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Characterizations of H_{x_1}-sufficiency of jets

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Abstract. The aim of this paper is to investigate germs of maps in manifolds with boundary. The main results are contained in Theorems 1, 2, 3.

1. Definitions and notation. Let $\boldsymbol{R}^n$ denote the Euclidean space of dimension n and $E(n, p)$ the set of all germs of maps $f\colon \boldsymbol{R}^n \to \boldsymbol{R}^p$, $f(0) = 0$, $J^r(n, p)$ is the space of jets.

DEFINITION 1. A homeomorphism $\sigma\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ is boundary preserving if

$$\sigma(\{0 \times \boldsymbol{R}^{n-1}\}) \subset \{0 \times \boldsymbol{R}^{n-1}\}.$$

The set of all homeomorphisms of $\boldsymbol{R}^n$ preserving boundaries will be denoted by H_{x_1}.

DEFINITION 2. An r-jet $w \in J^r(n, p)$ is called *H_{x_1}-sufficient* if for every $f\colon \boldsymbol{R}^n \to \boldsymbol{R}^p$, $f(0) = 0$, and $j^r(f) = w$ there exists $\sigma \in H_{x_1}$ such that $f_0 \sigma = w$.

Given $f\colon \boldsymbol{R}^n \to \boldsymbol{R}^p$, let us put

$$v_i^f = \left(x_1 \frac{\partial f_i}{\partial x_1}, \frac{\partial f_i}{\partial x_2}, \ldots, \frac{\partial f_i}{\partial x_n}\right), \quad i = 1, \ldots, p.$$

THEOREM 1. *An r-jet $w \in J^r(n, p)$ is H_{x_1}-sufficient if there exist a positive number c and a neighbourhood U of 0 such that*

(1) $$d(v_1^w, \ldots, v_p^w) \geqslant c|x|^{r-1}$$

for every $x \in U$.

THEOREM 2. *If $w \in J^r(n, p)$ is H_{x_1}-sufficient, then there exists a positive number c and a neighbourhood U of 0 such that*

(2) $$d(v_1^w, \ldots, v_p^w) \geqslant c|x|^{r+1} \quad \text{for every } x \in U.$$

THEOREM 3. *Let $w \in J^r(n, p)$. If there exist a positive number c and a neighbourhood U of 0 such that*

(3) $$d(\Phi_1, \ldots, \Phi_p) \geqslant c|x|^r \quad \text{for every } x \in U,$$

then w is H_{x_i}-sufficient for every i, where

$$(4)\qquad \Phi_i = \left(x_1\frac{\partial w_i}{\partial x_1}, x_2\frac{\partial w_i}{\partial x_2}, \dots, x_n\frac{\partial w_i}{\partial x_n}\right), \quad i = 1, \dots, p.$$

For manifolds without boundary, Bochnak [1] has shown that w is H_{x_1}-sufficient if and only if condition (1) is satisfied. The following example shows that for manifolds with boundary condition (1) is only sufficient and not necessary.

EXAMPLE. Let $w(x, y) = x^2 + y^2 \in J^2(2, 1)$. Then w is H_x- and H_y-sufficient.

2. Proof of Theorems 1, 2, 3.

2.1. Proof of Theorem 1. For every f: $(\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ and $j^r(f) = w$, let us put

$$F(x, t) = w(x) + t(f(x) - w(x)) \quad \text{for } (x, t) \in \mathbf{R}^n \times \mathbf{R}$$

and

$$(2.1)\qquad Y(x, t) = \sum_{i=1}^{p} ((0, 1)\cdot v_i^F)|N_i|^{-2} N_i = \sum_{i=1}^{p} (f_i - w_i)|N_i|^{-2} N_i,$$

where

$$v_i^F = \left(w_1\frac{\partial F_i(x, t)}{\partial x_1}, \frac{\partial F_i(x, t)}{\partial x_2}, \dots, \frac{\partial F_i(x, t)}{\partial x_n}, \frac{\partial F_i(x, t)}{\partial t}\right)$$

and N_i is the projection of v_i^F onto the space orthogonal to the space spanned by $v_j^F(x_1, \dots, x_n, t)$ with $j \neq i$. $Y(x, t)$ is the projection of $(0, \dots, 0, 1)$ onto the subspace spanned by $v_1^F, \dots, v_p^F$.

Then

$$(2.2)\qquad V(x, t) = (V_1, \dots, V_{n+1}) = (0, \dots, 0, 1) - Y(x, t)$$

is orthogonal to v_i^F for every i, i.e.,

$$x_1\frac{\partial F_i}{\partial x_1}\cdot V_1 + \frac{\partial F_i}{\partial x_2}\cdot V_2 + \dots + \frac{\partial F_i}{\partial t}\cdot V_{n+1} = 0.$$

Write

$$(2.3)\qquad X(x, t) = (x_1 V_1, V_2, \dots, V_{n+1}).$$

Then $X(x, t)$ is orthogonal to $(\partial F_i/\partial x_1, \dots, \partial F_i/\partial x_n, \partial F_i/\partial t)$, $i = 1, \dots, p$.

LEMMA 2.1 [2]. *In the notation of Theorem* 1, *we have*

$$d(v_1^F, \dots, v_p^F) \geqslant \tfrac{1}{2} c|x|^{r-1} \quad \textit{for every } x \in U.$$

Let us consider the differential equation

$$\frac{dx}{d\tau} = X(x, t), \tag{2.4}$$

where $X(x, t)$ is a map from an open subset of $\mathbf{R}^n \times \mathbf{R}$ containing $B^n_\varepsilon \times [0, 1]$ into $\mathbf{R}^{n+1}$,

$$B^n_\varepsilon = \{x \in \mathbf{R}^n: |x| \leqslant \varepsilon\}.$$

Let us note that $X(x, t)$ satisfies the following conditions:

(i) $X(x, t)$ is continuous for every $(x, t) \in B^n_\varepsilon \times [0, 2] = A_\varepsilon$,

(ii) $\lim_{x \to 0} \frac{|X(x, t) - X(x, 0)|}{|x|} = 0$ uniformly with respect to t,

(iii) $\langle X(x, t), l_{n+1}\rangle > 0$ for every $(x, t) \in A_\varepsilon$.

In fact, by Lemma 2.1 we have

$$|N_i| \geqslant d(v^F_1, \ldots, v^F_p) \geqslant \tfrac{1}{2} c|x|^{r-1}, \quad i = 1, \ldots, p.$$

This implies that

$$|Y(x, t)| \leqslant 2 \sum_{i=1}^{p} |f_i - w_i| \frac{1}{|N_i|} = o(|x|).$$

Hence $V(x, t)$ satisfies conditions (i)-(iii). Thus $X(x, t)$ does so. By [1], (2.4) has the unique solution $\eta(x, t; \tau)$ satisfying the condition $\eta(x, t, 0) = (x, t)$ and

(a) $\tau \to \eta(x, t; \tau)$ is the unique solution of (2.4) satisfying the condition $\eta(x, t; 0) = (x, t)$ in a neighbourhood W of $B^n_\varepsilon \times [0, 1] \times \{0\} \subset \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}$,

(b) $\eta(0, 0; \tau) = (0, \tau)$,

(c) $(\eta(\{x \times 0\} \times \mathbf{R})) \cap W \cap (\mathbf{R}^n \times \{1\})$ contains the point $(\sigma(x), 1)$,

(d) F is constant on $\eta((x, t) \times \mathbf{R}) \cap W$.

From conditions (a)-(d) it follows that the map σ is a homeomorphism such that $f \circ \sigma = w$. It remains to show that $\sigma \in H_{x_1}$.

In fact,

$$X(x, t)|_{x_1 = 0} = (0, V_2, \ldots, V_{n+1}) = X_1(x', t),$$

where $x' = (0, x_2, \ldots, x_n)$.

We have

$$\frac{\partial F_i(x', t)}{\partial x_2} \cdot V_2 + \ldots + \frac{\partial F_i(x', t)}{\partial t} \cdot V_{n+1} = 0.$$

On the other hand, since $X(x, t)$ satisfies (i)-(iii), $X_1(x', t)$ does so. This implies that the equation

$$\frac{dx'}{d\tau} = X_1(x', t) \tag{2.5}$$

has the unique solution $\eta_*(x', t; \tau)$ satisfying the condition $\eta_*(x', t; 0) = (x', t)$.

By the continuity of $\eta(x, t; \tau)$ we infer that $\eta(x, t; \tau)|_{x_1=0}$ is a solution of (2.5).

Thus

$$\eta(x, t; \tau)|_{x_1=0} = \eta_*. \tag{2.6}$$

Since η_* induces a homeomorphism σ_*: $\boldsymbol{R}^{n-1} \to \boldsymbol{R}^{n-1}$, we have by (2.6) $\sigma|_{x_1=0} = \sigma_*$. Thus σ preserves the boundary.

2.2. Proof of Theorem 2. We first prove the following claim, under the hypothesis of Theorem 2:

(d_{x_1}) For every f: $U \to \boldsymbol{R}^p$, $f \in C^r$, and $j^r(f) = w$ and for every sequence $\{a_i\}_{i \in N} \subset \boldsymbol{R}^n$, $a_i \to 0$, $a_i \neq 0$, $f^{-1}(f(a_i))$ is a topological manifold with boundary of codimension p and the boundary of $f^{-1}(f(a_i))$ is the intersection of $f^{-1}(f(a_i))$ with the hyperspace $x_1 = 0$.

To prove (d_{x_1}), we need the following lemma, which is an immediate consequence of Lemma 1 [1].

LEMMA. *Let U, V be open subsets of $\boldsymbol{R}^k$ and $\boldsymbol{R}^l$, respectively, F: $U \times V \to \boldsymbol{R}^p$ be a C^∞-map and B be a countable set of $\boldsymbol{R}^p$ contained in the set of all regular values of F and $F|_{x_1} = 0$.*

Then there exists a point $y_0 \in V$ such that B is contained in the set of regular values of both $F(x_1, \ldots, x_k, y_0)$ and $F(0, x_2, \ldots, x_k, y_0)$.

Applying this lemma we can prove (d_{x_i}).

Let $a_i \xrightarrow[i\to\infty]{} 0$, $a_i \in \boldsymbol{R}^n \setminus \{0\}$, and f: $\boldsymbol{R}^n \to \boldsymbol{R}^p$ be a C^r-map, with $j^r(f) = w$.

Consider the map $\boldsymbol{R}^n \times \boldsymbol{R}^p \to \boldsymbol{R}^p$ defined by

$$F_j(x, y) = w_j(x) + y_j|x|^{2r}, \quad j = 1, 2, \ldots, p;$$

obviously, $f(a_i)$ is a regular value of both $F(x, y)$ and $F(x, y)|_{x_1=0}$.

We now show that condition (d_{x_1}) implies Theorem 2.

Indeed, if it is not the case then there exists a sequence $\{a_i\}$; $a_i \neq 0$, $a_i \to 0$, such that

$$d(v_1^w(a_i), \ldots, v_p^w(a_i)) = o(|a_i|^{r+1}).$$

Then by [1] there are linearly independent vectors $\lambda_1'^i, \ldots, \lambda_p'^i$ satisfying the conditions:

(a) The vectors $v_2^w(a_i) + \lambda_2'^i, \ldots, v_p^w(a_i) + \lambda_p'^i$ are lineary independent,

(b) $v_1^w(a_i) \in \operatorname{Span}\{v_2^w(a_i) + \lambda_2'^i, \ldots, v_p^w(a_i) + \lambda_p'^i\}$,

(c) $|\lambda_k'^i| = o(|a_i|^{r+1})$, $k = 2, \ldots, p$.

Since $w(x)$ is H_{x_1}-sufficient, $w(x)|_{x_1=0}$ is H-sufficient in $\boldsymbol{R}^{n-1} = 0 \times \boldsymbol{R}^{n-1} \subset \boldsymbol{R}^n$.

Indeed; let $f\colon \mathbf{R}^{n-1} \to \mathbf{R}^p$ be a map with $J^r(f) = w(x)|_{x_1=0}$.

Write $w(x_1, x_2, \ldots, x_n)$ in the form

$$w(x_1, x_2, \ldots, x_n) = w(x_1, x_2, \ldots, x_n)|_{x_1=0} + x_1\,\varphi(x_1, x_2, \ldots, x_n).$$

Put $g = f(x_2, \ldots, x_n) + x_1\,\varphi(x_1, x_2, \ldots, x_n)$. Then

$$J^r(g) = J^r(f) + J^r(x_1\,\varphi) = w|_{x_1=0} + J^r(x_1\,\varphi) = w.$$

Since w is H_{x_1}-sufficient, there exists $\sigma \in H_{x_1}$ such that

$$(2.7)\qquad g\circ\sigma = w, \quad g\circ\sigma = f\circ\sigma + (x_1\,\varphi)\circ\sigma.$$

From (2.6) we get

$$(2.8)\qquad g\circ\sigma|_{x_1=0} = w(x_1, \ldots, x_n)|_{x_1=0}.$$

By (2.7)

$$(2.9)\qquad g\circ\sigma|_{x_1=0} = f\circ\sigma|_{x_1=0} + (x_1\,\varphi)\circ\sigma|_{x_1=0} = f\circ\sigma|_{x_1=0} = f\circ\sigma_*,$$

where $\sigma_* = \sigma|_{x_1=0}$.

From (2.8) and (2.9) we have

$$f\circ\sigma_* = w(x_1, \ldots, x_n)|_{x_1=0}.$$

Thus $w|_{x_1=0}$ is H-sufficient in $\mathbf{R}^{n-1}$.

Since $w|_{x_1=0}$ is H-sufficient, by a well-known result, there exist $c > 0$ and a neighbourhood U of 0 in $\mathbf{R}^{n-1}$ such that

$$d(\nabla_1 w(x'), \ldots, \nabla_p w(x')) \geqslant c\,|x'|^{r-1} \geqslant c\,|x'|^{r+1}, \qquad x' = (0, x_2, \ldots, x_n).$$

On the other hand,

$$d(\nabla_1 w(x'), \ldots, \nabla_p w(x)) = d(v_1^w, \ldots, v_p^w)|_{x_1=0}.$$

Therefore, considering d in the hyperspace $x_1 = 0$, we have

$$d(v_1^w, \ldots, v_p^w) \geqslant c\,|x|^{r+1} \qquad \forall x \in U.$$

Thus $a_{i1} \neq 0$ for every i.

Write the vectors $\lambda_k'^i$, $k = 2, \ldots, p$, in the form

$$\lambda_k'^i = (\alpha_{k,1}^i, \alpha_{k,2}^i, \ldots, \alpha_{k,n}^i).$$

Since

$$v_1^w(a_i) \in \operatorname{Span}\{v_2^w(a_i) + \lambda_2'^i, \ldots, v_p^w(a_i) + \lambda_p'^i\},$$

we have

$$\nabla_1 w(a_i) = \left(\frac{\partial w_1}{\partial x_1}(a_i), \ldots, \frac{\partial w_1}{\partial x_n}(a_i)\right)$$

$$\in \operatorname{Span}\left\{\left(\frac{\partial w_1}{\partial x_1}(a_i)+\frac{\alpha^i_{2,1}}{a_{ii}}, \frac{\partial w_2}{\partial x_2}(a_i)+\alpha^i_{2,2}, \ldots, \frac{\partial w_2}{\partial x_n}(a_i)+\alpha^i_{2,n}\right), \ldots\right.$$
$$\left.\ldots, \left(\frac{\partial w_p}{\partial x_1}(a_i)+\frac{\alpha^i_{p,1}}{a_{i,1}}, \frac{\partial w_p}{\partial x_2}(a_i)+\alpha^i_{p,2}, \ldots, \frac{\partial w_p}{\partial x_n}+\alpha^i_{p,n}\right)\right\}.$$

Let us put

$$\lambda^i_2 = \left(\frac{\alpha^i_{2,1}}{a_{i1}}, \alpha^i_{2,2}, \ldots, \alpha^i_{2,n}\right),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\lambda^i_p = \left(\frac{\alpha^i_{p,1}}{a_{i1}}, \alpha^i_{p,2}, \ldots, \alpha^i_{p,n}\right).$$

Then we have $|\lambda^i_k| = o(|a_i|^{r-1})$, $k = 2, \ldots, p$.

In fact, since $|\lambda'^i_k| = o(|a_i|^{r+1})$, $k = 2, \ldots, p$ and

$$\lambda'^i_k = (\alpha^i_{k,1}, \alpha^i_{k,2}, \ldots, \alpha^i_{k,n}),$$

we get $|\alpha^i_{k,j}| = o(|a_i|^{r+1})$, $j = 1, 2, \ldots, n$. On the other hand,

$$|\alpha^i_{k,1}| = |a_{i1}|\left|\frac{\alpha^i_{k,1}}{a_{i1}}\right| \quad \text{and} \quad |\alpha^i_{k,1}| = o(|a_i|^{r+1});$$

we get

$$\text{(2.10)} \qquad |a_{i1}|\left|\frac{\alpha^i_{k,1}}{a_{i,1}}\right| = o(|a_i|^{r+1}).$$

Since $|a_{i1}|/|a_i|^2 \leqslant |a_{i1}|/|a_{i1}|^2 = 1/|a_{i1}| \to \infty$, we have

$$\lim_{i\to\infty}\frac{|a_{i1}|}{|a_i|^2} = \infty.$$

Thus from (2.10) we obtain

$$\left|\frac{\alpha^i_{k,1}}{a_{i1}}\right| = o(|a_i|^{r-1}).$$

Since $|\alpha^i_{k,j}| = o(|a_i|^{r-1})$, we have $|\lambda^i_k| = o(|a_i|^{r-1})$, $k = 2, \ldots, p$. Finally, we get

$$\operatorname{grad} w_1(a_i) = \sum_{k=2}^{p} \alpha_k(\operatorname{grad} w_k(a_i)+\lambda^i_k), \qquad |\lambda^i_k| = o(|a_i|^{r-1}).$$

Repeating the proof of Theorem in [1], we conclude the proof of Theorem 2.

2.3. Proof of Theorem 3. Assume that $f\colon \boldsymbol{R}^n \to \boldsymbol{R}^p$, $f(0) = 0$ and $j^r(f) = w$. Let us show that there exists a homeomorphism

$$\sigma\colon \boldsymbol{R}^n \to \boldsymbol{R}^n, \quad \sigma\{x\in \boldsymbol{R}^n\colon x_1 = 0\} \subset \{x\in \boldsymbol{R}^n\colon x_1 = 0\}$$

such that $f_0 \circ \sigma = w$.

Put

(3.1) $$F(x, t) = w(x) + t(f(x) - w(x)).$$

We first show that the inequality $d(\Phi_1(w), \ldots, \Phi_p(w)) \geqslant c|x|^r$ for every $x \in U$ implies

(3.2) $$d(\Phi_1(F), \ldots, \Phi_p(F)) \geqslant \tfrac{1}{2} c|x|^r, \quad \forall x \in U \text{ and } \forall t,$$

where

$$\Phi_1(F) = \left(x_1 \frac{\partial F_1}{\partial x_1}, \ldots, x_n \frac{\partial F_1}{\partial x_n}, \frac{\partial F_1}{\partial t}\right),$$
$$\cdots\cdots\cdots$$
$$\Phi_p(F) = \left(x_1 \frac{\partial F_p}{\partial x_1}, \ldots, x_n \frac{\partial F_p}{\partial x_n}, \frac{\partial F_p}{\partial t}\right).$$

In fact, consider $\Phi_k(w)$ as a vector of the (x, t)-space. Since the last components of the vectors are zero, we get

(3.3) $$|\Phi_k(F) - \Phi_k(w)|$$
$$= \left|\left(tx_1\left(\frac{\partial f_k}{\partial x_1} - \frac{\partial w_k}{\partial x_1}\right), \ldots, tx_n\left(\frac{\partial f_k}{\partial x_n} - \frac{\partial w_k}{\partial x_n}\right), f_k - w_k\right)\right| \leqslant c_1 |x|^{r+1},$$

(3.4) $$\left|\sum_{i=1}^{p} \lambda_i \Phi_i(F)\right| \geqslant \left|\sum_{i=1}^{p} \lambda_i \Phi_i(w)\right| - \left|\sum_{i=1}^{p} \lambda_i (\Phi_i(F) - \Phi_i(w))\right|.$$

Fix k, $1 \leqslant k \leqslant p$, and $\lambda_k(x, t) = 0$; we have

$$\frac{|\lambda_k(\Phi_k(F) - \Phi_k(w))|}{\left|\sum_i \lambda_i \Phi_i(w)\right|} = \frac{|\Phi_k(F) - \Phi_k(w)|}{\left|\Phi_k(w) + \sum_{i \neq k} \lambda_k^{-1} \lambda_i \Phi_i(w)\right|}$$
$$\leqslant \frac{c_1 |x|^{r+1}}{d(\Phi_1(w), \ldots, \Phi_p(w))} \leqslant \frac{c_1 |x|^{r+1}}{c|x|^r} = \frac{c_1}{c}|x|.$$

For $\lambda_k(x, t) = 0$ the inequality is obvious. From (3.3) and (3.4) we infer that

(3.5) $$\lim \left|\sum_i \lambda_i \Phi_i(w)\right|^{-1} \left|\sum_i \lambda_i (\Phi_i(F) - \Phi_i(w))\right| = 0$$

uniformly with respect to t.

Thus by (3.4) we have

$$\left|\sum_i \lambda_i \Phi_i(F)\right| \geqslant \tfrac{1}{2}\left|\sum_i \lambda_i \Phi_i(w)\right|$$

when x is sufficiently small. Thus (3.2) is proved.

To continue the proof of the theorem, we put

(3.6) $$Y(x, t) = \sum_{i=1}^{p} ((0, 1) \cdot \Phi_i(F))|N_i|^{-2} N_i,$$

where N_i is the projection of the vector $\Phi_i(F)$ onto the space spanned by $\Phi_j(F)$, $j \neq i$.

Note that $Y(x, t)$ is the projection of $(0, \ldots, 0, 1)$ onto the subspace spanned by $\Phi_1(F), \ldots, \Phi_p(F)$.

Thus

$$X_1(x, t) = (0, \ldots, 0, 1) - Y(x, t)$$

is orthogonal to $\Phi_1(F), \ldots, \Phi_p(F)$. Note that

$$\Phi_k(F) = \left(x_1 \frac{\partial F_k}{\partial x_1}, \ldots, x_n \frac{\partial F_k}{\partial x_n}, \frac{\partial F_k}{\partial t}\right).$$

Thus if we put $X_1(x, t) = (X_{1,1}, \ldots, X_{1,n}, X_{1,n+1})$, we have

$$x_1 \frac{\partial F_k}{\partial x_1} \cdot X_{1,1} + \ldots + x_n \frac{\partial F_k}{\partial x_n} \cdot X_{1,n} + \frac{\partial F_k}{\partial t} \cdot X_{1,n+1} = 0$$

for $k = 1, \ldots, p$. Let

$$X(x, t) = (x_1 X_{1,1}, x_2 X_{1,2}, \ldots, x_n X_{1,n}, X_{1,n+1}).$$

Then $X(x, t)$ is orthogonal to ∇F_k, $k = 1, 2, \ldots, p$.

From (3.6) and (3.2) it follows that

$$\lim_{x \to 0} |x|^{-1} Y(x, t) = 0.$$

Thus the vector $X_1(x, t)$ has properties (i), (ii), (iii) occurring in the proof of Theorem 1. It is easy to see that $X(x, t)$ has properties (i), (ii), (iii).

Repeating the last part of the proof of Theorem 1, we obtain the proof of Theorem 3.

References

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