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Boundary value problems on $[a, b)$ and singular perturbations

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Abstract. In this paper we show the existence of a bounded solution for a non-linear differential equation with a general boundary condition. We assume that the non-linearity is not defined in $\{0\}$ and we use degree arguments, because the solutions belong to a set without the fixed point property with respect to compact and continuous operators.

The existence of a bounded solution of the boundary value problem

$$(*) \qquad \dot{x} = A(t)x+f(t, x), \qquad Tx = \varrho$$

on a non-compact interval has been studied by many authors. We recall the admissibility theory developed by J.L. Massera and J. J. Schäffer; further contributions are due to C. Corduneanu and W. A. Coppel. From among the results obtained in this theory we mention only the papers by Avramescu [1], [2] and by Corduneanu [8].

Boundary value problems like (*) on a compact interval can be solved by the alternative theory (L. Cesari [7]), the coincidence degree theory (Mawhin [12]), or by the more recent theory of the 0-epi functions (Furi, Martelli, Vignoli [9]). They can be more easily solved by the Leray–Schauder degree theory, from which the above-mentioned theories are derived. We mention here only the papers by Villari [14], [15], who used these methods in the study of ordinary differential equations. A rather complete bibliography is in Mawhin [13].

Problem (*) on a non-compact interval has been taken up, with use of topological methods, by Kartsatos in [10], [11], under the hypothesis that problem (*) for $f(t, x) \equiv 0$ has a unique solution, and by the authors in [3], [4], [5], without this hypothesis.

On the other hand, we assume here that $f(t, x)$ is defined in $\mathbf{R}^n - \{0\}$. Such a condition implies that the operator associated to (*) is defined on a non-convex set; so the Schauder theorem cannot be used while the degree theory may be applied.

In Section 1 notations and symbols are given; we construct an operator needed in the following section and we prove that it is compact and continuous. In Section 2 we prove an existence theorem using the Leray–Schauder degree theory.

1. Let us consider the linear boundary value problem

$$\dot{x} = A(t)x + f(t, x) \tag{1.1}$$

$$Tx = \varrho, \quad \varrho \in \boldsymbol{R}^n - \{0\}, \tag{1.2}$$

where: $t \in J = [a, b)$, $-\infty < a < b \leqslant +\infty$; $t \to A(t)$ is a continuous function from J into $\mathfrak{A}$ the algebra of endomorphisms of $\boldsymbol{R}^n$; $u \to f(t, u)$ is a continuous function from $\boldsymbol{R}^n - \{0\}$ into $\boldsymbol{R}^n$ for each $t \in J$; $x \to Tx$ is a continuous linear operator from $\operatorname{dom} T = BC$ (the space of bounded continuous functions from J into $\boldsymbol{R}^n$) into $\boldsymbol{R}^n$.

We are looking for solutions of (1.1)–(1.2) that belong to BC; this is a Banach space with the norm

$$\|x\| = \sup_{t \in J} |x(t)|.$$

Defining certain operators we shall identify $\boldsymbol{R}^n$ with the subspace of BC consisting of constant functions; we denote by I the identity operator in BC.

From now on we assume the following hypotheses:

(1) the linear system associated to (1.1)

$$\dot{y} = A(t)y \tag{1.3}$$

is strongly stable;

(2) there are two integrable functions $t \to p(t)$, $t \to q(t)$ from J into $\boldsymbol{R}_0^+ = [0, +\infty)$ such that

$$|f(t, u)| \leqslant p(t)\frac{1}{|u|} + q(t), \quad u \in \boldsymbol{R}^n - \{0\},\ t \in J$$

and

$$\int_a^b p(t)\,dt = \Gamma < +\infty, \quad \int_a^b q(t)\,dt = \Lambda < +\infty;$$

(3) T restricted to the space D of the solutions of (1.3) is onto $\boldsymbol{R}^n$.

Remark 1.1. The hypothesis of strong stability ensures that, for each $y \in D$, $y \neq 0$, the following cannot be true

$$\inf_{t \in J} |y(t)| = 0.$$

Moreover, from (3) we can deduce that the linear problem associated to (1.1)–(1.2),

$$\dot{y} = A(t)y, \quad Ty = \varrho, \quad \varrho \in \boldsymbol{R}^n - \{0\},$$

has a unique solution y such that, setting

$$\Phi_r^R = \{u \in \mathbf{R}^n\colon\ r \leqslant |u| \leqslant R\}, \qquad r,\, R \in \mathbf{R}^+ = (0,\, +\infty),$$

we have

$$y(t) \in \Phi_r^R$$

for suitably choosen r, R and $t \in J$.

Let $X(t)$ be a fundamental matrix of (1.3), principal at a; from (3) we see that the operator $T_0 = TX(t)$ (T applied to the columns of $X(t)$) belongs to $\mathfrak{A}$ and is invertible: let T_0^{-1} be its inverse.

Remark 1.2. If system (1.3) is strongly stable, the boundary value problem (1.1)–(1.2) is equivalent to

$$\dot{z} = g(t, z), \qquad \mathcal{S}z = \varrho, \qquad \varrho \in \mathbf{R}^n - \{0\}, \tag{1.4}$$

with the change of variable

$$z(t) = X^{-1}(t)\, x(t).$$

Our main theorem, stated without loss of generality for $A(t) \equiv 0$, is:

THEOREM 1.1. *Under hypotheses* (1)–(3), *if* $\varrho \in \mathbf{R}^n - \{0\}$ *is such that*

$$|T_0^{-1}\varrho| > \|I - T_0^{-1}T\|\, \Lambda + 2\sqrt{\|I - T_0^{-1}T\|\, \Gamma}, \tag{1.5}$$

then the boundary value problem (1.1)–(1.2) (*with* $A(t) \equiv 0$) *has at least one solution* x. *Moreover, there are* r, $R \in \mathbf{R}^+$ *such that* $x(t) \in \Phi_r^R$, $t \in J$.

We shall prove it in Section 2.

The solutions of (1.1)–(1.2) are (see [10]) the fixed points of the operator

$$M\colon \operatorname{dom} M \subset BC \to BC$$

defined by

$$t \to (Mx)(t) = X(t)\, T_0^{-1}\big(\varrho - TP(\cdot,\, x)\big) + P(t,\, x), \tag{1.6}$$

where

$$P(t,\, x) = \int_a^t X(t)\, X^{-1}(s) f\big(s,\, x(s)\big)\, ds$$

and

$$\operatorname{dom} M = \{x \in BC\colon\ \exists\, \varepsilon > 0\colon\ |x(t)| > \varepsilon,\ t \in J\}.$$

From (2) it follows that the operator M is well defined and that $\operatorname{Im} M \subset BC$.

Let $\mathscr{A} \subset \mathbf{R}^n$ be an open set, and $S \subset \mathscr{A}$ be a closed set with non-void interior. We write

$$S_1 = \{c \in \mathbf{R}_0^+\colon\ c = |d|,\ d \in S\};$$

the following theorem holds (see [6], Theorem 1):

THEOREM 1.2. *Suppose that, for each* $u \in \mathscr{A}$, $t \to h(t, u)$ *is a continuous function from* J *into* $\mathbf{R}^n$; *for each* $t \in J$, $u \to h(t, u)$ *is a continuous function from* $\mathscr{A}$ *into* $\mathbf{R}^n$ *such that*

(1.7) $$|h(t, u)| \leqslant g(t, |u|), \qquad t \in J,\ u \in S,$$

where $g\colon JxS_1 \to \mathbf{R}_0^+$ *satisfies the following conditions*:

(i) *for each* $t \in J$, $v \to g(t, v)$ *is continuous*, $v \in S_1$;

(ii) $t \to g_\mu(t) = \max\limits_{v \in [0,\mu] \cap S_1} g(t, v)$ *is integrable on* J, $\mu \in S_1$.

Let $L\colon BC \to BC$ *be a bounded linear operator and let*

$$K\colon \operatorname{dom} K \subset BC \to BC$$

be defined by $K = L \circ \tilde{K}$, *where* $\tilde{K}$ *is given by*

(1.8) $$t \to (\tilde{K}x)(t) = \int_a^t h(s, x(s))\, ds$$

and

$$\operatorname{dom} K = \{x \in BC\colon\ x(t) \in S,\ t \in J\}.$$

Then K *is a continuous and compact operator.*

The domain of the operator M is not closed; nevertheless, setting

$$\Delta_r^R = \{x \in BC\colon\ x(t) \in \Phi_r^R,\ t \in J\}$$

we have $\Delta_r^R \subset \operatorname{dom} M$ and:

LEMMA 1.1. Δ_r^R *is closed in the topology of* BC *and, setting*

$$B = \bigcup_{0 < \varepsilon < \frac{1}{2}(R-r)} \Delta_{r+\varepsilon}^{R-\varepsilon}, \qquad F = \{x \in \Delta_r^R\colon\ \inf_{t \in J} |x(t)| = r \ \text{or}\ \sup_{t \in J} |x(t)| = R\},$$

we have $B = \mathring{\Delta}_r^R$ *and* $F = \partial \Delta_r^R$, *where* $\mathring{A}$, ∂A *denote respectively the interior and the boundary of the set* A.

Proof. For $G \subset BC$, $G \neq \emptyset$, $x \in BC$, let $d(x, G)$ denote the distance from x to G: $d(x, G) = \inf\{\|x - y\|\colon\ y \in G\}$.

Let $H = \{x \in BC\colon\ |x(t)| = R,\ t \in J\}$. Furthermore, let $\bar{\Sigma}(G, \delta)$, $\Sigma(G, \delta)$ denote, respectively, the sets: $\{x \in BC\colon\ d(x, G) \leqslant \delta\}$ $\{x \in BC\colon\ d(x, G) < \delta\}$ $(\delta > 0)$. Clearly, $\bar{\Sigma}(G, \delta)$ is closed, $\Sigma(G, \delta)$ is open in BC for any $G \subset BC$, $G \neq \emptyset$.

Now, $\Delta_r^R = \bar{\Sigma}(H, R - r) \cap \bar{\Sigma}(\{0\}, R)$, which implies that Δ_r^R is closed. Moreover, for $\varepsilon_1 < \varepsilon$

$$\Delta_{r+\varepsilon}^{R-\varepsilon} \subset \Sigma(H, R - r - 2\varepsilon_1) \cap \Sigma(\{0\}, R - \varepsilon_1) \subset \mathring{\Delta}_{r+\varepsilon_1}^{R-\varepsilon_1};$$

therefore B is open and $B \subset \mathring{\Delta}_r^R$.

Let $x \notin B$; if $x \notin \Delta_r^R$, then $x \notin \mathring{\Delta}_r^R$; suppose: $x \in \Delta_r^R$. Then we have the following possibilities:

(a) $|x(t_1)| = R$ or $|x(t_1)| = r$ for some $t_1 \in J$,

(b) $\limsup_{t \to b^-} |x(t)| = R$ or $\liminf_{t \to b^-} |x(t)| = r$.

For example, let $\liminf_{t \to b^-} |x(t)| = r$. Let $\delta > 0$ be arbitrary, fixed, and let $x_0(t) = x(t) - \frac{1}{2}\delta$, $t \in J$. Obviously $x_0 \in \Sigma(\{x\}, \delta)$. We show that $x_0 \notin \Delta_r^R$. Since

$$\liminf_{t \to b^-} |x_0(t)| = r - \tfrac{1}{2}\delta,$$

there exists t_m such that $|x_0(t_m)| < r$, i.e., $x_0 \notin \Delta_r^R$; this implies $\Sigma(\{x\}, \delta) \not\subset \Delta_r^R$ for each δ, hence $x \notin \mathring{\Delta}_r^R$. In the other cases the proof is quite the same; thus $B = \mathring{\Delta}_r^R$. Hence we also deduce that $F = \partial \Delta_r^R$.

Remark 1.3. Condition (ii) of Theorem 1.2 is fulfilled if

$$g(t, v) = p(t)v + q(t), \qquad v \in S_1,\ t \in J,$$

with $t \to p(t)$, $t \to q(t)$ two integrable functions from J into $\boldsymbol{R}_0^+$; or if

$$g(t, v) = p(t)\frac{1}{v} + q(t), \qquad v \in S_1 - \{0\}, \qquad t \in J,$$

with the same hypotheses as above.

COROLLARY 1.1. *Under hypotheses* (1)–(3), *the operator M defined by* (1.6) *is continuous and compact on Δ_r^R, r, $R \in \boldsymbol{R}^+$.*

Proof. This follows from Theorem 1.2 and Lemma 1.1, on account of Remark 1.3.

Remark 1.4. The set Δ_r^R does not have the fixed point property with respect to continuous and compact operators. For example, the operator

$$Q\colon \Delta_r^R \to \Delta_r^R$$

defined by

$$t \to (Qx)(t) = -x(a)$$

is continuous and compact but has no fixed point.

2. In this section we suppose $A(t) \equiv 0$, $t \in J$. The operator M becomes

$$t \to (Mx)(t) = T_0^{-1}\big(\varrho - TP(\cdot, x)\big) + P(t, x), \tag{2.1}$$

where

$$P(t, x) = \int_a^t f(s, x(s))\,ds.$$

For $\lambda \in [0, 1]$, let $I - M_\lambda$ be the operator

$$t \to [(I - M_\lambda)x](t) = x(t) - T_0^{-1}\varrho - \lambda\big(P(t, x) - T_0^{-1}TP(\cdot, x)\big). \tag{2.2}$$

LEMMA 2.1. *Under hypotheses* (2) *and* (3), *if* $\varrho \in \mathbf{R}^n - \{0\}$ *is such that*

$$|T_0^{-1}\varrho| \geqslant \|I - T_0^{-1}T\| \Lambda + 2\sqrt{\|I - T_0^{-1}T\| \Gamma}, \tag{1.5}$$

it is possible to choose r, $R \in \mathbf{R}^+$ *such that*

$$\|(I - M_\lambda)x\| > 0$$

for $x \in \partial\Delta_r^R$, $\lambda \in [0, 1]$.

Proof. Let $x \in \partial\Delta_r^R$; from (2.2) we have

$$\|(I - M_\lambda)(x)\| \geqslant \|x - T_0^{-1}\varrho\| - \|\lambda(I - T_0^{-1}T)P(\cdot, x)\|; \tag{2.3}$$

x belongs to $\partial\Delta_r^R$ if $\|x\| < R$ and $\inf_{t \in J}|x(t)| = r$ or $\|x\| = R$. In the first case, choosing

$$r < |T_0^{-1}\varrho| \tag{2.4}$$

we have

$$\|x - T_0^{-1}\varrho\| = \sup_{t \in J}|x(t) - T_0^{-1}\varrho| \geqslant \big||T_0^{-1}\varrho| - |x(t)|\big| \geqslant |T_0^{-1}\varrho| - r > 0.$$

Since, for each $x \in \partial\Delta_r^R$, $1/\|x\| \leqslant 1/r$, from (2) we have

$$\|\lambda(I - T_0^{-1}T)P(\cdot, x)\| \leqslant \|I - T_0^{-1}T\|(\Gamma/r + \Lambda),$$

and then from (2.3)

$$\|(I - M_\lambda)x\| \geqslant |T_0^{-1}\varrho| - r - \|I - T_0^{-1}T\|(\Gamma/r + \Lambda). \tag{2.5}$$

Setting

$$\alpha = |T_0^{-1}\varrho| - \|I - T_0^{-1}T\| \Lambda, \qquad \beta = \|I - T_0^{-1}T\| \Gamma,$$

from (1.5) we have

$$\alpha > 2\sqrt{\|I - T_0^{-1}T\| \Gamma} = 2\beta^{\frac{1}{2}} > 0$$

and then

$$\alpha^2 - 4\beta > 0.$$

Formula (2.5) becomes

$$\|(I - M_\lambda)x\| \geqslant \alpha - r - \frac{\beta}{r} = \frac{-r^2 + \alpha r - \beta}{r},$$

and if we choose r such that

$$\tfrac{1}{2}(\alpha - (\alpha^2 - 4\beta)^{\frac{1}{2}}) < r < \tfrac{1}{2}(\alpha + (\alpha^2 - 4\beta)^{\frac{1}{2}}), \tag{2.6}$$

we get

$$\|(I - M_\lambda)x\| > 0$$

for

$$\tag{2.7} \tfrac{1}{2}(\alpha-(\alpha^2-4\beta)^{\frac{1}{2}}) < r < \tfrac{1}{2}(\alpha+(\alpha^2-4\beta)^{\frac{1}{2}}) \leqslant \alpha \leqslant |T_0^{-1}\varrho|.$$

In the second case from (2.3) we have

$$\begin{aligned}\|(I-M_\lambda)x\| &\geqslant \|x\| - \|T_0^{-1}\varrho + \lambda(I-T_0^{-1}T)P(\cdot, x)\| \\ &= R - \|T_0^{-1}\varrho + \lambda(I-T_0^{-1}T)P(\cdot, x)\|;\end{aligned}$$

but

$$\|T_0^{-1}\varrho + \lambda(I-T_0^{-1}T)P(\cdot, x)\| \leqslant |T_0^{-1}| + \|I-T_0^{-1}T\|(\Gamma/r+\Lambda)$$

and so, for

$$\tag{2.8} R > |T_0^{-1}\varrho| + \|(I-T_0^{-1}T)\|(\Gamma/r+\Lambda)$$

we get

$$\|(I-M_\lambda)x\| > 0.$$

We can now prove Theorem 1.1.

Proof. We show that the operator M_λ has a fixed point in Δ_r^R for $\lambda = 1$. This operator is continuous and compact in Δ_r^R by Corollary 1.1; moreover, by Lemma 2.1 we can choose r, $R \in \boldsymbol{R}^+$ such that

$$x \neq M_\lambda x, \quad x \in \partial\Delta_r^R \quad \text{and} \quad \lambda \in [0, 1];$$

then the Leray–Schauder degree

$$d_{LS}[I-M_\lambda, \Delta_r^R, 0]$$

is defined. From the homotopy invariance theorem we have

$$d_{LS}[I-M_1, \Delta_r^R, 0] = d_{LS}[I-M_0, \Delta_r^R, 0] = d_{LS}[I-T_0^{-1}\varrho, \Delta_r^R, 0].$$

$T_0^{-1}\varrho \in \Delta_r^R$ because of (2.7), (2.8), and so this degree is different from zero. Thus

$$d_{LS}[I-M_1, \Delta_r^R, 0] = d_{LS}[I-M, \Delta_r^R, 0] \neq 0,$$

i.e.,

$$\exists x \in \mathring{\Delta}_r^R \quad \text{such that} \quad Mx = x.$$

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