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On the extension of integrable solutions of a functional equation of n-th order*

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Abstract. Some theorems concerning the extension of integrable solutions φ of functional equation (1) are given, under suitable hypotheses on the given functions h and f_i, $i = 1,\dots,n$.

In this paper we study the problem of extending integrable solutions φ of the functional equation of n-th order

$$\varphi(x) = h(x, \varphi[f_1(x)],\dots,\varphi[f_n(x)]), \tag{1}$$

where h and f_i, $i = 1,\dots,n$, are given functions.

Integrable solutions of this equation have been investigated by Matkowski [5]; see also [4].

We start with a particular case of Baron's theorem [1], in which the following assumption plays an important role.

(i) f_i: $I \to I$, $i = 1,\dots,n$, where I is an arbitrary set; moreover, for a certain subset I_0 of I and for every $i = 1,\dots,n$, we have $f_i(I_0) \subset I_0$; and for every $x \in I$ there exists a positive integer k such that

$$f_{i_1} \circ \dots \circ f_{i_k}(x) \in I_0 \quad \text{for every } i_1,\dots,i_k = 1,\dots,n.$$

THEOREM 1 (Baron [1]). *Let I, Y and $I_0 \subset I$ be arbitrary sets and h: $I \times Y^n \to Y$ and f_i: $I \to I$, $i = 1,\dots,n$, arbitrary functions. If assumption* (i) *is fulfilled, then for every solution φ_0: $I_0 \to Y$ of equation* (1) *there exists exactly one solution φ: $I \to Y$ of* (1) *such that*

$$\varphi(x) = \varphi_0(x) \quad \textit{for } x \in I_0. \tag{2}$$

Since in the sequel we shall appeal to the method of the proof of this theorem, it seems reasonable to reproduce it here.

Proof of Theorem 1. We define the following sequence of sets

$$I_{k+1} = \bigcap_{i=1}^{n} f_i^{-1}(I_k), \quad k = 0, 1, 2,\dots \tag{3}$$

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As a direct consequence of the above definition we get

(4) $$f_i(I_{k+1}) \subset I_k, \quad i = 1,\ldots,n, \quad k = 0, 1, 2,\ldots$$

It follows from (i) that $I_0 \subset f_i^{-1}(I_0)$ for every $i = 1,\ldots,n$, and $I_0 \subset I_1$ in view of (3). By induction

(5) $$I_k \subset I_{k+1}; \quad k = 0, 1,\ldots$$

Moreover,

(6) $$I = \bigcup_{k=0}^{\infty} I_k$$

in virtue of hypothesis (i).

Now we define a sequence of functions $\{\varphi_k\}_{k=0}^{\infty}$, $\varphi_k\colon I_k \to Y$, $k = 0, 1,\ldots,$ by the formula

(7) $$\varphi_{k+1}(x) = h(x, \varphi_k[f_1(x)],\ldots,\varphi_k[f_n(x)]), \quad x \in I_{k+1}, \quad k = 0, 1,\ldots,$$

where $\varphi_0\colon I_0 \to Y$ is given solution of equation (1). It follows from (4) that this definition is correct. Recalling (5) and (6), we may define a function $\varphi\colon I \to Y$ by

(8) $$\varphi(x) := \varphi_k(x), \quad x \in I_k, \ k = 0, 1,\ldots$$

It is clear, in view of (7) and (8), that φ fulfils equation (1) in I and that it yields the unique extension of φ_0 to a solution of (1) in the whole of I.

In the sequel we denote by $L^p[I]$ the set of all functions which are Lebesgue integrable with p-th power on a measurable set $I \subset R$, $0 < p < \infty$. Moreover, by $\mathscr{L}[I]$ we denote the σ-algebra of all Lebesgue measurable subsets of the set $I \subset R$.

Regarding the functions h and f_i, $i = 1,\ldots,n$, we assume that

(ii) The function h maps $I \times R^n$ into R, where $I \in \mathscr{L}[R]$; for every fixed $(y_1,\ldots,y_n) \in R^n$, the function $h(\cdot, y_1,\ldots,y_n)\colon I \to R$ is measurable, and for almost every $x \in I$ and for every $i = 1,\ldots,n$, the function $h(x, y_1,\ldots,y_{i-1}, \cdot, y_{i+1},\ldots,y_n)\colon R \to R$ is a continuous function, the variables $y_j \in R$, $j = 1,\ldots,n$, $j \neq i$, being fixed arbitrarily.

(iii) For every $i = 1,\ldots,n$ and $A \in \mathscr{L}[I]$, $f_i^{-1}(A) \in \mathscr{L}[I]$.

Let us also quote the following

LEMMA 1 (Carathéodory [2]; see also [6]). *Suppose that the function h fulfils hypothesis* (ii). *If $g_i\colon I \to R$, $i = 1,\ldots,n$, are measurable functions, then the function $H\colon I \to I$ defined by*

$$H(x) = h(x, g_1(x),\ldots,g_n(x)), \quad x \in I,$$

is measurable.

Now we shall prove

THEOREM 2. *Suppose that the functions* $f_i\colon I \to I$, $i = 1,\ldots,n$, *and* $h\colon I \times R^n \to R$, *where* I *is a set, are given. If* (i) *holds with an* $I_0 \subset I$, *and if* $\varphi_0\colon I_0 \to R$ *is a solution of equation* (1), *then*

(a) *there exists exactly one solution* $\varphi\colon I \to R$ *of* (1) *such that condition* (2) *is satisfied;*

(b) *moreover, if hypotheses* (ii) *and* (iii) *are fulfilled and* φ_0 *is a measurable function, then* φ *is also measurable;*

(c) *finally, if there exists a function* $g \in L^p[I]$ *such that*

$$|h(x, y_1,\ldots,y_n)| \leqslant g(x), \quad (x, y_1,\ldots,y_n) \in I \times R^n,$$

then every measurable solution $\varphi\colon I \to R$ *of equation* (1) *belongs to the class* $L^p[I]$.

Proof. Assertion (a) is an immediate consequence of Theorem 1. For the proof of (b), let us observe that the functions $\varphi_k\colon I_k \to R$ defined by formula (7) are measurable, on account of Lemma 1, and we have

$$\varphi^{-1}([b, \infty)) = \bigcup_{k=0}^{\infty} \varphi_k^{-1}([b, \infty)) \quad \text{for every } b \in R.$$

Statement (c) follows from the inequality

$$\int_I |\varphi(x)|^p\,dx = \int_I \big|h\big(x, \varphi[f_1(x)],\ldots,\varphi[f_n(x)]\big)\big|^p\,dx \leqslant \int_I [g(x)]^p\,dx.$$

Let $I \subset R$ be an interval and let $\xi \in \bar{I}$, where $\bar{I}$ is the closure of I in $\bar{R} = [-\infty, +\infty]$. We shall denote by $S_\xi^n[I]$ (cf. [3], p. 20) the class of all functions $f\colon I \to R$ which have a continuous n-th derivative and fulfil the condition

$$0 < \frac{f(x)-\xi}{x-\xi} < 1 \quad \text{for } x \in I \setminus \{\xi\}. \tag{9}$$

For $\xi = +\infty$ or $\xi = -\infty$ condition (9) has to be replaced by $f(x) > x$ for $x \in I$, resp. $f(x) < x$ for $x \in I$.

Further, we denote by $R_\xi^n[I]$ the class of all functions $f \in S_\xi^n[I]$ which are strictly increasing.

Now we want to give a condition which guarantees that (i) holds.

LEMMA 2. *If* $f_i \in S_0^0[I]$, $i = 1,\ldots,n$, *where* $I = [0, a)$ *or* $I = [0, a]$ *with* a *finite and* $I_0 = [0, a_0)$, $0 < a_0 < a \leqslant \infty$, *then* (i) *holds.*

Proof (cf. Remark 2 in [1]). It is enough to prove only the second part of condition (i). To this end let us define a function $F\colon I \to R$ by the formula

$$F(x) = \max_{0 \leqslant t \leqslant x} \big[\max_{1 \leqslant i \leqslant n} \{f_i(t)\}\big], \quad x \in I. \tag{10}$$

It is easy to see that $F \in S_0^0[I]$ and so, by Theorem 0.4 from [3], for every $x \in I$ there exists a positive integer k such that

$$F^k(x) \in I_0, \tag{11}$$

where F^k denotes the k-th iterate of the function F.

Moreover, by induction, we have for every positive integer k and for every $i_1, \ldots, i_k \in \{1, \ldots, n\}$

$$0 \leqslant f_{i_1} \circ \ldots \circ f_{i_k}(x) \leqslant F^k(x), \qquad x \in I,$$

which in view of (11) ends the proof.

In the sequel we shall assume that:

(iv) $f_i \in R_0^1[I]$, where $I \subset R$ is an open interval, and the derivatives f_i' are positive almost everywhere in I, $i = 1, \ldots, n$;

(v) There exists a measurable function $g\colon I \to R$ such that

$$|h(x, y_1, \ldots, y_n) - h(x, 0, \ldots, 0)| \leqslant g(x) \cdot \sum_{i=1}^{n} |y_i|$$

for $(x, y_1, \ldots, y_n) \in I \times R^n$;

(vi) $h(\cdot, 0, \ldots, 0) \in L^p[I]$ for a certain $p \in (0, +\infty)$.

Write

$$M = \max_{1 \leqslant i \leqslant n} \operatorname{ess\,sup}_{I} \frac{[g(x)]^p}{f_i'(x)}. \tag{12}$$

THEOREM 3. *Let $I = [0, a]$ and $I_0 = [0, a_0)$, where $0 < a_0 < a < \infty$, and suppose that functions $f_i \in S_0^0[I]$, $i = 1, \ldots, n$, and $h\colon I \times R^n \to R$ are given and fulfil hypotheses* (ii)–(vi). *If the constant M defined by* (12) *is finite and the function $\varphi_0\colon I_0 \to R$ is a solution of equation* (1) *such that $\varphi_0 \in L^p[I]$, then there exists exactly one solution $\varphi\colon I \to R$ of* (1) *such that* (2) *holds; moreover, $\varphi \in L^p[I]$.*

Proof. The existence of a unique solution φ of equation (1) and its measurability follow Theorem 2, assertions (a) and (b), in view of Lemma 2.

Therefore it remains to prove that the integral of $|\varphi|^p$ on I is finite. To this end let us observe that the sets I_k defined by formula (3) are intervals having zero as the left endpoint; and since $a \in I$, it follows by (6) that there exists a positive integer k_0 such that $I = I_{k_0}$.

Hence it is enough to prove that

$$\int_{I_k} |\varphi(x)|^p \, dx < \infty; \qquad k = 0, 1, \ldots \tag{13}$$

In the proof of this fact we shall use the obvious inequality

$$\Big(\sum_{i=1}^{n} b_i\Big)^p \leqslant \big(n \cdot \max_{1 \leqslant i \leqslant n} b_i\big)^p \leqslant n^p \cdot \sum_{i=1}^{n} b_i^p, \qquad n \in N,\ 0 < p < \infty, \tag{14}$$

where b_i, $i = 1, \ldots, n$, are arbitrary non-negative real numbers.

For $k = 0$ inequality (13) is true by (2), φ_0 being integrable. Suppose that (13) holds for a certain $k \geqslant 0$. It follows from (1) and (14) that

$$\int_{I_{k+1}} |\varphi(x)|^p dx = \int_{I_{k+1}} |h(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)])|^p dx$$

$$= \int_{I_{k+1}} |h(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)]) - h(x, 0, \ldots, 0) + $$

$$+ h(x, 0, \ldots, 0)|^p dx$$

$$\leqslant 2^p \int_{I_{k+1}} |h(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)]) - h(x, 0, \ldots, 0)|^p dx +$$

$$+ 2^p \int_{I_{k+1}} |h(x, 0, \ldots, 0)|^p dx.$$

Thus, according to (vi), it suffices to prove that

$$\int_{I_{k+1}} |h(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)]) - h(x, 0, \ldots, 0)|^p dx < \infty. \tag{15}$$

Now, using accordingly hypothesis (v), inequality (14) with appriopriately chosen b_i, a change of variables in the integral, hypothesis (iv), relations (12) and (4) we obtain

$$\int_{I_{k+1}} |h(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)]) - h(x, 0, \ldots, 0)|^p dx$$

$$\leqslant \int_{I_{k+1}} |g(x)|^p \cdot [\sum_{i=1}^{n} |\varphi[f_i(x)]|]^p dx \leqslant n^p \cdot \int_{I_{k+1}} |g(x)|^p \sum_{i=1}^{n} |\varphi[f_i(x)]|^p dx$$

$$= n^p \cdot \sum_{i=1}^{n} \int_{I_{k+1}} |g(x)|^p \cdot |\varphi[f_i(x)]|^p dx$$

$$= n^p \cdot \sum_{i=1}^{n} \int_{I_{k+1}} \frac{[g(x)]^p}{f_i'(x)} \cdot |\varphi[f_i(x)]|^p \cdot f_i'(x)\, dx$$

$$= n^p \cdot M \sum_{i=1}^{n} \int_{f_i(I_{k+1})} |\varphi(x)|^p dx \leqslant n^p \cdot M \cdot \sum_{i=1}^{n} \int_{I_k} |\varphi(x)|^p dx,$$

which after taking into account the induction hypothesis, gives (15).

We have also the following

THEOREM 4. *Let $I = [0, a)$ and $I_0 = [0, a_0)$, where $0 < a_0 < a \leqslant \infty$, and suppose that functions $f_i \in S_0^0[I]$, $i = 1, \ldots, n$, and $h\colon I \times R^n \to R$ are given and fulfil hypotheses* (ii)–(vi), *the constant M in* (12) *being finite. If*

$$b := \max_{1 \leqslant i \leqslant n} \sup_{x \in I} f_i(x) < a, \tag{16}$$

and $\varphi_0 \in L^p[I_0]$ *is a solution of equation* (1), *then there exists exactly one solution* $\varphi\colon I \to R$ *of* (1) *which fulfils* (2); *moreover,* $\varphi \in L^p[I]$.

Proof. Let us observe that by (16) we have $I = I_{k_0}$ for some positive integer k_0, and we can just repeat the argument of the proof of Theorem 3.

In the sequel we shall use the following

LEMMA 3. *If* $\{b_k\}_{k=1}^{\infty}$ *is a sequence of non-negative real numbers which fulfils*

$$b_{k+1} \leqslant s \cdot b_k + c, \qquad k = 1, 2, \ldots, \tag{17}$$

where $s \in (0, 1)$, $c \in R$, *then the sequence* $\{b_k\}_{k=1}^{\infty}$ *is bounded.*

Proof. By induction the following inequality is true

$$b_{k+1} \leqslant s^k \cdot b_1 + c \cdot \sum_{j=0}^{k-1} s^j, \qquad k = 1, 2, \ldots,$$

and hence the assertion.

Now we define a function $\alpha\colon (0, \infty) \to (0, 1]$ by the formula

$$\alpha(p) = \begin{cases} 1 & \text{for } p \in (0, 1], \\ 1/p & \text{for } p \in (1, \infty). \end{cases} \tag{18}$$

A certain property of this function is expressed by the following

LEMMA 4. *Let* $X \in \mathcal{L}[R]$. *If* f *and* g *are measurable real functions, defined on the set* X, *then*

$$\Big[\int_X |f(x)+g(x)|^p dx\Big]^{\alpha(p)} \leqslant \Big[\int_X |f(x)|^p dx\Big]^{\alpha(p)} + \Big[\int_X |g(x)|^p dx\Big]^{\alpha(p)}, \quad p \in (0, \infty). \tag{19}$$

Proof. For $p \in (0, 1]$ inequality (19) follows immediately from the inequality

$$|f(x)+g(x)|^p \leqslant |f(x)|^p + |g(x)|^p, \qquad x \in X,$$

in view of (18); and for $p \in (1, \infty)$, (19) is simply Minkowski's inequality.

Our last theorem reads as follows.

THEOREM 5. *Let* $I = [0, a)$ *and* $I_0 = [0, a_0)$, *where* $0 < a_0 < a \leqslant \infty$, *and suppose that functions* $f_i \in S_0^0[I]$, $i = 1, \ldots, n$, *and* $h\colon I \times R^n \to R$ *are given and fulfil hypotheses* (ii)–(vi), *the constant* M *given by* (12) *being finite. If there exists a number* $b \in (0, a)$ *such that*

$$s := \sum_{i=1}^{n} \operatorname*{ess\,sup}_{x \in [b, a)} \left[\frac{[g(x)]^p}{f_i'(x)}\right]^{\alpha(p)} < 1, \tag{20}$$

where α *is the function defined by* (18), *and if* $\varphi_0 \in L^p[I_0]$ *is a solution of equation* (1), *then there exists exactly one solution* $\varphi\colon I \to R$ *of* (1) *which fulfils* (2); *moreover,* $\varphi \in L^p[I]$.

Proof. The existence and the measurability of φ results from Theorem 2, assertion (b), in view of hypotheses (ii) and (iii). Thus it remains to prove that $\int_I |\varphi(x)|^p dx < \infty$. Since the sets I_k, $k = 0, 1, \ldots$, are intervals having zero as the left endpoints, we infer from (6) that there exists a positive integer k_0 such that $[0, b] \subset I_{k_0}$. Moreover, recalling the proof of Theorem 3, we have

$$\varphi|_{I_k} \in L^p[I_k], \quad k = 0, 1, \ldots \tag{21}$$

We introduce the following notation:

$$K = \int_{I_{k_0}} |\varphi(x)|^p dx < \infty, \tag{22}$$

$$U_k = I_{k_0+k} \setminus I_{k_0}, \quad k = 1, 2, \ldots, \tag{23}$$

and

$$b_k = \Big[\int_{U_k} |\varphi(x)|^p dx\Big]^{\alpha(p)}, \quad k = 1, 2, \ldots, \tag{24}$$

where α is the function defined by (18).

It follows from (24), (23), (21) and (6) that for the proof of the integrability of $|\varphi|^p$ it suffices to show that the sequence $\{b_k\}_{k=1}^{\infty}$ is bounded. This can be achieved with use of Lemma 3. Namely, we shall show that (17) holds with s defined by (20) and a number c which will be defined later on.

Using (24), (1) and Lemma 4 we have for every $k = 1, 2, \ldots$

$$\begin{aligned} b_k &= \Big[\int_{U_k} |\varphi(x)|^p dx\Big]^{\alpha(p)} \\ &= \Big\{\int_{U_k} |h(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)]) - h(x, 0, \ldots, 0) + h(x, 0, \ldots, 0)|^p dx\Big\}^{\alpha(p)} \\ &\leqslant \Big\{\int_{U_k} |h(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)]) - h(x, 0, \ldots, 0)|^p dx\Big\}^{\alpha(p)} + \\ &\qquad\qquad + \Big[\int_{U_k} |h(x, 0, \ldots, 0)|^p dx\Big]^{\alpha(p)}. \end{aligned}$$

Write

$$K_1 = \Big[\int_I |h(x, 0, \ldots, 0)|^p dx\Big]^{\alpha(p)}. \tag{25}$$

By hypothesis (vi) the number K_1 is finite. Using hypothesis (v), notation (25), Lemma 4, hypothesis (iv), relation (20), a change of variables in the integral, definition (25) and inclusion (4) accordingly, we obtain

$$\begin{aligned} b_k - K_1 &\leqslant \Big\{\int_{U_k} \Big[g(x) \cdot \sum_{i=1}^{n} |\varphi[f_i(x)]|\Big]^p dx\Big\}^{\alpha(p)} \\ &\leqslant \sum_{i=1}^{n} \left\{\int_{U_k} \frac{[g(x)]^p}{f_i'(x)} \cdot |\varphi[f_i(x)]|^p \cdot f_i'(x)\, dx\right\}^{\alpha(p)} \end{aligned}$$

$$\leqslant \sum_{i=1}^{n} \operatorname*{ess\,sup}_{x\in[b,a)} \left\{\frac{[g(x)]^p}{f_i'(x)}\right\}^{\alpha(p)} \cdot \Big\{\int_{U_k} |\varphi[f_i(x)]|^p \cdot f_i'(x)\,dx\Big\}^{\alpha(p)}$$

$$\leqslant s \cdot \max_{1\leqslant i\leqslant n} \Big\{\int_{f_i(U_k)} |\varphi(x)|^p\,dx\Big\}^{\alpha(p)} \leqslant s \cdot \Big[\int_{U_{k-1}\cup I_{k_0}} |\varphi(x)|^p\,dx\Big]^{\alpha(p)}.$$

Taking into account (22), (24) and the fact that $\alpha(p) \leqslant 1$, we hence get

$$b_k - K_1 \leqslant s\cdot(K + b_{k-1}^{1/\alpha(p)})^{\alpha(p)} \leqslant s\cdot(K^{\alpha(p)} + b_{k-1}) = s\cdot b_{k-1} + s\cdot K^{\alpha(p)}.$$

This shows that the sequence $\{b_k\}_{k=1}^{\infty}$ fulfils (17) with

$$c = s\cdot K^{\alpha(p)} + K_1,$$

which ends the proof.

Remark. The choice of zero as the common fixed point of the functions f_i, $i = 1,\ldots,n$, as well as the fact that it is the left endpoint of the intervals considered, is not essential.

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