theorem (7.1).)

THEOREM 1. *If $f(x)$ is a finite real function of a real variable such that*

(i) $\limsup_{h\to 0+} f(x-h) \leqslant f(x) \leqslant \limsup_{h\to 0+} f(x+h)$ *at every point x, and*

(ii) *the values assumed by $f(x)$ at the points x where either $f'(x)$ exists and is < 0, or*

$$D^+f(x) = D_-f(x) < 0\,, \quad D_+f(x) = -\infty\,, \quad D^-f(x) = +\infty\,,$$

form a set whose interior measure is zero,

then the function $f(x)$ is monotone non-decreasing.

Let, again, $f(x)$ be a finite real function which satisfies the relation (1) at every point x, and let E_1 and E_2 be the sets as defined in (5) and (6).

Let, for $i = 1$ and 2, $E_{i\infty}$ denote the set of those points of E_i where all the four derivates of $f(x)$ are infinite.

Since at any point $x \in (E_1 - E_{1\infty})$, or $(E_2 - E_{2\infty})$, $f(x)$ has at least one derivate finite, it follows from a well known theorem (see [4], p. 271, Theorem (4.6)) of S. Saks that $f(x)$ fulfils the Lusin's condition (N) (7) on either of the sets $(E_1 - E_{1\infty})$ and $(E_2 - E_{2\infty})$.

Hence, in case $m(E_1 + E_2) = 0$, we have

$$mf(E_1 - E_{1\infty}) = 0 = mf(E_2 - E_{2\infty})\,,$$

so that

$$m_i f(E_1 + E_2) = m_i f(E_{1\infty} + E_{2\infty})\,. \tag{7}$$

If $m_i f(E_{1\infty} + E_{2\infty}) > 0$, the image-set $f(E_{1\infty} + E_{2\infty})$ has the power $\mathfrak{c}$ (8) and so the set $(E_{1\infty} + E_{2\infty})$ then also has the power $\mathfrak{c}$.

Hence, in case the set $(E_{1\infty} + E_{2\infty})$ has a power less than $\mathfrak{c}$, we have

$$m_i f(E_{1\infty} + E_{2\infty}) = 0\,. \tag{8}$$

Combining the equations (7) and (8) it follows that, in case $m(E_1 + E_2) = 0$ and the set $(E_{1\infty} + E_{2\infty})$ has a power less than $\mathfrak{c}$, we have

$$m_i f(E_1 + E_2) = 0\,,$$

which in turn implies with the help of Theorem 1 that the function $f(x)$ is then monotone non-decreasing.

We have thus proved the following

THEOREM 2. *If $f(x)$ is a finite real function of a real variable such that*

(i) $\limsup_{h\to 0+} f(x-h) \leqslant f(x) \leqslant \limsup_{h\to 0+} f(x+h)$ *at every point x,*

---

(7) For the definition of Lusin's condition (N) see Saks [4], p. 224.

(8) For, as the interior measure of a set, E is the least upper bound of the measures of all closed sets contained in E, in case $m_i E > 0$, E contains a closed set F which has its measure > 0. Clearly, F is then an unenumerable closed set, and so has the power $\mathfrak{c}$. (See Natanson [2], p. 53.) The set E then evidently has the power $\mathfrak{c}$.

(ii) *the points where* $f'(x)$ *exists and is* < 0, *or*

$$D^+f(x) = D_-f(x) < 0\,, \quad D_+f(x) = -\infty\,, \quad D^-f(x) = +\infty\,,$$

form a set of measure zero, and

(iii) *the points where* $f'(x)$ *exists and is* $= -\infty$, *or*

$$D^+f(x) = D_-f(x) = -\infty\,, \quad D_+f(x) = -\infty\,, \quad D^-f(x) = +\infty\,,$$

form a set whose power is less than that of the continuum, then the function $f(x)$ *is monotone non-decreasing.*

2. Functions fulfilling Banach's condition (T_1). We first observe that the condition (i) of Theorem 1 is automatically satisfied in case the function $f(x)$ is continuous.

Moreover, if a continuous function $f(x)$ also fulfils the Banach's condition (T_1), the values assumed by $f(x)$ at the points x where it has no derivative (finite or infinite) form a set of measure zero. (See [3], p. 130, Theorem 1, or [4], p. 278, Theorem (6.2)). Hence, in this case Theorem 1 gives

THEOREM 3. *Let* $f(x)$ *be a continuous function which fulfils the Banach's condition* (T_1). *If the values assumed by* $f(x)$ *at the points* x *where* $f'(x)$ *exists and is* < 0 *form a set whose interior measure is zero, then the function* $f(x)$ *is monotone non-decreasing.*

Let, again, $f(x)$ be a continuous function which fulfils the condition (T_1), and let

$$E_1 = \{x;\ f'(x) \text{ exists and is } < 0\}\,,$$

$$E_{1\infty} = \{x;\ f'(x) \text{ exists and is } = -\infty\}\,.$$

We have already observed in the proof of Theorem 2 that in case $mE_1 = 0$, and the set $E_{1\infty}$ has a power $< \mathfrak{c}$, we have

$$m_i f(E_1) = 0\,.$$

But, according to the above Theorem 3, the last equation implies that the function $f(x)$ is monotone non-decreasing.

Hence, we have the following

THEOREM 4. *Let* $f(x)$ *be a continuous function which fulfils the Banach's condition* (T_1). *If* (i) *the points where* $f'(x)$ *exists and is* < 0 *form a set of measure zero, and if* (ii) *the points where* $f'(x) = -\infty$ *form a set whose power is* $< \mathfrak{c}$, *then the function* $f(x)$ *is monotone non-decreasing.*

As a continuous function of bounded variation always fulfils the condition (T_1) (see [4], p. 279, Theorem (6.3)), the above Theorems 3 and 4 also hold for functions of *bounded variation.*

3. Functions fulfilling Banach's condition (T_2). In case of a continuous function which fulfils the Banach's condition (T_2), we have a slightly weaker result than that of Theorem 3, viz.:

THEOREM 5. *Let $f(x)$ be a continuous function which fulfils the Banach's condition* (T_2). *If the values assumed by $f(x)$ at the points x where $f'(x)$ exists and is < 0 form a set whose measure is zero, then the function $f(x)$ is monotone non-decreasing.*

Proof. S. Saks proved (see [3], p. 133, Theorem 5 or [4], p. 280, Theorem (6.6)) in 1931 that

"If a continuous function $f(x)$ fulfils the condition (T_2) in an interval $[a, b]$, then

$$-m_e f(N) \leqslant f(b)-f(a) \leqslant m_e f(P),$$

where P and N denote respectively the sets of points of the interval $[a, b]$ where $f(x)$ possesses a unique non-negative, non-positive, derivative."

Let $f(x)$ be a continuous function which fulfils the condition (T_2) in an interval I.

Let, if possible, a and b be two points of I for which

$$a < b, \quad f(a) > f(b). \tag{9}$$

Since the function $f(x)$ is continuous and fulfils the Condition (T_2) in $[a, b]$, it follows from the above theorem of Saks that

$$-m_e f(N\cdot[a, b]) \leqslant f(b)-f(a), \tag{10}$$

where N denotes the set of points in I where $f(x)$ has a non-positive derivative.

Denoting by E_1 the set of points in I where $f(x)$ has a derivative < 0, we evidently have

$$f'(x) = 0 \quad \text{for} \quad x \in N-E_1. \tag{11}$$

This implies, according to another well known theorem (9) of Saks, that

$$mf(N-E_1) = 0. \tag{12}$$

Thus, in case there exist points a, b in I for which (9) holds, we have, with the help of (10) and (12),

$$-m_e f(E_1\cdot[a, b]) \leqslant f(b)-f(a) < 0,$$

i.e.

$$m_e f(E_1\cdot[a, b]) > 0. \tag{13}$$

Hence, if we have $m_e f(E_1) = 0$, there exist no points a, b in I for which the relation (9) holds, and so the function $f(x)$ is then monotone non-decrasing.

(9) Viz., "If one of the four Dini derivates of a function $f(x)$ vanishes at every point of a set E, then $mf(E) = 0$". See Saks [4], p. 272.

This completes the proof of Theorem 5.

In case a continuous function $f(x)$ fulfils the *Lnsin's condition* (N), it also fulfils the condition (T_2) (see [4], p. 284, Theorem (7.3)), and since in this case

$$mf(E) = 0 \quad \text{whenever } mE = 0 ,$$

it follows from Theorem 5

COROLLARY ([10]). *Let $f(x)$ be a continuous function which fulfils the Lusin's condition* (N). *If the points where $f'(x)$ exists and is < 0 form a set of measure zero, then the function $f(x)$ is monotone non-decreasing.*

Let, once again, $f(x)$ be a continuous function which fulfils the condition (T_2). Let, as before,

$$E_1 = \{x;\ f'(x) \text{ exists and is } < 0\}$$

and

$$E_{1\infty} = \{x;\ f'(x) \text{ exists and is } = -\infty\} .$$

Since $f'(x)$ is finite at each point of $(E_1 - E_{1\infty})$, the function $f(x)$ fulfils the condition (N) on $(E_1 - E_{1\infty})$ (see [4], p. 271, Theorem (4.6)).

Hence, in case $mE_1 = 0$, we have

$$mf(E_1 - E_{1\infty}) = 0 ,$$

i.e.

$$m_e f(E_1) = m_e f(E_{1\infty}) .$$

In case $m_e f(E_{1\infty}) > 0$, the image-set $f(E_{1\infty})$ is unenumerable, and so the set $E_{1\infty}$ is also unenumerable.

Thus, in case $mE_1 = 0$ and the set $E_{1\infty}$ is at most enumerable, we have

$$m_e f(E_1) = m_e f(E_{1\infty}) = 0 ,$$

which implies with the help of Theorem 5 that $f(x)$ is then monotone non-decreasing.

The following result has thus been proved:

THEOREM 6. *Let $f(x)$ be a continuous function which fulfils the Banach's condition* (T_2). *If* (i) *the points where $f'(x)$ exists and is < 0 form a set of measure zero, and if* (ii) *the points where $f'(x) = -\infty$ are at most enumerable, then the function $f(x)$ is monotone non-decreasing.*

([10]) This corollary strengthens the following theorem of T. Ważewski (see [5], p. 118, Theorem 2 and the following remark 2): "If a function $f(x)$, continuous in an interval I, satisfies in this interval the Lusin's condition (N), and if the inequality $D^+ f(x) \geqslant 0$ holds almost everywhere in I, then the function $f(x)$ is non-decreasing in the interval."

The above corollary is, however, not new. (See Saks [4], p. 286.) This corollary is also known to hold in case the function $f(x)$, instead of fulfilling the condition (N), possesses a derivative (finite or infinite) with the exception of points of an at most enumerable set. (See. Z. Zahorski [6], p. 19, Theorem 2.)

References

[1] K. M. Garg, *An analogue of Denjoy's theorem*, Ganita 12 (1961), pp. 9-14.

[2] I. P. Natanson, *Theory of Functions of a Real Variable*, vol. I, New York 1955.

[3] S. Saks, *Sur certaines classes de fonctions continues*, Fund. Math. 17 (1931), pp. 124-151.

[4] — *Theory of the Integral*, Warszawa-Lwów 1937.

[5] T. Ważewski, *Sur une condition nécessaire et suffisante pour qu'une fonction continue soit monotone*, Ann. Soc. Polonaise Math. 24 (1951), pp. 111-119.

[6] Z. Zahorski, *Sur la première dérivée*, Trans. Amer. Math. Soc. 69 (1950), pp. 1-54.

DEPARTMENT OF MATHEMATICS & ASTRONOMY,
UNIVERSITY OF LUCKNOW, INDIA.

Reçu par la Rédaction le 21. 5. 1962