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On the tangency of multifunctions

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Abstract. We present a definition of tangency of multifunctions and relate it with various notions of differentiability of multifunctions. We define a characteristic number on each multifunction with the property that it remains the same for two tangent multifunctions. The sign of this number is proved to give a sufficient condition for the asymptotic stability of multivalued differential equations and ordinary differential equations which cannot be locally linearized. At last we obtain some perturbation results.

0. The purpose of this work is to define the notion of α-tangent multifunctions at the origin, where the multifunctions vanish. We relate α-tangency with various definitions of differentiability of multifunctions ([2], [4], [5]). Furthermore, to each multifunction F we associate a characteristic number $\chi_\alpha(F)$, which does not change on α-tangent multifunctions. This characteristic number is a multivalued and Hilbert space version of the characteristic exponents of non-linear single-valued functions in a Banach space, which were introduced in [3]. Also we show that the negativeness of $\chi_\alpha(F)$ implies the asymptotic stability of the multivalued differential equation $x' \in F(x)$. Such a result is applied to ordinary differential equations which cannot be locally linearized. Finally we obtain some perturbation results.

1. Let H a real Hilbert space with norm $|\cdot|$ and inner product $(\cdot, \cdot)$. We denote by $c(H)$ the collection of all compact convex non-empty subsets of H. One can endow $c(H)$ with the following metric, usually called *Hausdorff distance*,

$$\delta(A, B) = \inf\{\lambda > 0\colon\ A \subset B + \lambda S,\ B \subset A + \lambda S\},$$

where S is the unit ball around 0 in H.

Given two upper semi-continuous multifunctions $F, G\colon H \to c(H)$, such that $F(0) = G(0) = 0$, and a number $\alpha > 0$, we say that F and G are *α-tangent* (at the origin) if

$$\lim_{x \to 0} \frac{\delta(F(x), G(x))}{|x|^\alpha} = 0.$$

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Clearly α-tangency is a relation of equivalence, because of the properties of the Hausdorff distance (e.g. [2]).

Now, given $x \in H$, $A \in c(H)$, the following function is well defined ([1])

$$\sigma(x, A) = \sup_{y \in A}(x, y).$$

$\sigma(x, A)$ is called the *support function* of the convex set A.

If $F\colon H \to c(H)$ is upper semi-continuous, $F(0) = 0$, and $\alpha > 0$, we define the following (extended real) number

$$\chi_\alpha(F) = \limsup_{x \to 0} \frac{\sigma(x, F(x))}{|x|^{\alpha+1}}.$$

Obviously $\chi_\alpha(F)$ is finite, if F is quasi-bounded of order α at the origin, i.e.

$$\limsup_{x \to 0} \frac{1}{|x|^\alpha} \sup_{y \in F(x)} |y| < \infty.$$

It is a fundamental property of the number χ_α which depends on the equivalence class of α-tangent multifunctions only. This is seen by the following proposition:

PROPOSITION 1. *Let F, $G\colon H \to c(H)$ upper semi-continuous, $F(0) = G(0) = 0$, and $\alpha > 0$. If F and G are α-tangent, then $\chi_\alpha(F) = \chi_\alpha(G)$.*

Proof. For any $\varepsilon > 0$ there exists a neighbourhood of the origin where

$$F(x) \subset G(x) + \varepsilon |x|^\alpha S, \qquad G(x) \subset F(x) + \varepsilon |x|^\alpha S.$$

Therefore we get

$$\sigma(x, F(x)) \leqslant \sigma(x, G(x)) + \varepsilon |x|^{\alpha+1}, \qquad \sigma(x, G(x)) \leqslant \sigma(x, F(x)) + \varepsilon |x|^{\alpha+1}.$$

Thus it is implied

$$|\chi_\alpha(F) - \chi_\alpha(G)| \leqslant \varepsilon$$

and the proof is completed, as ε is arbitrary.

PROPOSITION 2. *Let F, $G\colon H \to c(H)$ upper semi-continuous, $F(0) = G(0) = 0$, $\alpha \geqslant 1$, $J(x) = x|x|^{\alpha-1}$, $\lambda \geqslant 0$ and $k \in \mathbf{R}$. Then we have*

$$\chi_\alpha(\lambda F) = \lambda \chi_\alpha(F), \qquad \chi_\alpha(F+G) \leqslant \chi_\alpha(F) + \chi_\alpha(G), \qquad \chi_\alpha(F + kJ) = \chi_\alpha(F) + k.$$

Proof. It is a direct consequence of the following relations: $\sigma(x, \lambda F(x)) = \lambda \sigma(x, F(x))$, $\sigma(x, F(x) + G(x)) = \sigma(x, F(x)) + \sigma(x, G(x))$, $\sigma(x, F(x) + kJ(x)) = \sigma(x, F(x)) + k|x|^{\alpha+1}$.

We say that the multifunction $G\colon H \to c(H)$ is *α-order homogeneous* ($\alpha > 0$), if $G(\lambda x) = \lambda^\alpha G(x)$, $\lambda \geqslant 0$, $x \in H$.

PROPOSITION 3. *If G is α-order homogeneous, then*

$$\chi_\alpha(G) = \sup_{|x|=1} \sigma(x, G(x)).$$

Proof. It follows from the definition of lim sup and the homogeneity of $G(x)$ and $\sigma(x, G(x))$.

2. In order to fix the ideas we assume in this section that H is finite dimensional, e.g. $H = \boldsymbol{R}^n$. In the sequel we assume $\alpha \geqslant 1$.

A multifunction $F: \boldsymbol{R}^n \to c(\boldsymbol{R}^n)$ is called *α-order Lipschitzian* at the origin if there exist constants $L \geqslant 0$ and $\delta > 0$ such that for all $y \in F(x)$, $|x| \leqslant \delta$, we have

$$|y| \leqslant L|x|^\alpha.$$

An upper semi-continuous α-order homogeneous multifunction $\Phi: \boldsymbol{R}^n \to c(\boldsymbol{R}^n)$ is called *α-order upper differential* of the α-order Lipschitzian multifunction F if there exists $\delta > 0$ such that for all $|x| < \delta$

$$F(x) \subset \Phi(x).$$

Note that, if F is α-order Lipschitzian, an α-order upper differential always exists; it is $\Phi(x) = \{y \in \boldsymbol{R}^n: |y| \leqslant L|x|^\alpha\}$.

We define the *α-order differential* at the origin of the α-order Lipschitzian multifunction F by

$$D_0^\alpha F(x) = \bigcap \{\Phi(x): \Phi \text{ is an } \alpha\text{-order differential of } F\}.$$

The above definition of α-order differentiability of multifunctions generalizes the definition of Lasota and Strauss [4] concerning the existence of multivalued (first order, i.e., $\alpha = 1$) differentials of non-differentiable Lipschitzian at 0 single-valued functions (cf. the first order generalization of De Blasi in [2]).

The following result shows that $D_0^\alpha F$ is well-behaved.

PROPOSITION 4. *If $D_0^\alpha F$ is the α-order differential of F at 0, then* (i) *the range of $D_0^\alpha F$ is $c(\boldsymbol{R}^n)$*, (ii) *the multifunction $D_0^\alpha F: \boldsymbol{R}^n \to c(\boldsymbol{R}^n)$ is upper semi-continuous α-order homogeneous and* (iii) *there exists a sequence $\{\Phi_n\}$ of α-order upper differentials such that $\Phi_{n+1}(x) \subset \Phi_n(x)$, $x \in \boldsymbol{R}^n$, $n = 1, 2, \ldots$, and $D_0^\alpha F(x) = \bigcap_{n=1}^{\infty} \Phi_n(x)$.*

Proof. (i) The only fact which requires a proof is that $D_0^\alpha F(x) \neq \emptyset$ for $x \neq 0$ (the case $x = 0$ is trivial). For sufficiently large integer n we have for all $y_n \in F(x/n)$ that

$$|y_n|/|x/n|^\alpha \leqslant L.$$

Thus there exists a limit point z of $y_n/|x/n|^\alpha$ as $n \to \infty$. Let $n_k \to \infty$ such that $y_{n_k}/|x/n_k|^\alpha \to z$. Let Φ any α-order upper differential of F. Then for n_k sufficiently large

$$\frac{y_{n_k}}{|x/n_k|^\alpha} \in \frac{F(x/n_k)}{|x/n_k|^\alpha} \subset \frac{\Phi(x/n_k)}{|x/n_k|^\alpha} = \Phi\left(\frac{x}{|x|}\right).$$

Letting $n_k \to \infty$, we obtain $z \in \Phi(x/|x|)$. Thus $|x|^\alpha z \in \Phi(x)$ for every α-order upper differential Φ, i.e. $|x|^\alpha z \in D_0^\alpha F(x)$.

The proof of (ii) and (iii) is similar to the one in [4] and so it is omitted.

The next result relates α-order differentiability with α-tangency.

PROPOSITION 5. *Let $F\colon \mathbf{R}^n \to c(\mathbf{R}^n)$ α-order Lipschitzian at the origin. If there exists an α-order homogeneous upper semi-continuous multifunction $G\colon \mathbf{R}^n \to c(\mathbf{R}^n)$ such that F and G are α-tangent at the origin, then $G = D_0^\alpha F$, i.e. F and $D_0^\alpha F$ are α-tangent at the origin.*

Proof. First we remark that G, as defined above, is the multivalued differential of F at 0 in the sense of De Blasi [2] (if $\alpha = 1$) and of [5] (if $\alpha > 1$). Thus the conclusion of the proposition follows from a direct extension of the proof of Theorem 4.8 of [2] for any α.

Let us remark that the existence of G in Proposition 5 is an indispensable assumption. Indeed, the (single-valued) function $f\colon \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^\alpha \sin(1/x)$, $x \neq 0$, $f(0) = 0$, is α-order Lipschitzian at 0, $D_0^\alpha f(x) = x^\alpha S$, but f and $D_0^\alpha f$ are not α-tangent at 0.

3. Now we are going to give some applications of the above ideas to the stability of multivalued differential equations.

PROPOSITION 6. *Let $F\colon \mathbf{R}^n \to c(\mathbf{R}^n)$ upper semi-continuous, $F(0) = 0$, and $\alpha \geqslant 1$. If $\chi_\alpha(F) < 0$, then the zero solution of the multivalued differential equation*

$$x' \in F(x) \tag{1}$$

is asymptotically stable.

Proof. Let $\varepsilon > 0$ be such that $k = \chi_\alpha(F) + \varepsilon < 0$ and $\delta > 0$ be such that $\sigma(x, F(x)) \leqslant k|x|^{\alpha+1}$ whenever $|x| < \delta$. We consider any solution $x(t)$ of (1) (in its right maximal interval of existence $[0, T)$) such that $|x(0)| < \delta$. If there exists $0 < t_1 \leqslant T$ such that $|x(t)| < \delta$, $t \in [0, t_1)$, $|x(t_1)| = \delta$, then we would have in $[0, t_1)$

$$\frac{d}{dt}|x(t)|^2 = 2(x(t), x'(t)) \leqslant 2\sigma(x(t), F(x(t))) \leqslant 2k|x(t)|^{\alpha+1},$$

i.e., by integrating the above differential inequality in $[0, t_1)$ we get

$$\delta = |x(t_1)| \leqslant |x(0)| e^{kt_1} < \delta, \qquad \text{if } \alpha = 1,$$

$$\delta = |x(t_1)| \leqslant \frac{|x(0)|}{[1 - k(\alpha-1)|x(0)|^{\alpha-1} t_1]^{1/(\alpha-1)}} < \delta, \qquad \text{if } \alpha > 1,$$

a contradiction. Therefore for all $t \in [0, T)$ we have $|x(t)| < \delta$. A standard argument shows that $T = \infty$. So, since for all $t \geqslant 0$

$$|x(t)| \leqslant |x(0)| e^{kt}, \qquad \text{if } \alpha = 1,$$

$$|x(t)| \leqslant \frac{|x(0)|}{[1 - k(\alpha-1)|x(0)|^{\alpha-1} t]^{1/(\alpha-1)}}, \qquad \text{if } \alpha > 1,$$

it follows that the zero solution of (1) is asymptotically stable.

Combining the above proposition with Propositions 1 and 3 we get the following result.

Corollary 7. *Let F: $\mathbf{R}^n \to c(\mathbf{R}^n)$ upper semi-continuous, $F(0) = 0$, $a \geqslant 1$. Suppose that there exists an α-order homogeneous upper semi-continuous multifunction G: $\mathbf{R}^n \to c(\mathbf{R}^n)$ such that F and G are α-tangent at the origin. If for any x, $|x| = 1$, we have $\sigma(x, G(x)) < 0$, then the zero solution of (1) is asymptotically stable.*

Now we give an example of a scalar single-valued differential equation with right-hand side not linearized at 0. For any $\alpha \geqslant 1$ we consider the function f_α: $\mathbf{R} \to \mathbf{R}$ defined by $f_\alpha(x) = x^\alpha(\theta \sin(1/x) - 1)$, $x \neq 0$, $f_\alpha(0) = 0$, for some fixed $\theta \in (0, 1)$. Clearly f_1 is not differentiable at 0 and for any $\alpha > 1$ the differential of f_α at 0 vanishes. An easy computation shows that if α is an odd integer, then $\chi_\alpha(f_\alpha) < 0$, which implies that the zero solution of the differential equation

$$x' = x^\alpha\left(\theta \sin\frac{1}{x} - 1\right)$$

is asymptotically stable.

Finally we obtain some perturbation results applying the previous propositions.

Proposition 8. *Let F: $\mathbf{R}^n \to c(\mathbf{R}^n)$ upper semi-continuous, $F(0) = 0$, $\alpha \geqslant 1$. If $\chi_\alpha(F) < 0$ and $\varepsilon > 0$ sufficiently small, then the zero solution of the perturbed multivalued differential equation*

$$x' \in F(x) + \varepsilon|x|^\alpha S \tag{2}$$

is asymptotically stable.

Proof. Let $G(x) = F(x) + \varepsilon|x|^\alpha S$. Then we have

$$\chi_\alpha(G) = \limsup_{x \to 0} \frac{\sigma(x, F(x)) + \delta(x, \varepsilon|x|^\alpha S)}{|x|^{\alpha+1}} \leqslant \chi_\alpha(F) + \varepsilon.$$

Since $\chi_\alpha(F) < 0$, we can take $\varepsilon > 0$ such that $\chi_\alpha(G) = \chi_\alpha(F) + \varepsilon < 0$. Then the conclusion follows from Proposition 6.

Combining Propositions 3 and 8 we get the following result.

Corollary 9. *Let F: $\mathbf{R}^n \to c(\mathbf{R}^n)$ α-order homogeneous upper semi-continuous, $\alpha \geqslant 1$. If for any x, $|x| = 1$, we have $\sigma(x, F(x)) < 0$ and $\varepsilon > 0$ sufficiently small, then the zero solution of (2) is asymptotically stable.*

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