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Application of the extremal points method to some variational problems in the theory of schlicht functions

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The aim of this paper is to show the application of Leja's method to some variational problems in the theory of schlicht functions. Let $z = \varphi(w)$, $\varphi(\infty) = \infty$ be the conformal mapping of the exterior of the unit circle $|w| > 1$ onto the unbounded domain D which contains the point $z = \infty$. The complementary set E is a bounded continuum. Suppose that the capacity $d(E)$ of E is 1. Denote by $w = \Phi(z)$ the inverse function to $z = \varphi(w)$. We have

$$\log\Phi(z) = G(z, \infty) + iH(z, \infty), \tag{1}$$

where $G(z, \infty)$ is the Green function of D with the pole at $z = \infty$, $H(z, \infty)$ is its conjugate.

If E^* denotes a new continuum obtained from E by a variation, we shall later specify the kind of variation, and Φ^*, G^*, H^* the corresponding functions to Φ, G, H defined above, then

$$\log\Phi^*(z) = G^*(z, \infty) + iH^*(z, \infty). \tag{2}$$

Let J be a Jordan curve which contains E strictly inside, i.e. the distance from J to E is positive. For small enough variation of E the continuum E^* lies inside J. If we put

$$h(z) = G^*(z, \infty) - G(z, \infty) + i\big(H^*(z, \infty) - H(z, \infty)\big) \tag{3}$$

then for $|h(z)| < \varepsilon$, $z \in J$ is $|h(z)| < \varepsilon$ for z outside of J.

From (1), (2) and (3) follows $\Phi^*(z) = \Phi(z)e^{h(z)}$. Therefore

$$z = \Phi^{*-1}\big(\Phi(z)e^{h(z)}\big) = \Phi^{*-1}\big(w + wh + O(h^2)\big) \tag{4}$$

where $O(h^2)$ denotes the terms which contain h in higher powers.

If one extends the right side of (4) with respect to h one obtains ([1], p. 302)

$$z = z^* + whz' + O(h^2). \tag{5}$$

Under the transformation $w = 1/x$, $z = 1/y$ the formula (5) takes the form

$$y^* = y + xhy' + O(h^2) . \tag{6}$$

Here $y = y(x)$ denotes the conformal mapping of the unit circle $|x| < 1$ onto Δ, where Δ is the image of D under $y = 1/z$. Now we consider the variation of D defined by

$$z^* = z/(1 + z\varrho e^{i\alpha}) , \tag{7}$$

where $\varrho \geqslant 0$ and α is an arbitrary real number. Let D^* be the image of D under (7). For $\varrho = 0$ is $D^* = D$; for $\varrho > 0$ and $|\alpha|$ small enough D^* represents small variation of D.

According to (7) is $(z = \infty) \Longleftrightarrow (z^* = 1/\varrho e^{i\alpha})$ and $(z^* = \infty) \Longleftrightarrow (z = -1/\varrho e^{i\alpha})$. We want to compute (3) in the case of variation (7). Observe that

$$\zeta = \exp[G(z; -1/\varrho e^{i\alpha}) + iH(z; -1/\varrho e^{i\alpha})]$$

and

$$\zeta^* = \exp[G(z; \infty) + iH(z; \infty)]$$

are related by the homography

$$\zeta = \frac{1 - \bar{\zeta}_0^* \zeta^*}{\zeta^* - \zeta_0^*} e^{i\theta} , \qquad \zeta_0^* = \exp\left[G\left(-\frac{1}{\varrho e^{i\alpha}}; \infty\right) + iH\left(-\frac{1}{\varrho e^{i\alpha}}; \infty\right)\right],$$

$$\exp\left\{i\theta + G\left(-\frac{1}{\varrho e^{i\alpha}}; \infty\right) + iH\left(-\frac{1}{\varrho e^{i\alpha}}, \infty\right)\right\}$$

$$= -\exp\left\{G\left(\infty; -\frac{1}{\varrho e^{i\alpha}}\right) + iH\left(\infty; -\frac{1}{\varrho e^{i\alpha}}\right)\right\},$$

$$\theta = 2H\left(-\frac{1}{\varrho e^{i\alpha}}; \infty\right) .$$

Therefore

$$\exp\left\{G\left(z; -\frac{1}{\varrho e^{i\alpha}}\right) + iH\left(z; -\frac{1}{\varrho e^{i\alpha}}\right)\right\} \tag{8}$$

$$= e^{i\theta} \frac{1 - \exp\left[G\left(-\frac{1}{\varrho e^{i\alpha}}; \infty\right) - iH\left(-\frac{1}{\varrho e^{i\alpha}}; \infty\right) + G(z; \infty) + iH(z, \infty)\right]}{\exp[G(z, \infty) + iH(z, \infty)] - \exp\left[G\left(-\frac{1}{\varrho e^{i\alpha}}; \infty\right) + iH\left(-\frac{1}{\varrho e^{i\alpha}}; \infty\right)\right]}$$

and

$$h(z) = G\left(\eta; -\frac{1}{\varrho e^{i\alpha}}\right) + iH\left(\eta; -\frac{1}{\varrho e^{i\alpha}}\right) - G(z, \infty) - iH(z, \infty) , \tag{9}$$

where

$$\eta = z/(1 - z\varrho e^{i\alpha}) . \tag{10}$$

From (8) and (9) follows for small $|h|$

$$1+h \approx \exp h$$
$$= e^{i\theta}\frac{f(z)}{\exp[G(\eta;\infty)+iH(\eta;\infty)]-\exp\left[G\left(-\frac{1}{\varrho e^{i\alpha}};\infty\right)+iH\left(-\frac{1}{\varrho e^{i\alpha}};\infty\right)\right]},$$

where

$$f(z)=\exp[-G(z,\infty)-iH(z,\infty)]-\exp\left[G\left(-\frac{1}{\varrho e^{i\alpha}};\infty\right)-iH\left(-\frac{1}{\varrho e^{i\alpha}};\infty\right)+\right.$$
$$\left.+G(\eta;\infty)+iH(\eta;\infty)-G(z,\infty)-iH(z,\infty)\right].$$

For small $\varrho>0$, we neglect the terms with ϱ in higher power than 1, is

$$\exp\left[-G\left(-\frac{1}{\varrho e^{i\alpha}};\infty\right)-iH\left(-\frac{1}{\varrho e^{i\alpha}};\infty\right)\right]\approx-\varrho e^{i\alpha}.$$

Therefore using the fact that (see [2])

$$G(z;\infty)+iH(z;\infty)=\int\log(z-\xi)\,d\mu(\xi),$$

where μ is the natural mass-distribution on E, we have

$$1+h(z)\approx\frac{1+\int\log(\eta-\xi)\,d\mu(\xi)-\int\log(z-\xi)\,d\mu(\xi)+\varrho e^{i\alpha}\exp\left[-\int\log(z-\xi)\,d\mu\right]}{1+\varrho e^{i\alpha}\exp\left[\int\log(\eta-\xi)\,d\mu(\xi)\right]}$$

and

(11) $$h(z)\approx\int\log(\eta-\xi)\,d\mu(\xi)-\int\log(z-\xi)\,d\mu(\xi)+$$
$$+\varrho e^{i\alpha}\exp\left[-\int\log(z-\xi)\,d\mu\right]-\varrho e^{i\alpha}\exp\left[\int\log(\eta-\xi)\,d\mu\right].$$

On the other hand, for $\xi\in E$ and $z\in D$ such that $|\xi/z|<1$ is

$$\int\log(z-\xi)\,d\mu=\log z+\int\log\left(1-\frac{\xi}{z}\right)d\mu=\log z-\frac{s_1}{z}-\frac{s_2}{2z^2}-\frac{s_3}{3z^3}-\dots,$$

where $s_k=\int\xi^k d\mu$, see [3], $k=1,2,\dots$

From (10) and (11) follows that

(12) $$h(z)=\varrho\left\{e^{i\alpha}\left[\left(z+s_1+\frac{s_2}{z}+\frac{s_3}{z^2}+\dots\right)-z\left(1-\frac{s_1}{z}-\frac{s_2}{2z^2}-\frac{s_3}{3z^3}-\dots\right)\right]+\right.$$
$$\left.+e^{-i\alpha}\frac{1}{z}\left(1+\frac{s_1}{z}+\frac{s_2}{2z^2}+\dots\right)\right\}+O(\varrho^2).$$

As the function $y(x)$ belongs to the class S of all schlicht functions in the unit circle with the extension

(13) $$y=x+a_2x^2+a_3x^3+\dots$$

we can use the variational formula (6) to compare the coefficients of (13) with those of

$$y^* = a_1^* x + a_2^* x^2 + a_3^* x^3 + \dots \tag{14}$$

If we put (13) and (14) in (6), using (12) and compare the coefficients on both sides, we obtain

$$a_1^* = 1 + 2\varrho e^{i\alpha} s_1 , \tag{15}$$

$$a_2^* = a_2 + \varrho \{e^{i\alpha}[-4a_2^2 + \tfrac{3}{2} s_2 - \tfrac{1}{2} s_1^2] + e^{-i\alpha}\} , \tag{16}$$

$$a_3^* = a_3 + \varrho \{e^{i\alpha}[-4a_4] + 2a_2 e^{-i\alpha}\} , \dots \tag{17}$$

The function $y^*(x)/a_1^*$ belongs to S. In [3] are given formulas which permit to express the coefficients a_k by s_k. For example

$$a_2 = -s_1 ; \quad a_3 = \frac{3s_1^2 - s_2}{2} ; \quad a_4 = \frac{-8s_1^3 + 6s_1 s_2 - s_3}{3} , \quad \dots \tag{18}$$

Suppose that $y(x)$ is this function of S with $a_2 > 0$ which realise the maximum of $|a_2|$ in S. For this function is

$$|a_2^*/a_1^*| \leqslant a_2 . \tag{19}$$

Using (15) and (16) we obtain from (19)

$$\begin{gathered} \frac{a_2^*}{a_1^*} = a_2 + \varrho \{e^{i\alpha}[-3a_3 + 2a_2^2] + e^{-i\alpha}\} , \\ |a_2 + \varrho \{e^{i\alpha}[-3a_3 + 2a_2^2] + e^{-i\alpha}\}| \leqslant a_2 \end{gathered} \tag{20}$$

for every $\varrho > 0$ and α. (20) can be rewrite in the form

$$\mathrm{Re}\{e^{i\alpha}(-3a_3 + 2a_2^2 + 1)\} \leqslant 0 .$$

Therefore

$$3a_3 = 2a_2^2 + 1 . \tag{21}$$

The condition (21) is the special case of Marty's formula for $n = 2$ (see [4]). In the similar way we obtain from (15) and (17) for the function of S with $a_3 > 0$ which realises the maximum of $|a_3|$

$$\frac{a_3^*}{a_1^*} = a_3 + \varrho \{e^{i\alpha}[-4a_4 + 2a_2 a_3] + 2a_2 e^{-i\alpha}\} \tag{22}$$

and in consequence ($a_3 > 0$)

$$4a_4 = 2a_2 a_3 + 2\bar{a}_2 . \tag{23}$$

This is Marty's formula for $n = 3$.

If we suppose that the second coefficient of the extremal function has the argument φ_2, i.e. if $a_2 = |a_2|e^{i\varphi_2}$, then (21) takes the form

$$(21^*) \qquad 3a_3 = 2a_2^2 + e^{2i\varphi_2}.$$

In the similar way if the third coefficient a_3 of the extremal function is equal $a_3 = |a_3|e^{i\varphi_3}$, then

$$(22^*) \qquad 4a_4 = 2a_2 a_3 + 2\bar{a}_2 e^{2i\varphi_3}.$$

Denote by b_k the coefficients of the inverse function $x = x(y)$

$$x = y + b_2 y^2 + b_3 y^3 + \ldots$$

Using (20) and (22) we obtain easily the following conditions for the coefficients b_2, b_3, b_4 of the function $x(y)$ which realises the maximum of $|b_2|$ or $|b_3|$ respectively

$$3b_3 = 4b_2^2 - e^{2i\varphi_2}, \qquad b_2 = |b_2|e^{i\varphi_2},$$
$$4b_4 = 6b_2 b_3 + 2\bar{b}_2 e^{2i\varphi_3}, \qquad b_3 = |b_3|e^{i\varphi_3}.$$

References

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