ANNALES
POLONICI MATHEMATICI
XLVI (1985)

Envelope of Dirichlet problems on a domain in C^n

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Abstract. The upper envelope of plurisubharmonic functions with given boundary values φ on a domain $\Omega \subset\subset C^n$ is characterized as the infimum of all functions on Ω with boundary values φ which are harmonic with respect to some Kähler metric on Ω. On a convex domain $\Omega \subset R^n$ the lower envelope of Dirichlet problems is shown to be the upper envelope of convex functions.

We will show that the infimum of the solutions of a certain family of Dirichlet problems with fixed boundary values is equal to a supremum of plurisubharmonic (psh.) functions. We consider a domain $\Omega \subset\subset C^n$ such that $\partial\Omega$ is C^2 and strongly pseudoconvex, and we consider a Hermitian matrix-valued function $(a_{ij}(z))$ on Ω with the property that $a_{ij}(z)$ is measurable, and there exists $\varepsilon > 0$ such that

$$\varepsilon I \leqslant (a_{ij}(z)) \leqslant \frac{1}{\varepsilon} I \tag{1}$$

holds a.e. on Ω. Let us define the divergence form operator

$$P_a(v) = \sum_{i,j=1}^{n} \frac{\partial}{\partial z_i}\left(a_{ij}(z)\frac{\partial v(z)}{\partial \bar{z}_j}\right). \tag{2}$$

We will use the notation

$$L_1^2(\Omega) = \{\varphi \in L^2(\Omega):\ \nabla\varphi \in L^2(\Omega)\}$$

to denote the Sobolev space of functions with gradient in $L^2(\Omega)$. For $\varphi \in L_1^2(\Omega)$, we say that

$P_a\varphi = f$ holds in a generalized sense if

$$-\sum_{i,j=1}^{n} \int_{\Omega} \frac{\partial\psi}{\partial z_i} a_{ij}(z) \frac{\partial\varphi}{\partial \bar{z}_j} = \int_{\Omega} \psi f$$

holds for all $\psi \in C_0^\infty(\Omega)$. A standard argument involving the Hermitian form

$$\langle \varphi, \psi \rangle = \sum \int_\Omega \frac{\partial \psi}{\partial z_i} a_{ij}(z) \frac{\partial \varphi}{\partial \bar{z}_j}$$

yields a generalized solution of the Dirichlet problem:

(3) If $f \in C^1(\partial\Omega)$, then there is a unique $v \in L_1^2(\Omega)$ such that $v = f$ on $\partial\Omega$, and $P_a v = 0$,

where P and (a_{ij}) are defined by (1, 2) (see [8]).

Given a Hermitian matrix-valued function $(a_{ij}(z))$, we also define the $(n-1, n-1)$ form

$$\eta(a) = \sum_{i,j=1}^{n} a_{ij} * (dz_i \wedge d\bar{z}_j)$$

where the $*$-operator is defined so that

$$*(dz_i \wedge d\bar{z}_j) \wedge \sqrt{-1}\, dz_i \wedge d\bar{z}_j = \bigwedge_{j=1}^{n} \sqrt{-1}\, dz_j \wedge d\bar{z}_j.$$

Let us consider the class

$\mathscr{A} = \{(a_{ij}(z))$: a_{ij} is bounded and measurable on Ω, (a_{ij}) is Hermitian a.e., there is a constant $\varepsilon > 0$ such that $\varepsilon I \leqslant (a_{ij}(z)) \leqslant \frac{1}{\varepsilon} I$ holds a.e., $\det a_{ij}(z) = 1$ a.e., and $d\eta(a) = 0$ in the sense of distributions$\}$.

Our main result concerns the following envelopes

$$I(f) = \inf\{v_a\colon a \in \mathscr{A} \text{ and } v_a \text{ solves (3)}\}$$

and

(4) $U(f) = \sup\{w\colon w$ is psh. on Ω, and $\limsup_{\zeta \to z} w(\zeta) \leqslant f(z)$ for $z \in \partial\Omega\}$.

THEOREM 1. *Let $\Omega \subset\subset C^n$ be strongly pseudoconvex with C^2 boundary. Then for $f \in C^1(\partial\Omega)$, $I(f) = U(f)$.*

Remarks.

1. It was shown in [2] that the right-hand side of (4) is the generalized solution of equation (7) with $\varepsilon = 0$.

2. Other dual characterizations of the upper envelope of psh. functions were given in [3], [4], [6], [7] where different families are used in the infimum.

Proof. Let U_0 denote the right-hand side of (4). Then $U_0 \in C(\bar{\Omega}) \cap P(\Omega)$, and thus $U_0 \in L_1^2(\Omega, \text{loc})$. Thus for $a \in \mathscr{A}$, $P_a(U_0)$ is well defined as a distribution of order -1. We will show that $P_a(U_0) \geqslant 0$, and so the inequality $\geqslant$ in (4) will follow from the maximum principle.

Let $\{U^\delta\} \subset C^\infty(\Omega) \cap P(\Omega)$ be a sequence converging to U_0 in $L_1^2(\Omega, \text{loc})$. If $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqslant 0$, then

$$\begin{aligned}\int_\Omega \varphi P_a(U_0) &= \sum_{i,j=1}^n \int_\Omega -\frac{\partial\varphi}{\partial z_i}\left[a_{ij}(z)\frac{\partial U_0}{\partial \bar{z}_j}\right] \\ &= \lim_{\delta\to 0} \sum_{i,j=1}^n \int_\Omega -\frac{\partial\varphi}{\partial z_i}\left[a_{ij}(z)\frac{\partial U^\delta}{\partial \bar{z}_j}\right] \\ &= \lim_{\delta\to 0} \int_\Omega -\partial\varphi \wedge \sqrt{-1}\,\eta(a) \wedge \bar{\partial} U^\delta \\ &= \lim_{\delta\to 0} \int_\Omega \varphi\eta(a) \wedge \sqrt{-1}\,\partial\bar{\partial} U^\delta \geqslant 0.\end{aligned}$$

The integration by parts is justified since $d\eta(a) = 0$, and the inequality arises from the positivity of $\eta(a)$.

To consider the reverse inequality, we consider the envelope

$$I_\varepsilon(f) = \inf\{v_a^\varepsilon\colon a \in \mathscr{A}\}, \tag{5}$$

where v_a^ε is the unique solution $v \in L_1^2(\Omega)$ to $v|_{\partial\Omega} = f$, $P_a v = \varepsilon$ on Ω. By the maximum principle, it is evident that

$$I_\varepsilon(f) \leqslant I_0(f).$$

Let $\varrho \in C^2(\bar{\Omega})$ be a function such that $\{\varrho < 0\} = \Omega$, $\{\varrho = 0\} = \partial\Omega$, and

$$\sqrt{-1}\,\partial\bar{\partial}\varrho \geqslant \sum_{j=1}^n \sqrt{-1}\,dz_j \wedge d\bar{z}_j.$$

(Such a function exists by the strict pseudoconvexity of Ω.) For a competitor v_a^ε in the infimum (5) and the corresponding competitor v_a in (4), we may argue as above to conclude that

$$P_a(v+\varepsilon\varrho) \geqslant \varepsilon\eta(a) \wedge \sqrt{-1}\,\partial\bar{\partial}\varrho \geqslant \varepsilon\,\text{trace}(a_{ij}) \geqslant n\varepsilon.$$

By the maximum principle, then,

$$v_a + \varepsilon\varrho \leqslant v_a^\varepsilon,$$

and so we conclude that

$$I(f) - \varepsilon \max_\Omega |\varrho| \leqslant I_\varepsilon(f). \tag{6}$$

Now let us consider the solution U_ε of the Dirichlet problem:

(7) $$U_\varepsilon \in P(\Omega) \cap C(\bar{\Omega}), \quad U_\varepsilon|_{\partial\Omega} = f, \quad (dd^c U_\varepsilon)^n = \varepsilon^n \bigwedge_{j=1}^{n} \sqrt{-1}\, dz_j \wedge d\bar{z}_j.$$

For arbitrary $\delta > 0$, there exists U_ε^δ such that

(8) $$U_\varepsilon^\delta \in P(\Omega) \cap C^{1,1}(\bar{\Omega}), \quad \max_{\Omega} |U_\varepsilon(z) - U_\varepsilon^\delta(z)| < \delta,$$
$$(dd^c U_\varepsilon^\delta)^n = \varepsilon^n \bigwedge_{j=1}^{n} \sqrt{-1}\, dz_j \wedge d\bar{z}_j.$$

(This follows from Theorem 7.3 of [1].) Now we let $(\tilde{a}_{ij})$ denote the Hermitian matrix-valued function such that

(8′) $$\eta(\tilde{a}) = \frac{\varepsilon^{-n+1}}{n!} (dd^c U_\varepsilon^\delta)^{n-1}.$$

One may check that $\tilde{a} \in \mathscr{A}$. Further by (8), it follows that $P_{\tilde{a}}(U_\varepsilon^\delta) = \varepsilon$. Thus we see that

(9) $$I_\varepsilon(f_\delta) \leqslant U_\varepsilon^\delta,$$

where $f_\delta = U_\varepsilon^\delta|_{\partial\Omega}$. By the maximum principle it is evident that

$$|I_\varepsilon(f_\delta) - I_\varepsilon(f)| < \delta,$$

and so from (6), (8) and (9), we have

$$I(f) - \varepsilon \max_{\Omega} |\varrho| \leqslant U_\varepsilon^\delta + \delta \leqslant U_\varepsilon + 2\delta.$$

Letting $\varepsilon \to 0$, we obtain

$$I(f) \leqslant U_0 + 2\delta,$$

which gives the reverse inequality for (4), since δ is arbitrary.

Remark. It is evident from the proof that the infimum on the left-hand side in (4) may be taken only over the family $P_{\tilde{a}}$, where $\tilde{a}$ is given in (8′).

We will now give two reformulations of this result. The first involves Kähler metrics.

Recall that if $g = \sum g_{ij}\, dz_i\, d\bar{z}_j$ is a Kähler metric on a manifold Ω, then the associated Laplacian is given by

$$\Delta_g = \sum \frac{1}{\sqrt{g}} \frac{\partial}{\partial z_j} \sqrt{g}\, g^{ij} \frac{\partial}{\partial \bar{z}_j},$$

where $g = \det(g_{ij})$ and (g^{ij}) is the inverse matrix of (g_{ij}).

We use now a result of Evans [5] which says that the solution U_ε^δ of (8) is in fact C^∞ smooth. (Another proof is given by Trudinger [9].) Now let g_ε^δ

denote the Kähler metric with $(g^\delta_\varepsilon)_{ij} = \partial^2 (U^\delta_\varepsilon)/\partial z_i \partial \bar{z}_j$. It follows, then, that

$$\Delta_{g^\delta_\varepsilon} = P_{\tilde{a}},$$

where $\tilde{a}$ is defined by (8′).

We will let $\mathscr{K}$ denote the set of Kähler metrics on Ω such that $g_{ij} \in C^\infty(\Omega)$, and

$$\varepsilon I \leqslant (g_{ij}) \leqslant \frac{1}{\varepsilon} I$$

for some $\varepsilon > 0$. For $f \in C^1(\partial\Omega)$ and $g \in \mathscr{K}$, there exists a unique solution $v_g \in L^2_1(\Omega)$ such that

$$\Delta_g v_g = 0 \quad \text{and} \quad v_g|_{\partial\Omega} = f.$$

We may define

$$\tilde{I}(f) = \inf\{v_g \colon g \in \mathscr{K}\}.$$

Our first reformulation shows that the upper envelope of psh. functions coincides with the lower envelope of all functions that are harmonic with respect to some Kähler metric.

THEOREM 2. *If $\Omega \subset\subset C^n$ is strongly pseudoconvex with C^2 boundary, and if $f \in C^1(\partial\Omega)$, then*

$$\tilde{I}(f) = U(f).$$

Our second reformulation concerns the corresponding problem in $\mathbf{R}^n$. If $\omega \subset\subset \mathbf{R}^n$ is a convex domain with C^2 boundary and nowhere vanishing Gauss–Kronecker curvature, then we may identify $\omega + i\mathbf{R}^n$ with the logarithmic image of a strongly pseudoconvex Reinhardt domain $\Omega \subset\subset C^n$. If we are given $f \in C^1(\partial\omega)$, then this corresponds to a rotation invariant function $f \in C^1(\partial\Omega)$. We may take the function U^δ_ε in (8) and the operator $P_{\tilde{a}}$ to be invariant. Thus we will have $\partial U^\delta_\varepsilon/\partial\theta_j = 0$, $j = 1, \ldots, n$, where we write $z_j = r_j e^{i\theta_j}$.

If we use the variable $t = (t_1, \ldots, t_n)$ to denote (real) coordinates on $\omega \subset \mathbf{R}^n$, then $t \in \omega$ corresponds to $z \in \Omega$ with $r_j = e^{t_j}$. The function U^δ_ε on Ω induces a function w^δ_ε on ω via

$$w^\delta_\varepsilon(t) = U^\delta_\varepsilon(e^t).$$

Let us replace ε on the right-hand side of (8) by $\varepsilon/(|z_1|^2 \ldots |z_n|^2)$. With this new choice of $U^\delta_\varepsilon(z)$ and $w^\delta_\varepsilon(t)$, we have

$$\det\left(\frac{\partial^2 w^\delta_\varepsilon}{\partial t_i \partial t_j}\right) = 4^n e^{2(t_1 + \ldots + t_n)} \det\left(\frac{\partial^2 U^\delta_\varepsilon}{\partial z_i \partial \bar{z}_j}\right) = \frac{1}{n!}\varepsilon.$$

Thus, taking (b_{ij}) to be a constant times the cofactor matrix of $\partial^2 w^\delta_\varepsilon/\partial t_i \partial t_j$, we have:

$$P_b(w^\delta_\varepsilon) = \sum_{i,j=1}^{n} b_{ij}(t) \frac{\partial^2}{\partial t_i \partial t_j}(w^\delta_\varepsilon) = \varepsilon,$$

and P_b is in divergence form:

$$P_b = \sum_{i,j=1}^{n} b_{ij}(t)\frac{\partial^2}{\partial t_i\, \partial t_j} = \sum_{i,j=1}^{n} \frac{\partial}{\partial t_i} b_{ij}(t)\frac{\partial}{\partial t_j}.$$

Now we consider the real version of $\mathcal{A}$ above:

$\mathcal{A}_R = \{(b_{ij}(t))$: (b_{ij}) is bounded, measurable, real, and symmetric for $t \in \omega$, P_b is divergence form, $\varepsilon I \leqslant (b_{ij}) \leqslant \frac{1}{\varepsilon} I$ for some $\varepsilon > 0$, and $\det(b_{ij}) = 1\}$.

In a similar way we may define the lower envelope

$$I_R(f) = \inf\{v_b\colon\ b \in \mathcal{A}_R \text{ and } v_b \text{ solves } (3)\}.$$

Theorem 1 in this case yields the following:

THEOREM 3. *Let $\omega \subset\subset \mathbf{R}^n$ be a smoothly bounded convex domain such that the principle curvatures of $\partial\Omega$ are all positive. If $f \in C^1(\partial\omega)$, then the infimum of Dirichlet solvers is the convex envelope, i.e.*

$$I_R(f) = \sup\{v\colon\ v \textit{ is convex on } \omega, \textit{ and } \limsup_{\zeta\to x} v(\zeta) \leqslant \varphi(x) \textit{ for all } x \in \partial\omega\}.$$

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Reçu par la Rédaction le 14.02.1984