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On a generalization of the sum form functional equation. VI

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Dedicated to Professor M. Kuczma on his 50th birthday

Abstract. Information measures which are either additive such as Shannon's entropy, the directed divergence, etc., or non-additive such as the entropy of degree β can be characterized with the help of functional equations. Here we solve two functional equations (1.3) and (1.4) connected with these measures.

1. Introduction. Shannon's entropy, the inaccuracy, the directed divergence, etc., are additive. However, there are information measures such as the entropy of degree β [3] which are not additive. The sum form representation of these measures (which is a common algebraic property of most of these measures) together with the additivity or the non-additivity property resulted in the study of many interesting functional equations and their generalizations, for example (1.1) and (1.2) ([1], [2], [4], [6], [7], [11]):

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i)+\sum_{j=1}^{m} f(q_j), \tag{1.1}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} x_i^{\alpha}\cdot\sum_{j=1}^{m} f(q_j)+\sum_{j=1}^{m} y_j^{\beta}\cdot\sum_{i=1}^{n} f(p_i), \tag{1.2}$$

for $P\in\Gamma_n$, $Q\in\Gamma_m$, where $\Gamma_n = \{P=(p_1, p_2, \ldots, p_n)\colon p_i \geqslant 0,\ \sum_{i=1}^{n} p_i = 1\}$.

Our aim is to obtain the 'measurable' solutions of the following functional equations (which are generalizations of (1.1) and (1.2)) connected with the above information measures and more:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f_{ij}(x_i y_j) = \sum_{i=1}^{n} x_i^{\alpha}\cdot\sum_{j=1}^{m} g_j(y_j)+\sum_{j=1}^{m} y_j^{\beta}\cdot\sum_{i=1}^{n} h_i(x_i), \tag{1.3}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i y_j, u_i v_j) = \sum_{i=1}^{n} x_i^{\alpha}\cdot\sum_{j=1}^{m} f(y_j, v_j)+\sum_{j=1}^{m} y_j^{\beta}\cdot\sum_{i=1}^{n} f(x_i, u_i) \tag{1.4}$$

for $X, Y \in \Gamma_n$, $U, V \in \Gamma_m$, α, β non-zero reals. Here we solve (1.3) and (1.4) for 'measurable' functions, holding for some fixed pair n, m ($n = 2$, $m \geqslant 3$) by simple and direct methods. In the process we obtain some interesting and elegant results (like Results 3 and 4) which are quite useful here, but could also be useful elsewhere under similar circumstances. In [8], equation (1.4) is solved for 'measurable' f, holding for some fixed pair m, n, where both m and n are $\geqslant 3$.

2. Notation and auxiliary results. Let $I = [0, 1]$, R, reals, $J = I \times]0, 1[\cup \cup \{(0, 0), (1, 1)\}$, $s(x) = -x \log x - (1-x)\log(1-x)$. We follow the convention: $0 \cdot \log 0 = 0 = 0^\gamma$ ($\gamma \neq 0$) and whenever $v_j = 0$ the corresponding $y_j = 0$ too. In the next two sections we make use of the following auxiliary results.

RESULT 1 ([5], [8]). *Let f_i: $I \to R$ ($i = 1, 2, \ldots, n$, $n \geqslant 3$ (fixed)) and any one of the f_i's be measurable. Then the f_i's satisfy*

$$\sum_{i=1}^{n} f_i(x_i) = c, \qquad X \in \Gamma_n \ (\text{fixed } n \geqslant 3), \tag{2.1}$$

where c is a constant if and only if

$$f_i(x) = Ax + b_i, \qquad x \in I \ (i = 1, 2, \ldots, n), \tag{2.2}$$

where A, b_i are constants with $A + \sum_{i=1}^{n} b_i = c$.

RESULTS 2 [8]. *Let F_i: $I \times I$ (or J) $\to R$ ($i = 1, 2, \ldots, n$, $n \geqslant 3$ (fixed)) and any one of the F_i's be measurable in each variable. Then the F_i's satisfy*

$$\sum_{i=1}^{n} F_i(x_i, y_i) = d, \qquad X, Y \in \Gamma_n, \tag{2.3}$$

where d is a constant if and only if

$$F_i(x, y) = Ax + By + c_i \qquad (i = 1, 2, \ldots, n), \tag{2.4}$$

where A, B, c_i are constants with $A + B + \sum_{i=1}^{n} c_i = d$.

RESULTS 3 [9]. *Let L: $I \to R$ satisfy the functional equation*

$$L(x) + L(y) = \frac{x^\alpha + y^\alpha}{(x+y)^\alpha} L(x+y) \qquad (x, y \geqslant 0, \ 0 < x+y \leqslant 1), \tag{2.5}$$

where α ($\neq 1$) $\in R$. Then

$$L(x) = Cx^\alpha, \qquad \text{where } C \text{ is a real constant}. \tag{2.6}$$

RESULT 4. *Suppose $l, m, n, k\colon I \to \mathbf{R}$ and let one of l, m, n be measurable and let α, β be non-zero reals. Then l, m, n and k satisfy the functional equation*

$$l(xy)+m((1-x)y)-(x^\alpha+(1-x)^\alpha)n(y) = y^\beta k(x), \quad x, y \in I, \tag{2.7}$$

if and only if they are of the form

$$\begin{aligned} l(x) &= Ax\log x+(B+C)x+a, \\ m(x) &= Ax\log x+Cx+b \quad (\alpha = 1,\ \beta = 1), \\ n(x) &= Ax\log x+(C-D)x+a+b, \\ k(x) &= A[x\log x+(1-x)\log(1-x)]+Bx+D, \end{aligned} \tag{2.8}$$

or

$$\begin{aligned} &l(x) = Ax^\beta+Dx+a, \quad m(x) = Bx^\beta+Dx-a+b \quad (\alpha = 1,\ \beta \neq 1), \\ &n(x) = Cx^\beta+Dx+b, \quad k(x) = Ax^\beta+B(1-x)^\beta-C, \end{aligned} \tag{2.9}$$

or

$$\begin{aligned} &l(x) = Ax^\alpha+Bx^\beta+a, \quad m(x) = Ax^\alpha+Cx^\beta-a \quad (\alpha \neq 1,\ \alpha \neq \beta), \\ &n(x) = Ax^\alpha+Dx^\beta, \quad k(x) = -D(x^\alpha+(1-x)^\alpha)+Bx^\beta+C(1-x)^\beta, \end{aligned} \tag{2.10}$$

or

$$\begin{aligned} l(x) &= Ax^\alpha\log x+(B+D)x^\alpha+a, \\ m(x) &= Ax^\alpha\log x+(C+D)x^\alpha-a \quad (\alpha \neq 1,\ \alpha = \beta), \\ n(x) &= Ax^\alpha\log x+Dx^\alpha, \\ k(x) &= A(x^\alpha\log x+(1-x)^\alpha\log(1-x))+Bx^\alpha+C(1-x)^\alpha, \end{aligned} \tag{2.11}$$

where A, B, C, D, a and b are arbitrary constants.

Proof. For the case $\alpha = 1$ refer to [5]. Now, we treat the case $\alpha \neq 1$.

First of all, $y = 0$ in (2.7) gives $n(0) = 0 = l(0)+m(0)$. Then $x = 1$ in (2.7) gives

$$n(y) = l(y)-l(0)-k(1)y^\beta. \tag{2.12}$$

From (2.12) and (2.7) with $x = 0$, we get

$$m(y) = l(y)+(k(0)-k(1))y^\beta-2l(0). \tag{2.13}$$

From (2.12) and (2.13) it is obvious that if any one of l, m, n is measurable, so are the others and from (2.7) so is k.

Setting $y = 1$ in (2.7) and using (2.13), we have

$$k(x) = l(x)+l(1-x)-2l(0)+(k(0)-k(1))(1-x)^{\beta}-n(1)(x^{\alpha}+(1-x)^{\alpha}). \tag{2.14}$$

In the above replacing x by $1-x$ and then adding the two, we obtain

$$k(x)+k(1-x) = 2(l(x)+l(1-x)-2l(0))+(k(0)-k(1))(x^{\beta}+(1-x)^{\beta})-2n(1)(x^{\alpha}+(1-x)^{\alpha}). \tag{2.15}$$

From (2.7), (2.12) and (2.13) results

$$l(xy)+l((1-x)y)-2l(0)-(x^{\alpha}+(1-x)^{\alpha})(l(y)-l(0)-k(1)y^{\beta}) = y^{\beta}[k(x)-(k(0)-k(1))(1-x)^{\beta}]. \tag{2.16}$$

Now, define

$$L(x) = 2(l(x)-l(0)) \quad \text{for } x \in I, \text{ so that } L(0) = 0. \tag{2.17}$$

Replacing x by $1-x$ in (2.16) and adding the two using (2.17), and (2.15), we get

$$L(xy)+L((1-x)y) = (x^{\alpha}+(1-x)^{\alpha})(L(y)-Cy^{\beta})+y^{\beta}(L(x)+L(1-x)), \tag{2.18}$$

where C is a constant. Putting $y = 1$ in (2.18), it is easy to see that $C = L(1)$.

Now, put $x = \frac{1}{2}$ and $y = \frac{1}{2}$ separately in (2.18), to get

$$L(\tfrac{1}{2}y) = \frac{1}{2^{\alpha}}(L(y)-L(1)y^{\beta})+L(\tfrac{1}{2})y^{\beta}, \tag{2.19}$$

and

$$L(\tfrac{1}{2}x)+L(\tfrac{1}{2}(1-x)) = \left(L(\tfrac{1}{2})-L(1)\frac{1}{2^{\beta}}\right)(x^{\alpha}+(1-x)^{\alpha})+\frac{1}{2^{\beta}}(L(x)+L(1-x)). \tag{2.20}$$

From (2.19) and (2.20), when $\alpha \neq \beta$, we obtain

$$L(x)+L(1-x) = D(x^{\alpha}+(1-x)^{\alpha})+B(x^{\beta}+(1-x)^{\beta}), \tag{2.21}$$

for some constants D and B.

Now, $x = 0$ in (2.21) gives $D+B = L(1)$, since $L(0) = 0$. Writing

$$H(x) = L(x)-Bx^{\beta} \quad \text{for } x \in I, \tag{2.22}$$

using (2.21), (2.18) can be rewritten as,

$$(2.22') \qquad H(xy)+H\big((1-x)y\big) = \big(x^\alpha+(1-x)^\alpha\big)H(y) \qquad (\alpha \neq 1).$$

By setting $xy = u$, $(1-x)y = v$ in (2.22′), it is obvious to see that (2.22′) takes the form (2.5), so by Result 3, we have $H(x) = Ax^\alpha$ (here we do not need H to be measurable), so that $L(x) = Bx^\beta + Ax^\alpha$. From (2.17), (2.13), (2.12) and (2.14), it follows that we indeed have the form given by (2.10) when $\alpha \neq 1$ and $\alpha \neq \beta$.

Remark. Replacing y by $1-y$ in (2.18) and adding the two and using (2.21), we obtain $M(xy)+M\big((1-x)y\big)+M\big(x(1-y)\big)+M\big((1-x)(1-y)\big) = 0$, where $M(x) = L(x)-Dx^\alpha-Bx^\beta$. From [10], we have $M(x) = ax-a/4$. Since $M(0) = L(0) = 0$, we again get $L(x) = Dx^\alpha+Bx^\beta$ as above.

Let us finally consider the case $\alpha = \beta$ $(\alpha \neq 1)$.

Letting $xy = u$, $(1-x)y = v$, (2.18) can be put in the form

$$(2.23) \qquad L(u)+L(v) = \frac{u^\alpha+v^\alpha}{(u+v)^\alpha}\big(L(u+v)-L(1)(u+v)^\alpha\big)+(u+v)^\alpha\left(L\left(\frac{u}{u+v}\right)+L\left(\frac{v}{u+v}\right)\right)$$

for $u, v \geqslant 0$ with $0 < u+v \leqslant 1$, $\alpha \neq 1$.

Replacing u by tu and v by tv $(t \in I)$ in (2.23) and utilizing (2.23), we have

$$(2.24) \qquad L(tz)+L(tv) = \frac{u^\alpha+v^\alpha}{(u+v)^\alpha}L\big(t(u+v)\big)+t^\alpha\big(L(u)+L(v)\big)-t^\alpha\frac{u^\alpha+v^\alpha}{(u+v)^\alpha}L(u+v).$$

For fixed $t \in I$, define

$$(2.25) \qquad N(x) = L(tx)-t^\alpha L(x) \quad \text{for } x \in I.$$

Then (2.24) takes the form (2.5), so that from Result 3, we obtain $N(x) = C(t)x^\alpha$, that is, from (2.25) we have

$$(2.26) \qquad L(tx) = t^\alpha L(x)+x^\alpha C(t), \qquad x, t \in I,$$

from which it follows that $L(x) = Ax^\alpha \log x+Bx^\alpha$, $x \in I$. Now (2.17), (2.13), (2.12) and (2.14) yield the form given by (2.11). This proves Result 4.

Remark. N in (2.25) need not be measurable. L in (2.25) indeed has the asserted form above can be seen by interchanging x and t in (2.26) to get $C(t) = L(t)+Bt^\alpha$ and then recognizirg the resultant equation as the logarithmic functional equation.

3. Solution of equation (1.3).

THEOREM 1. *Let* $f_{ij}, g_j, h_i: I \to \mathbf{R}$ $(i = 1, 2,\ j = 1, 2, \ldots, m,\ m$ *(fixed)* $\geq 3)$ *be such that for each* $j = 1, 2, \ldots, m$, *one of the functions* f_{1j}, f_{2j}, g_j *be measurable. Then these functions satisfy the functional equation*

$$\sum_{j=1}^{m} f_{1j}(xy_j) + \sum_{j=1}^{m} f_{2j}((1-x)y_j) = \left(x^\alpha + (1-x)^\alpha\right) \sum_{j=1}^{m} g_j(y_j) + \sum_{j=1}^{m} y_j^\beta \cdot \left(h_1(x) + h_2(1-x)\right) \tag{3.1}$$

for $x \in I$, $Y = (y_j) \in \Gamma_m$, α, β *non-zero reals and* m *a fixed integer* ≥ 3 *if and only if they are given by*

$$\begin{aligned} h_1(x) + h_2(1-x) &= A\left(x \log x + (1-x)\log(1-x)\right) + Bx + a, \\ f_{1j}(x) &= Ax\log x + (B + B_j)x + a_{1j}, \\ f_{2j}(x) &= Ax\log x + B_j x + a_{2j}, \\ g_j(x) &= Ax\log x + (D + B_j)x + a_j, \end{aligned} \tag{3.2}$$

with $\sum_{i=1}^{2}\sum_{k=1}^{m} a_{ik} - \sum_{k=1}^{m} a_k = D + a$, *in the case*

$$\alpha = 1, \quad \beta = 1$$

or

$$\begin{aligned} h_1(x) + h_2(1-x) &= Ax^\beta + B(1-x)^\beta - C, \\ f_{1j}(x) &= Ax^\beta + B_j x + a_{1j}, \\ f_{2j}(x) &= Ax^\beta + B_j x + a_{2j}, \\ g_j(x) &= Cx^\beta + (D + B_j)x + a_j, \end{aligned} \tag{3.3}$$

with

$$D = \sum_{k=1}^{m} a_k - \sum_{i=1}^{2}\sum_{k=1}^{m} a_{ik},$$

in the case

$$\alpha = 1, \quad \beta \neq 1;$$

$$\begin{aligned} h_1(x) + h_2(1-x) &= Ax^\beta + B(1-x)^\beta - D\left(x^\alpha + (1-x)^\alpha\right), \\ f_{1j}(x) &= Ax^\beta + A_j x^\alpha + Cx + a_{1j}, \\ f_{2j}(x) &= Bx^\beta + A_j x^\alpha + Cx + a_{2j}, \\ g_j(x) &= Dx^\beta + A_j x^\alpha + Ex + a_j, \end{aligned} \tag{3.4}$$

with

$$C+\sum_{i=1}^{2}\sum_{k=1}^{m} a_{ik} = 0 = E+\sum_{k=1}^{m} a_k,$$

in the case

$$\alpha \neq 1, \quad \alpha \neq \beta;$$

or

$$(3.5)\quad \begin{aligned} h_1(x)+h_2(1-x) &= A\,[x^\alpha \log x+(1-x)^\alpha \log(1-x)]+Bx^\alpha+D(1-x)^\alpha, \\ f_{1j}(x) &= Ax^\alpha \log x+(B+E_j)\,x^\alpha+Cx+a_{1j}, \\ f_{2j}(x) &= Ax^\alpha \log x+(D+E_j)\,x^\alpha+Cx+a_{2j}, \\ g_j(x) &= Ax^\alpha \log x+E_j\,x^\alpha+Ex+a_j, \end{aligned}$$

with

$$C+\sum_{i=1}^{2}\sum_{k=1}^{m} a_{ik} = 0 = E+\sum_{k=1}^{m} a_k,$$

in the case

$$\alpha \neq 1, \quad \alpha = \beta,$$

for $x \in I$, $j = 1, 2, \ldots, m$, *where* $A, B, C, D, E, A_j, B_j, E_j, a_{ij}, a_j$ *and* a *are constants.*

Proof. For fixed $x \in I$, writing

$$(3.6)\quad f_j(y) = f_{1j}(xy)+f_{2j}((1-x)\,y)- \\ -\big(x^\alpha+(1-x)^\alpha\big)g_j(y)-y^\beta\big(h_1(x)+h_2(1-x)\big)$$

for $y \in I$, $j = 1, 2, \ldots, m$, (3.1) can be reduced to (2.1) with $c = 0$, so that from Result 1, we have

$$(3.7)\qquad f_j(y) = Ay+b_j \quad \text{with} \quad A+\sum_{j=1}^{m} b_j = 0.$$

Then

$$b_j = f_j(0) = f_{1j}(0)+f_{2j}(0)-\big(x^\alpha+(1-x)^\alpha\big)g_j(0),$$

$$A = -\sum_{j=1}^{m} b_j = -\sum_{i=1}^{2}\sum_{j=1}^{m} f_{ij}(0)+\sum_{j=1}^{m} g_j(0)\cdot\big(x^\alpha+(1-x)^\alpha\big) = d+c\big(x^\alpha+(1-x)^\alpha\big),$$

so that (3.6) and (3.7) yield

$$(3.8)\quad \{f_{1j}(xy)-f_{1j}(0)-dxy\}+\{f_{2j}((1-x)\,y)-f_{2j}(0)-d(1-x)\,y\}- \\ -\big(x^\alpha+(1-x)^\alpha\big)\{g_j(y)+cy-g_j(0)\} = y^\beta\{h_1(x)+h_2(1-x)\},$$

which is of the form (2.7). First from Result 4, it is easy to see that if, for example, f_{1j} is measurable so are f_{2j}, g_j and $h_1(x)+h_2(1-x)$. Secondly, noting that $h_1(x)+h_2(1-x)$ is the same for all j, starting off with $h_1(x)+$ $+h_2(1-x)$ (i.e., $k(x)$ in Result 4), from Result 4, we obtain the required forms (3.2) to (3.5). The converse is easy to verify. This proves Theorem 1.

4. Solution of equation (1.4). Let $f\colon J \to \boldsymbol{R}$ be measurable in each variable and satisfy the functional equation

$$\text{(4.1)} \quad \sum_{j=1}^{n} \left[f(xy_j, uv_j)+f\left((1-x)y_j, (1-u)v_j\right)\right] = \left(x^\alpha+(1-x)^\alpha\right) \sum_{j=1}^{n} f(y_j, v_j)+\sum_{j=1}^{n} y_j^\beta \cdot \left(f(x, u)+f(1-x, 1-u)\right)$$

for $x, u \in [0, 1]$, $Y=(y_j)$, $V=(v_j)\in \Gamma_n$, α, β non-zero reals and n a fixed integer $\geqslant 3$.

For fixed $(x, u)\in J$, define

$$\text{(4.2)} \quad g(y, v) = f(xy, uv)+f\left((1-x)y, (1-u)v\right)-\left(x^\alpha+(1-x)^\alpha\right)f(y, v)- -y^\beta\left(f(x, u)+f(1-x, 1-u)\right) \quad \text{for } (y, v)\in J.$$

Then (4.1) becomes $\sum_{j=1}^{n} g(y_j, v_j)=0$ $(n \geqslant 3)$ with g measurable in each variable, so that by Result 2,

$$g(y, v) = A(x, u)y+B(x, u)v-\frac{1}{n}(A+B)(x, u)$$

and that

$$\text{(4.3)} \quad f(xy, uv)+f\left((1-x)y, (1-u)v\right)- -\left(x^\alpha+(1-x)^\alpha\right)f(y, v)-y^\beta\left(f(x, u)+f(1-x, 1-u)\right) = A(x, u)y+B(x, u)v-\frac{1}{n}(A+B)(x, u).$$

Now $y=0$ in (4.3) gives

$$\text{(4.4)} \quad f(0, uv)+f\left(0, (1-u)v\right)-\left(x^\alpha+(1-x)^\alpha\right)f(0, v) = B(x, u)v-\frac{1}{n}(A+B)(x, u).$$

Let us first treat the case $\alpha \neq 1$.

In (4.4), put $x = x_1$ and $x = x_2$, holding u, v constant, first to obtain $f(0, v) = D(u)v + C(u)$ and then noticing the independence of u of the left-hand side, we obtain

$$f(0, v) = Dv + a, \tag{4.5}$$

where a, D are constants.

Use (4.4) and (4.5) to get

$$B(x, u) = D\big(1 - x^{\alpha} - (1-x)^{\alpha}\big), \qquad A(x, u) = -(D + na)\big(1 - x^{\alpha} - (1-x)^{\alpha}\big) - na,$$

so that (4.3) now becomes

$$\begin{aligned} f(xy, uv) + f\big((1-x)y, (1-u)v\big) - \\ -\big(x^{\alpha} + (1-x)^{\alpha}\big) f(y, v) - y^{\beta}\big(f(x, u) + f(1-x, 1-u)\big) \\ = \big(x^{\alpha} + (1-x)^{\alpha}\big)(Dy + nay - Dv - a) - (D + 2na)\,y + Dv + 2a. \end{aligned} \tag{4.6}$$

Setting $x = 0$ in (4.6) and using (4.5), we have

$$f\big(y, (1-u)v\big) - f(y, v) + Duv - y^{\beta}\,[f(1, 1-u) + Du + a] = -nay,$$

that is, (changing u into $1-u$)

$$\{f(y, uv) - Duv\} - \{f(y, v) - Dv\} = y^{\beta}\,\{f(1, u) - Du + D + a\} - nay. \tag{4.7}$$

Letting $y = 1$, (4.7) becomes

$$\begin{aligned} & f(1, uv) - Duv + D - (n-1)\,a \\ & \qquad = \{f(1, v) - Dv + D - (n-1)\,a\} + \{f(1, u) - Du + D - (n-1)\,a\}, \end{aligned}$$

from which (which is logarithmic), since f is measurable in each variable, we get

$$f(1, u) = A \log u + Du + (n-1)\,a - D \qquad \text{for } u \in \,]0, 1].$$

Putting this back into (4.7), we get

$$f(y, uv) - Duv = f(y, v) - Dv + y^{\beta}(A \log u + na) - nay$$

also

$$= f(y, u) - Du + y^{\beta}(A \log v + na) - nay,$$

so that

$$f(y, v) - Dv - Ay^{\beta} \log v = a \quad \text{function of } y \text{ (say) } k(y),$$

that is,

$$f(y, v) = Ay^{\beta} \log v + Dv + k(y) \qquad \text{for } (y, v) \in J. \tag{4.8}$$

Substituting (4.8) into (4.6), we have

$$(4.9)\quad [k(xy)+Dxy+2naxy-a]+[k((1-x)y)+D(1-x)y+2na(1-x)y-a]- \\ -(x^\alpha+(1-x)^\alpha)(k(y)+Dy+nay-a)-y^\beta(k(x)+k(1-x)+D) \\ = Ay^\beta[x^\alpha+(1-x)^\alpha-x^\beta-(1-x)^\beta]\log v.$$

If $\alpha \neq \beta$, then $A = 0$ since the left-hand side is independent of v. If $\alpha = \beta$, the term on the right-hand side drops out. In either case (4.9) is of the form (2.7). Then, by Result 4, in the case $\alpha \neq 1$, $\alpha \neq \beta$, (2.10) gives (use $n(x)$)

$$k(y)+Dy+nay-a = By^\alpha+Cy^\beta,$$

so that

$$f(y, v) = By^\alpha+Cy^\beta-Dy-nay+a+Dv.$$

This f is a solution of (4.1) provided $B = -C$, $a = 0$, that is,

$$(4.10)\qquad f(y, v) = B(y^\alpha-y^\beta)-D(y-v) \quad (\alpha \neq 1,\ \alpha \neq \beta).$$

Again by Result 4, in the case $\alpha \neq 1$, $\alpha = \beta$, (2.11) gives (use $n(x)$)

$$k(y)+Dy+nay-a = By^\alpha \log y+Cy^\alpha,$$

that is

$$f(y, v) = Ay^\alpha \log v+By^\alpha \log y+Cy^\alpha-Dy-nay+a+Dv.$$

This f is a solution of (4.1) if $C = 0 = a$, that is,

$$(4.11)\qquad f(y, v) = Ay^\alpha \log v+By^\alpha \log v-D(y-v) \quad (\alpha \neq 1,\ \alpha = \beta).$$

Now, let us treat the case $\alpha = 1$.

For the case $\beta = 1 = \alpha$ we refer to [7]. So, let us assume that $\beta \neq 1$.

First, putting $x = 0$, $u = 0$ in (4.1), we get

$$nf(0, 0) = \sum_{j=1}^{n} y_j^\beta \cdot (f(0, 0)+f(1, 1))$$

that is,

$$f(0, 0) = 0 = f(1, 1).$$

Now (4.4) becomes

$$f(0, uv)+f(0, (1-u)v)-f(0, v) = B(x, u)v-\frac{1}{n}(A+B)(x, u).$$

First with $v = 0$ we get $(A+B)(x, u) = 0$. Then observing the independence of

x of the left-hand side, we can conclude, so should be the right-hand side, and get $B(x, u) = B(u)$ (say), and then $A(u) = -B(u)$, so that

$$f(0, uv) + f(0, (1-u)v) - f(0, v) = B(u)v,$$

where B is symmetric ($B(u) = B(1-u)$). Now, the application of Result 4, yields

$$f(0, u) = -Du \log u + Eu, \tag{4.12}$$

$$B(u) = -A(u) = -D\big(u \log u + (1-u)\log(1-u)\big) = Ds(u). \tag{4.13}$$

Now, (4.3) can be rewritten as, using (4.13),

$$\begin{aligned} f(xy, uv) + f\big((1-x)y, (1-u)v\big) - f(y, v) \\ = y^{\beta}\big(f(x, u) + f(1-x, 1-u)\big) + D(v-y)s(u). \end{aligned} \tag{4.14}$$

Setting $x = 1$ in (4.14) and using (4.12), we have

$$\begin{aligned} f(y, uv) - f(y, v) = D(1-u)v \log(1-u)v - E(1-u)v + \\ + y^{\beta} \cdot \{f(1, u) - D(1-u)\log(1-u) + E(1-u)\} + (v-y)Ds(u). \end{aligned} \tag{4.15}$$

With $y = 1$, (4.15) can be put in the form

$$\begin{aligned} &[f(1, uv) + Duv \log uv + E(1-uv)] \\ &\quad = [f(1, u) + Du \log u + E(1-u)] + [f(1, v) + Dv \log v + E(1-v)], \end{aligned}$$

so that, since f is measurable in each variable, we obtain

$$f(1, u) = -Du \log u + E(u-1) + C \log u, \qquad u \in]0, 1]. \tag{4.16}$$

Putting this value of $f(1, u)$ into (4.15), we have

$$\begin{aligned} &f(y, uv) + Dav \log uv - Ebv \\ &\quad = f(y, v) + Dv \log v - Ev - Dys(u) + y^{\beta}[Ds(u) + C \log u] \end{aligned}$$

also by symmetry in u and v

$$= f(y, u) + Du \log u - Eu - Dys(v) + y^{\beta}[Ds(v) + C \log v],$$

from which, by equating the right-hand sides, we obtain

$$f(y, u) + Du \log u - Eu + Dys(u) - y^{\beta}[Ds(u) + C \log u] = k(y) \quad \text{(say)},$$

that is,

$$f(y, u) = k(y) - Du \log u + Eu - Dys(u) + y^{\beta}[Ds(u) + C \log u]. \tag{4.17}$$

Putting $x = 0$ in (4.14) and substituting this value of f into the resultant and also using (4.12) and (4.16) we have $D(y - y^{\beta})\big[s(u) + s(v) - s\big((1-u)v\big)\big] = 0$.

Since $\{y, y^\beta\}$ $(\beta \neq 0, 1)$ is linearly independent and $s(\frac{1}{2})+s(\frac{1}{2})-s(\frac{3}{4}) = \frac{3}{4}\log 3 \neq 0$, it follows that $D = 0$. Now, (4.17) becomes

$$f(y, u) = k(y)+Eu+Cy^\beta \log u. \tag{4.18}$$

Substituting this value of f into (4.14), noting the linear independence of 1, $\log v$ and equating the coefficient of $\log v$, we get $Cy^\beta(x^\beta+(1-x)^\beta-1) = 0$. Since $\beta \neq 1$, this implies $C = 0$, so that (4.14) and (4.18) yield,

$$k(xy)+k((1-x)y)-k(y) = y^\beta[k(x)+k(1-x)+E],$$

which is a special case of (2.7). From Result 4, it follows that $k(x) = Ax^\beta+Dx$ and from (4.18), $f(y, u) = Ay^\beta+Dy+Eu$. This f satisfies (4.1) provided, $A+D+E = 0$, that is,

$$f(y, u) = A(y^\beta-u)+D(y-u) \quad (\alpha = 1,\ \beta \neq \alpha). \tag{4.19}$$

Thus, we have proved the following theorem.

THEOREM 2. *Let $f\colon J \to \mathbf{R}$ be measurable in each variable. Then f satisfies the functional equation* (4.1) *for some fixed $n \geq 3$, α, β non-zero reals if and only if*

$$f(x, y) = \begin{cases} Ax\log x+Bx\log y+Cy\log y+Dx+Ey+a & (\alpha = 1 = \beta), \\ Ax^\alpha \log x+Bx^\alpha \log y+C(x-y) & (\alpha = \beta,\ \alpha \neq 1), \\ A(x^\alpha-x^\beta)+B(x-y) & (\alpha \neq \beta), \end{cases}$$

where A, B, C, D, E and a are constants with $D+E = (n-2)a$.

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