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On a recurrence relation

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1. The object of the present paper is the recurrence relation

$$x_{n+m}+\ldots+x_n = b_n\,, \quad m \geqslant 1\,, \tag{1}$$

in which x_n denotes the required real sequence and b_n is a known real sequence. We denote by Δa_n the difference

$$\Delta a_n \overset{\text{df}}{=} a_{n+1}-a_n$$

and further

$$\Delta^0 a_n \overset{\text{df}}{=} a_n\,, \quad \Delta^{v+1} a_n \overset{\text{df}}{=} \Delta(\Delta^v a_n)\,, \quad v = 0, 1, 2, \ldots$$

We shall seek the sequences for which the differences of order p $\Delta^p x_n$ have a constant sign. In the case of $m = 1$ this problem has been investigated by M. Kuczma in [6].

We shall also give some applications of the results obtained to the theory of functional equations. We shall establish some conditions (compare [4]) of the uniqueness and existence of solutions satisfying equation (23). Equation (23) is a special case of the equation discussed in [2]. Formulas (33), (34), and (38) are a direct generalization of the formulas in [3] and [5] (compare also [1]).

2. At first we shall prove the following

LEMMA 1. *There may exist at most one sequence x_n satisfying relation* (1) *and such that for a certain $p \geqslant 0$*

$$\lim_{n\to\infty} \Delta^p x_n = 0\,. \tag{2}$$

If it does exist, then the differences $\Delta^p x_n$ are given by the formula

$$\Delta^p x_n = -\sum_{v=0}^{\infty} \Delta^{p+1} b_{n+(m+1)v}\,, \quad n = 0, 1, 2, \ldots \tag{3}$$

Proof. Applying the operation Δ^p to both sides of relation (1) we obtain

$$\Delta^p x_{n+m}+\ldots+\Delta^p x_n = \Delta^p b_n\,.$$

Putting $y_n \overset{\text{df}}{=} \Delta^p x_n$, $c_n \overset{\text{df}}{=} \Delta^p b_n$, we have

$$y_{n+m} + \ldots + y_n = c_n \tag{4}$$

whence

$$y_n = c_n - y_{n+1} - \ldots - y_{n+m} .$$

According to (4)

$$y_{n+1} = c_{n+1} - y_{n+2} - \ldots - y_{n+m+1}$$

and hence

$$y_n = -\Delta c_n + y_{n+m+1} .$$

By induction one can obtain the relation

$$y_n = -\sum_{v=0}^{k} \Delta c_{n+(m+1)v} + y_{n+(m+1)(k+1)} , \qquad k = 0, 1, 2, \ldots \tag{5}$$

According to (2) $\lim\limits_{n\to\infty} y_n = 0$, whence by (5) relation (3) follows. Putting $z_n \overset{\text{df}}{=} \Delta^{p-1} x_n$, we have

$$z_{n+m} + \ldots + z_n = \Delta^{p-1} b_n$$

and consequently

$$z_n = \frac{1}{m+1}\left[\Delta^{p-1} b_n - \sum_{v=0}^{m-1} (m-v) y_{n+v}\right] . \tag{6}$$

Analogously we obtain $\Delta^{p-2} x_n$ and next $\Delta^{p-3} x_n, \ldots, \Delta x_n, x_n$, whence the uniqueness of the sequence x_n follows. This completes the proof.

Lemma 1 easily implies

COROLLARY 1. *The only sequence satisfying the homogeneous relation*

$$x_{n+m} + \ldots + x_n = 0 \tag{7}$$

and fulfilling condition (2) *is* $x_n = 0$, $n = 0, 1, 2, \ldots$

LEMMA 2. *If for a certain integer* $p \geqslant 0$

$$\lim_{n\to\infty} \Delta^p b_n = 0 , \tag{8}$$

then there exists at most one sequence x_n *satisfying relation* (1) *and such that the terms* $\Delta^p x_n$ *have a constant sign.*

Proof. According to Lemma 1, it is enough to prove that condition (2) holds. The proof of this fact is analogous to the proof of Lemma 3 in [6].

Now we shall prove the following

THEOREM 1. *If the terms Δb_n have a constant sign and*

$$\lim_{n\to\infty} b_n = 0 \,, \tag{9}$$

then there exists exactly one sequence x_n satisfying relation (1) *and such that the terms x_n have a constant sign. This sequence is given by the formula*

$$x_n = -\sum_{v=0}^{\infty} \Delta b_{n+(m+1)v} \,. \tag{10}$$

Proof. Since b_n is a monotonic sequence, the series

$$\sum_{v=0}^{\infty} (-1)^v d_{nv}$$

where

$$d_{nv} = \begin{cases} b_{n+(m+1)k} & \text{for} \quad v = 2k \,, \\ b_{n+(m+1)k+1} & \text{for} \quad v = 2k+1 \,, \end{cases}$$

$k = 0, 1, 2, \ldots$, converges in view of (9). Grouping its terms, we obtain the formula (10), whence it follows that x_n have a constant sign. It can easily be verified that sequence (10) satisfies relation (1). The uniqueness follows from Lemma 2 for $p = 0$.

Remark 1. If we require that

$$\Delta^p x_n \geqslant 0 \qquad (\Delta^p x_n \leqslant 0) \,,$$

then the condition (8) can be replaced by

$$\limsup_{n\to\infty} \Delta^p b_n = 0 \qquad (\liminf_{n\to\infty} \Delta^p b_n = 0) \,. \tag{11}$$

Actually, if b_n satisfies the condition

$$\limsup_{n\to\infty} \Delta^p b_n = 0 \,,$$

then

$$0 \leqslant \liminf_{n\to\infty} \{\Delta^p x_{n+m} + \ldots + \Delta^p x_n\} \leqslant \limsup_{n\to\infty} \{\Delta^p x_{n+m} + \ldots + \Delta^p x_n\} = 0 \,,$$

whence (since $\Delta^p x_n \geqslant 0$) we have

$$\lim_{n\to\infty} \Delta^p x_n = 0$$

and the uniqueness follows from Lemma 1. If we assume the conditions

$$\Delta^p x_n \leqslant 0 \,, \qquad \liminf_{n\to\infty} \Delta^p b_n = 0 \,,$$

the proof is analogous.

Theorem 1 can be proved for differences of order p. To avoid burdensome calculations and to make the argument clear, we shall consider only the relation

$$x_{n+2}+x_{n+1}+x_n = b_n . \tag{12}$$

We shall prove the following

THEOREM 2. *If for a certain $r \geqslant 1$ the terms $\Delta^{r+1}b_n$ have a constant sign and for a certain integer $0 < p \leqslant r$ condition* (8) *holds, then there exists exactly one sequence x_n satisfying relation* (12) *and such that the terms $\Delta^r x_n$ have a constant sign* (*opposite to the sign of $\Delta^{r+1}b_n$*). *This sequence is given by the formula*

$$x_n = \sum_{k=0}^{p-1}\sum_{v=k}^{p-1}(-1)^v\frac{\binom{v}{k}2^{v-k}}{3^{v+1}}\Delta^v b_{n+k}+\frac{(-1)^p}{3^p}\sum_{v=0}^{p}\binom{p}{v}2^{p-v}y_{n+v} , \tag{13}$$

where

$$y_n = \Delta^p x_n = -\sum_{i=0}^{\infty}\Delta^{p+1}b_{n+3i} .$$

The proof of the above theorem will be based on some lemmas.

LEMMA 3. *For an arbitraty sequence b_n we have*

$$\sum_{k=0}^{p-1}\sum_{v=k}^{p-1}(-1)^v\frac{\binom{v}{k}2^{v-k}}{3^{v+1}}\{\Delta^v b_{n+k}+\Delta^v b_{n+k+1}+\Delta^v b_{n+k+2}\} \tag{14}$$
$$= b_n-\frac{(-1)^p}{3^p}\sum_{v=0}^{p}\binom{p}{v}2^{p-v}\Delta^p b_{n+v} ,\qquad p = 1, 2, \ldots$$

Proof. The proof will be by induction. For $p=1$ formula (14) follows from the definition of $\Delta^p b_n$. Assuming its validity for $p > 1$, we have for $p+1$

$$\sum_{k=0}^{p}\sum_{v=k}^{p}(-1)^v\frac{\binom{v}{k}2^{v-k}}{3^{v+1}}\{\Delta^v b_{n+k}+\Delta^v b_{n+k+1}+\Delta^v b_{n+k+2}\}$$
$$= b_n-\frac{(-1)^p}{3^p}\sum_{k=0}^{p}\binom{p}{k}2^{p-k}\Delta^p b_{n+k}+\frac{(-1)^p}{3^{p+1}}\sum_{k=0}^{p}\binom{p}{k}2^{p-k}\times$$
$$\times\{\Delta^p b_{n+k}+\Delta^p b_{n+k+1}+\Delta^p b_{n+k+2}\} .$$

Next let us add and subtract the sum

$$X = \frac{(-1)^{p+1}}{3^{p+1}}\sum_{k=0}^{p+1}\binom{p+1}{k}2^{p+1-k}\Delta^{p+1}b_{n+k} .$$

It is enough to prove that

$$I = X + \frac{(-1)^{p+1}}{3^{p+1}} \sum_{k=0}^{p} 3\binom{p}{k} 2^{p-k} \Delta^p b_{n+k} -$$
$$- \frac{(-1)^{p+1}}{3^{p+1}} \sum_{k=0}^{p} \binom{p}{k} 2^{p-k} \{\Delta^p b_{n+k} + \Delta^p b_{n+k+1} + \Delta^p b_{n+k+2}\} = 0 .$$

Replacing $\Delta^{p+1} b_n$ by $\Delta^p b_{n+1} - \Delta^p b_n$, we have

$$I = \frac{(-1)^{p+1}}{3^{p+1}} \sum_{k=0}^{p+2} A_k \Delta^p b_{n+k} ,$$

where

$$A_k \stackrel{\text{df}}{=} \binom{p+1}{k-1} 2^{p+1-(k-1)} - \binom{p+1}{k} 2^{p+1-k} + 3\binom{p}{k} 2^{p-k} -$$
$$- \binom{p}{k-2} 2^{p-(k-2)} - \binom{p}{k-1} 2^{p-(k-1)} - \binom{p}{k} 2^{p-k} ,$$

$$\binom{n}{i} \stackrel{\text{df}}{=} 0 \quad \text{for} \quad i < 0 \text{ and } i > n .$$

Using the formula

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k} \tag{15}$$

we easily obtain

$$A_k = 0 \quad \text{for} \quad k = 0, 1, 2, \ldots, p+2 ,$$

which completes the proof of Lemma 3.

Let p be written as $3s+q$, $0 \leqslant q \leqslant 2$; let us write

$$B_i \stackrel{\text{df}}{=} 3\binom{p-1}{i} 2^{p-1-i} + \binom{p}{i-1} 2^{p-i+1} - \binom{p}{i} 2^{p-i} , \quad i = 0, 1, 2, \ldots, p+1 ,$$

$$C_{3i+q} = -\sum_{k=0}^{i} \Delta B_{3k+q-1} , \quad i = 0, 1, \ldots, s , \; q = 0, 1, 2 .$$

LEMMA 4. *We have*

$$\sum_{i=0}^{s} B_{3i} = \sum_{i=0}^{s} B_{3i+1} = \sum_{i=0}^{s-1} B_{3i+2} = 3^{p-1} , \tag{16}$$

$$C_i = -\binom{p-1}{i} 2^{p-1-i} , \quad i = 0, 1, 2, \ldots, p-1 . \tag{17}$$

Proof. Suppose that $p = 3s$. Then

$$\sum_{i=0}^{s} B_{3i} = \left[3\binom{p-1}{0}2^{p-1}+3\binom{p-1}{3}2^{p-4}+\ldots+3\binom{p-1}{p-3}2^{2}\right]+$$
$$+\left[\binom{p}{2}2^{p-2}+\binom{p}{5}2^{p-5}+\ldots+\binom{p}{p-1}2\right]-\left[\binom{p}{0}2^{p}+\binom{p}{3}2^{p-3}+\ldots+\binom{p}{p}\right]$$

and applying (15) to the last two brackets, we obtain

$$\binom{p-1}{0}2^{p-1}+\binom{p-1}{1}2^{p-2}+\ldots+\binom{p-1}{p-1} = 3^{p-1}.$$

If $p = 3s+1$ or $p = 3s+2$ the proof is analogous. The proof of relation (17) is analogous.

Proof of Theorem 2. Let x_n be any sequence satisfying relation (12) and such that the terms $\Delta^r x_n$ have a constant sign. Since $p \leqslant r$, the terms $\Delta^p x_n$ also have a constant sign for n sufficiently large. According to (8) and Lemma 1 the relation

$$y_n = \Delta^p x_n = -\sum_{v=0}^{\infty} \Delta^{p+1} b_{n+3v} \tag{18}$$

follows and next we obtain (according to (12))

$$\Delta^{p-1}x_n = \tfrac{1}{3}\Delta^{p-1}b_n-\tfrac{2}{3}y_n-\tfrac{1}{3}y_{n+1}. \tag{19}$$

We shall prove the formula

$$\Delta^{p-i}x_n = \sum_{k=0}^{i-1}\sum_{v=k}^{i-1}(-1)^v\frac{\binom{v}{k}2^{v-k}}{3^{v-1}}\Delta^{p+v-i}b_{n+k}+\frac{(-1)^i}{3^i}\sum_{v=0}^{i}\binom{i}{v}2^{i-v}y_{n+v} \tag{20}$$

for a certain i, $1 \leqslant i \leqslant p$. For $i = 1$ formula (20) is valid according to (19). Assuming its validity for $1 < i \leqslant p-1$ we have for $i+1$

$$\Delta^{p-(i+1)}x_n = \tfrac{1}{3}\Delta^{p-(i+1)}b_n-\tfrac{2}{3}\Delta^{p-i}x_n-\tfrac{1}{3}\Delta^{p-i}x_{n+1}$$

and next according to (20) (grouping the terms)

$$\Delta^{p-(i+1)}x_n = \sum_{k=0}^{i}\sum_{v=k}^{i}(-1)^v\frac{\binom{v}{k}2^{v-k}}{3^{v+1}}\Delta^{p+v-(i+1)}b_{n+k}+$$
$$+\frac{(-1)^{i+1}}{3^{i+1}}\sum_{v=0}^{i+1}\binom{i+1}{v}2^{i+1-v}y_{n+v}.$$

We have proved that formula (20) is valid for $1 \leqslant i \leqslant p$. If $i = p$, formula (20) is equivalent to formula (13).

Since $\Delta^{r+1}b_n$ have a constant sign and $1 \leqslant p \leqslant r$, the terms $\Delta^{p+1}b_n$ also have a constant sign for n sufficiently large. Thus the series

$$\sum_{v=0}^{\infty} (-1)^v d_{nv} ,$$

where

$$d_{nv} = \begin{cases} \Delta^p b_{n+3v} & \text{for} \quad v = 2k , \\ \Delta^p b_{n+3v+1} & \text{for} \quad v = 2k+1 , \end{cases}$$

$k = 0, 1, 2, \ldots$, converges and formula (13) actually defines a sequence x_n. We have

$$\Delta x_n = \sum_{k=0}^{p-1} \sum_{v=k}^{p-1} (-1)^v \frac{\binom{v}{k} 2^{v-k}}{3^{v+1}} \Delta^{v+1} b_{n+k} + \frac{(-1)^p}{3^p} \sum_{i=0}^{p} \binom{p}{i} 2^{p-i} \Delta y_{n+i} .$$

We shall prove that formula (20) for sequence (13) and $i = p-1$ holds. It is enough to prove that

$$S = \frac{(-1)^p}{3^p} \sum_{i=0}^{p} \binom{p}{i} 2^{p-i} \Delta y_{n+i} + \frac{(-1)^p}{3^{p-1}} \sum_{i=0}^{p-1} \binom{p-1}{i} 2^{p-1-i} y_{n+i} \tag{21}$$

$$= \frac{(-1)^p}{3^p} \sum_{i=0}^{p-1} \binom{p-1}{i} 2^{p-1-i} \Delta^p b_{n+i} .$$

The sum S may also be written in the form

$$S = \frac{(-1)^p}{3^p} \sum_{i=0}^{p+1} B_i y_{n+i} .$$

If $p = 3s$, then

$$S = \frac{(-1)^p}{3^p} \left\{ \sum_{i=0}^{s-1} B_{3i+2} y_{n+p-1} + \sum_{i=0}^{s} B_{3i} y_{n+p} + \sum_{i=0}^{s} B_{3i+1} y_{n+p+1} \right\} + \tag{22}$$

$$+ \frac{(-1)^{p+1}}{3^p} \left\{ \sum_{k=0}^{s-2} \sum_{i=0}^{k} B_{3i+2} \Delta^{p+1} b_{n+3i+2} + \sum_{k=0}^{s-1} \sum_{i=0}^{k} B_{3i} \Delta^{p+1} b_{n+3i} + \right.$$

$$\left. + \sum_{k=0}^{s-1} \sum_{i=0}^{k} B_{3i+1} \Delta^{p+1} b_{n+3i+1} \right\} .$$

According to (16) and the condition

$$\Delta^{p+1} b_k = \Delta^p b_{k+1} - \Delta^p b_k ,$$

we obtain

$$S = \frac{(-1)^p}{3^p} \sum_{i=0}^{s-1} B_{3i+2} \Delta^p b_{n+p-1} + \frac{(-1)^{p+1}}{3^p} \sum_{i=0}^{p-2} C_i \Delta^p b_{n+i} +$$

$$+ \frac{(-1)^{p+1}}{3^p} \sum_{i=0}^{s-1} B_{3i+1} \Delta^p b_{n+p-1}$$

$$= \frac{(-1)^{p+1}}{3^p} \sum_{i=0}^{p-1} C_i \Delta^p b_{n+i} ,$$

whence, on account of relation (17) we obtain formula (21). If $p = 3s+1$ or $p = 3s+2$, the proof is analogous. Applying this procedure succesively for Δx_n, $\Delta^2 x_n$, etc. p times, we finally obtain

$$\Delta^p x_n = -\sum_{v=0}^{\infty} \Delta^{p+1} b_{n+3v} .$$

Applying the operation Δ^{r-p} to both sides of the above equality we get

$$\Delta^r x_n = -\sum_{v=0}^{\infty} \Delta^{r+1} b_{n+3v} .$$

Since the terms $\Delta^{r+1} b_n$ have a constant sign, the terms $\Delta^r x_n$ also have a constant sign (opposite to that of $\Delta^{r+1} b_n$). According to Lemma 3 we easily verify that sequence (13) satisfies relation (12). The uniqueness of such a sequence follows from Lemma 2 in view of the fact that condition (8) and the inequality $r \geqslant p$ imply the relation

$$\lim_{n\to\infty} \Delta^r b_n = 0 ,$$

which was to be proved.

3. The results obtained can be applied to establish some conditions of uniqueness and existence of solutions of the functional equation

$$\varphi[f^m(x)] + \ldots + \varphi[f(x)] + \varphi(x) = F(x) \tag{23}$$

where $\varphi(x)$ is the unknown function, $f(x)$ and $F(x)$ are given functions defined in an interval Y (finite or not). We shall assume that function $f(x)$ fulfils the conditions

$$f(x) \neq x , \quad f(Y) \subset Y .$$

Let us introduce the notation:

$$f^0(x) \overset{\mathrm{df}}{=} x\,, \quad f^{n+1}(x) \overset{\mathrm{df}}{=} f[f^n(x)]\,,$$
$$x_n \overset{\mathrm{df}}{=} \varphi[f^n(x)]\,, \quad F_n \overset{\mathrm{df}}{=} F[f^n(x)]\,,$$
$$\Delta^0_{(f)}\varphi(x) \overset{\mathrm{df}}{=} \varphi(x)\,, \quad \Delta_{(f)}\varphi(x) \overset{\mathrm{df}}{=} \varphi[f(x)] - \varphi(x)\,,$$
$$\Delta^{n+1}_{(f)}\varphi(x) \overset{\mathrm{df}}{=} \Delta_{(f)}\{\Delta^n_{(f)}\varphi(x)\}\,, \quad n = 0, 1, 2, \ldots, \quad x \in Y\,.$$

THEOREM I. *The only function satisfying the homogeneous equation*

$$\varphi[f^m(x)] + \ldots + \varphi(x) = 0 \tag{24}$$

and fulfilling the condition

$$\lim_{n\to\infty} \Delta^p_{(f)}\varphi[f^n(x)] = 0 \tag{25}$$

for every $x \in Y$ and a certain $p \geqslant 0$, is

$$\varphi(x) \equiv 0 \quad \text{for} \quad x \in Y\,.$$

This theorem is an immediate consequence of Corollary 1.

Lemma 1 implies, however, the following

THEOREM II. *There exists at most one solution of equation* (23) *satisfying condition* (25).

An analogue of Lemma 2 is also true:

THEOREM III. *If for every $x \in Y$ and $p \geqslant 0$*

$$\lim_{n\to\infty} \Delta^p_{(f)}F[f^n(x)] = 0 \tag{26}$$

then there may exist at most one solution of equation (23) *such that for every $x \in Y$ the differences*

$$\Delta^p_{(f)}\varphi[f^n(x)]\,, \quad n = 0, 1, 2, \ldots \tag{27}$$

have a constant sign.

If the function $\varphi(x)$ is semimonotonic $\{f\}$ (cf. [4]) or monotonic, then the differences (27) have a constant sign for $p = 1$ and we obtain

COROLLARY 2. *If*

$$\lim_{n\to\infty} \Delta_{(f)}F[f^n(x)] = 0 \tag{28}$$

for every $x \in Y$, then there may exist at most one solution of equation (23) *semimonotonic $\{f\}$ (monotonic).*

Remark 2. If we require that

$$\Delta^p_{(f)}\varphi[f^n(x)] \geqslant 0 \quad (\Delta^p_{(f)}\varphi[f^n(x)] \leqslant 0)$$

then condition (26) can be replaced by

$$\limsup_{n\to\infty} \Delta^p_{(f)} F[f^n(x)] = 0 \qquad (\liminf_{n\to\infty} \Delta^p_{(f)} F[f^n(x)] = 0)\,. \tag{29}$$

THEOREM IV. *If for every $x \in Y$ the differences*

$$\Delta_{(f)} F[f^n(x)]\,, \qquad n = 0, 1, 2, \ldots \tag{30}$$

have a constant sign and for every $x \in Y$

$$\lim_{n\to\infty} F[f^n(x)] = 0\,, \tag{31}$$

then there exists exactly one solution $\varphi(x)$ of equation (23) *such that for every $x \in Y$ the terms $\varphi[f^n(x)]$, $n = 0, 1, 2, \ldots$ have a constant sign. This solution is given by the formula*

$$\varphi(x) = -\sum_{v=0}^{\infty} \Delta_{(f)} F[f^{(m+1)v}(x)]\,. \tag{32}$$

The proof follows immediately from Theorem I.

THEOREM V. *If the function $F(x)$ is continuous and the function $f(x)$ is continuous and strictly increasing in the interval $\langle a, b\rangle$, $f(x) > x$ for $x \in (a, b)$, $f(a) = a$, $f(b) = b$, and if there exist functions $\varphi(x)$ and $\psi(x)$ which satisfy equation* (23) *and are continuous in the intervals $(a, b\rangle$ and $\langle a, b)$ respectively, then*

$$\varphi(x) = \frac{1}{m+1} F(b) + \sum_{v=0}^{\infty} (-1)^v [G_v(x) - F(b)]\,, \tag{33}$$

$$\psi(x) = \frac{1}{m+1} F(a) + \sum_{v=0}^{\infty} (-1)^v [H_v(x) - F(a)]\,, \tag{34}$$

where

$$G_v(x) = \begin{cases} F[f^{(m+1)k}(x)] & \textit{for} \quad v = 2k\,, \\ F[f^{(m+1)k+1}(x)] & \textit{for} \quad v = 2k+1; \end{cases} \tag{35}$$

$$H_v(x) = \begin{cases} F[f^{-(m+1)k-m}(x)] & \textit{for} \quad v = 2k\,, \\ F[f^{-(m+1)(k+1)}(x)] & \textit{for} \quad v = 2k+1 \end{cases} \tag{36}$$

$k = 0, 1, 2, \ldots$

Proof. The sequence

$$x_n = \varphi[f^n(x)] - \frac{1}{m+1} F(b)$$

fulfils the relation

$$x_{n+m}+\ldots+x_n = F_n - F(b) \overset{\text{df}}{=} F_n^* .$$

Hence, according to Lemma 1 ($p = 0$ and $n = 0$), we get formulas (33) and (35). For the sequence

$$x_n = \psi[f^{-n}(x)] - \frac{1}{m+1}F(a)$$

formulas (34) and (36) can be obtained in a similar manner.

THEOREM VI. *Let us suppose that function $f(x)$ is continuous and strictly increasing in an interval $(a, b\rangle$ and that $f(x) > x$ in (a, b), $f(b) = b$. If a function $\varphi(x)$ satisfies equation* (23) *and for every $x \in (a, b)$ fulfils the condition*

$$\lim_{n\to\infty} \Delta_{(f)}\varphi[f^n(x)] = 0 , \tag{37}$$

then

$$\varphi(x) = \frac{1}{m+1}\left\{F(x) - \sum_{v=0}^{m-1} (m-v)g[f^v(x)]\right\} \tag{38}$$

where

$$g(x) = \sum_{v=0}^{\infty} (-1)^v \{G_v[f(x)] - G_v(x)\} .$$

The proof follows from the proof of Lemma 1 (according to (6)) for $p = 1$, $n = 0$,

$$z_n = \varphi[f^n(x)] \quad \text{and} \quad c_n = \Delta_{(f)}F[f^n(x)] .$$

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