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On the selector problems for the partitions of Polish spaces and for the compact-valued mappings

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Abstract. It is shown that under suitable assumptions on the partition there exists a selector with properties of some interest in Topology, or in the Measure Theory or in the Descriptive Set Theory.

The last section contains a very simple proof of the selection-theorem of Ryll-Nardzewski and the author for the case when the set-valued mappings are supposed compact-valued and the σ-lattice S, under consideration, is a σ-algebra.

1. Introduction. Terminology. There are essentially two types of problems concerning selections. The first (which corresponds to the axiom of choice in its classical form) consists in the following. Given a partition $\boldsymbol{Q}$ of a topological space X into closed non-empty (disjoint) subsets, one has to define a selector for $\boldsymbol{Q}$, i.e. a set S such that $S\cap E$ is a singleton for each $E\in\boldsymbol{Q}$.

In the second type of problems (corresponding to the general principle of choice) we have given a set-valued mapping $F\colon X\to 2^Y$ and we are looking for a choice function (called also a selector for F) $f\colon X\to Y$ such that $f(x)\in F(x)$ for each $x\in X$.

We will use the following terminology (see also section 4).

Given a family $\boldsymbol{L}$ of subsets of X, we will say that $\boldsymbol{Q}$ is an $\boldsymbol{L}^-$*-partition* if

$$\Big\{\bigcup_{E\in\boldsymbol{Q}} : E\cap G\neq\emptyset\Big\}\in\boldsymbol{L}\quad\text{for each open } G\subset X, \tag{1}$$

and we say that the set-valued mapping F is an $\boldsymbol{L}^-$*-mapping* (see [6], p. 273, and [5]), if

$$\{x\colon F(x)\cap G\neq\emptyset\}\in\boldsymbol{L}\quad\text{for each open } G\subset Y. \tag{2}$$

Accordingly, given a point-valued mapping $f\colon X\to Y$ (case where $F(x)$ is a singleton for each x), we say that f is an $\boldsymbol{L}$*-mapping* if

$$f^{-1}(G)\in\boldsymbol{L}\quad\text{for each open } G\subset Y. \tag{3}$$

In what follows (except in section 5), we shall assume that $X = Y$ is a ***Polish space*** (i.e. complete metric and separable) and that $\boldsymbol{L}$ is a *σ-algebra* of subsets of X (i.e. $X \in \boldsymbol{L}$ and $\boldsymbol{L}$ is closed under the operations of subtraction and of countable unions and intersections).

Under these assumptions (and in fact, under some slighter assumptions on $\boldsymbol{L}$) the following statement has been shown (in [7], see also [8] and [9]).

THEOREM A. *Every set-valued $\boldsymbol{L}^-$-mapping $F: X \to 2^Y$ admits a selector $f: X \to Y$ which is an $\boldsymbol{L}$-mapping.*

As seen, this is a statement on selectors of the second type. The corresponding statement of the first type — which we are going to prove here — is the

THEOREM B. *Under the same assumptions on X (here $X = Y$) and on $\boldsymbol{L}$ and assuming, moreover, that $\boldsymbol{L}$ contains all open subsets of X, every $\boldsymbol{L}^-$-partition $\boldsymbol{Q}$ admits a selector which is a member of $\boldsymbol{L}$.*

The following notation states the relation of Theorem B to A.

Denote by P the projection of X onto $\boldsymbol{Q}$ defined by the condition

$$x \in P(x) \in \boldsymbol{Q}.$$

We have the following equivalence

(4) *$\boldsymbol{Q}$ is an $\boldsymbol{L}^-$-partition iff P is an $\boldsymbol{L}^-$-mapping.*

Because, as easily seen,

$$\Big\{\bigcup_{E \in \boldsymbol{Q}} : E \cap Z \neq \emptyset\Big\} = \{x: P(x) \cap Z \neq \emptyset\}, \tag{5}$$

whatever is $Z \subset X$.

One sees also easily that *if f is a selector for P and f is constant on every $E \in \boldsymbol{Q}$, then the set $f(X)$ is a selector for $\boldsymbol{Q}$.*

Consequently, the proof of Theorem B reduces to proving the two following propositions.

PROPOSITION 1. *If P is an $\boldsymbol{L}^-$-mapping, then P admits a selector f which is an $\boldsymbol{L}$-mapping and is constant on each $E \in \boldsymbol{Q}$.*

PROPOSITION 2. $f(X) \in \boldsymbol{L}$.

2. Proof of Proposition 1. Restart the proof given in [7], p. 399.

X being assumed separable, let $R = (r_1, r_2, \ldots)$ be a countable set dense in X. We may suppose, of course, that the diameter of X is less than 1.

We shall define $f: X \to R$ as the limit of mappings $f_n: X \to R$ satisfying the following conditions

$(\mathrm{i})_n$ *f_n is an $\boldsymbol{L}$-mapping,* (ii)$_n$ $\quad f_n$ *is constant on each* $E \in \boldsymbol{Q}$,

(iii)$_n$ $\quad \varrho[f_n(x), P(x)] < 1/2^n$,

(iv)$_n$ $\quad |f_n(x) - f_{n-1}(x)| < 1/2^{n-1}$ *for* $n > 0$.

We proceed by induction. Put $f_0(x) = r_1$ for each $x \in X$. Thus conditions (i)$_0$–(iii)$_0$ are fulfilled.

Let us assume, for a given $n > 0$, that f_{n-1} satisfies conditions (i)$_{n-1}$–(iii)$_{n-1}$.

Denote G_i^n the open ball $\{x\colon |x - r_i| < 1/2^n\}$ and put

(6) $$C_i^n = \{x\colon P(x) \cap G_i^n \neq \emptyset\},$$

(7) $$D_i^n = f_{n-1}^{-1}(G_i^{n-1}),$$

(8) $$A_i^n = C_i^n \cap D_i^n.$$

Since $\boldsymbol{Q}$ is an $\boldsymbol{L}^-$-partition, so P is an $\boldsymbol{L}^-$-mapping (see (4)) and hence (by (2)) $C_i^n \in \boldsymbol{L}$. Since f_{n-1} is an $\boldsymbol{L}$-mapping (by (i)$_{n-1}$), we have $D_i^n \in \boldsymbol{L}$. Therefore

(9) $$A_i^n \in \boldsymbol{L}.$$

We shall show that

(10) $$X = A_1^n \cup A_2^n \cup \ldots$$

Let $x \in X$. By (iii)$_{n-1}$ there is $y \in P(x)$ such that

$$|y - f_{n-1}(x)| < 1/2^{n-1}.$$

Since R is dense in X, there is i such that

$$|r_i - y| < 1/2^n \quad \text{and} \quad |r_i - y| + |y - f_{n-1}(x)| < 1/2^{n-1}.$$

It follows that $x \in A_i^n$. Hence (10) is true.

Now put

$$B_1^n = A_1^n \quad \text{and} \quad B_i^n = A_i^n - (A_1^n \cup \ldots \cup A_{i-1}^n) \quad \text{for } i > 1.$$

By (10) and (9) we have

(11) $$X = B_1^n \cup B_2^n \cup \ldots, \quad B_i^n \cap B_{i'}^n = \emptyset \quad \text{for } i \neq i', \text{ and } B_i^n \in \boldsymbol{L}.$$

Let us define $f_n(x)$ for $x \in X$ as follows

(12) $$f_n(x) = r_i \quad \text{if } x \in B_i^n, \quad \text{i.e.} \quad f_n^{-1}(r_i) = B_i^n.$$

We have to show that f_n satisfies conditions (i)$_n$–(iv)$_n$.

Since R is countable we have by (12), $f_n^{-1}(Z) \in \boldsymbol{L}$ for each $Z \subset R$. In particular $f_n^{-1}(G) \in \boldsymbol{L}$ for G open in X; hence (i)$_n$ is fulfilled.

In order to show (ii)$_n$, denote by $\boldsymbol{L}^*$ the family of those $Z \in \boldsymbol{L}$ which are unions of some members of $\boldsymbol{Q}$. Obviously $\boldsymbol{L}^*$ is a σ-algebra (containing X) and condition $(\mathrm{ii})_n$ means that

$$(13)_n \qquad f_n^{-1}(r_i) \in \boldsymbol{L}^* \quad \text{for } i = 1, 2, \ldots$$

So we may assume that $(13)_{n-1}$ is true. It follows at once that $D_i^n \in \boldsymbol{L}^*$. We have also $C_i^n \in \boldsymbol{L}^*$; for suppose that $x \in C_i^n$, i.e. $P(x) \cap G_i^n \neq \emptyset$, and let $x' \in P(x)$. Then $P(x') = P(x)$ and hence $P(x') \cap G_i^n \neq \emptyset$, which means that $x' \in C_i^n$. Thus $P(x) \subset C_i^n$. Hence $C_i^n \in \boldsymbol{L}^*$, and since $D_i^n \in \boldsymbol{L}^*$, it follows by (8) that $A_i^n \in \boldsymbol{L}^*$ and consequently $B_i^n \in \boldsymbol{L}^*$. This implies $(13)_n$ by (12).

To show $(\mathrm{iii})_n$ and $(\mathrm{iv})_n$, put $x \in B_i^n$. Then $f_n(x) = r_i$ and by (8)

$$x \in C_i^n, \quad \text{i.e.,} \ \varrho\big(r_i, P(x)\big) < 1/2^n, \quad \text{and} \quad x \in D_i^n, \quad \text{i.e.,} \ |f_{n-1}(x) - r_i| < 1/2^{n-1},$$

and our conclusion follows.

By $(\mathrm{iv})_n$ and by the completeness of the space X, the sequence $f_1, f_2, \ldots$ converges uniformly to a mapping f: $X \to X$. Since the limit of a uniformly convergent $\boldsymbol{L}$-mappings is an $\boldsymbol{L}$-mapping (see e.g. [7], p. 398, lemma), it follows by $(\mathrm{i})_n$ that f is an $\boldsymbol{L}$-mapping.

It follows also from $(\mathrm{ii})_n$ that f is constant on each $E \in \boldsymbol{Q}$.

Finally $(\mathrm{iii})_n$ implies that $f(x) \in P(x)$, i.e. that f is a selector for P.

3. Proof of Proposition 2.

LEMMA. *Assume that the space X contains a countable open base and that the σ-lattice $\boldsymbol{L}$* (i.e. *closed under countable unions*) *contains all open subsets of X. Let f: $X \to X$ be an $\boldsymbol{L}$-mapping and put*

$$(14) \qquad I = \{x \colon f(x) = x\}.$$

Then $I \in (-\boldsymbol{L})$, i.e. $(X - I) \in \boldsymbol{L}$.

Proof. Let $G_1, G_2, \ldots$ be an open base of X. Obviously

$$(x = y) \equiv \forall_n \colon (x \in G_n \Rightarrow y \in \bar{G}_n) \quad \text{for each } x, y \in X.$$

Hence

$$\big(x = f(x)\big) \equiv \forall_n \colon \big[(x \notin G_n) \vee \big(f(x) \in \bar{G}_n\big)\big],$$

and therefore, by (14), we have

$$I = \bigcap_n \left[(X - G_n) \cup f^{-1}(\bar{G}_n)\right].$$

By assumption the sets $X - G_n$ and $f^{-1}(\bar{G}_n)$ belong to $(-\boldsymbol{L})$. Hence so does I.

This completes the proof of the Lemma.

It remains to show that $f(X) = I$, where the mapping f: $X \to X$ is assumed to be constant on each $E \in \boldsymbol{Q}$.

So let $y \in f(X)$. Put $y = f(x_0)$ and $x_0 \in E$. Since f is a selector for P, we have $f(x_0) \in P(x_0)$, i.e. $y \in E$, and since f is constant on E, it follows that $f(y) = f(x_0)$, i.e. $f(y) = y$, and thus $y \in I$.

4. Particular cases. Remarks.

1. Theorem B can be applied to the cases where $\boldsymbol{L}$ denotes the families: of Borel sets, of sets having the Baire property, of measurable sets. In the last case, Theorem B reads as follows: *every measurable partition admits a measurable selector.*

2. In the definitions of $\boldsymbol{L}^-$-partitions and $\boldsymbol{L}^-$-mappings we have referred to *open sets* G. Symmetrically $\boldsymbol{Q}$ is said an $\boldsymbol{L}^+$-*partition* if

$$(1') \qquad \{\bigcup_{E\in \boldsymbol{Q}}: E\cap K \neq \emptyset\}\in \boldsymbol{L} \quad \text{for each closed } K\subset X,$$

and F is an $\boldsymbol{L}^+$-*mapping* if ([6], p. 273):

$$(2') \qquad \{x\colon F(x)\cap K \neq \emptyset\}\in \boldsymbol{L} \quad \text{for each closed } K \subset Y.$$

For point-valued mappings $f\colon X\to Y$ condition (3) and condition

$$(3') \qquad f^{-1}(K)\in(-\boldsymbol{L}) \quad \text{for each closed } K \subset Y$$

are obviously equivalent (thus, if $\boldsymbol{L} = -\boldsymbol{L}$, it is legitimate to call f briefly an $\boldsymbol{L}$-mapping if (3), or equivalently (3′), is fulfilled).

3. Let us note that our terminology for partitions and set-valued mappings agrees with the usual one in case of semi-continuity, case when $\boldsymbol{L}$ represents the family (not an algebra!) of open, respectively closed sets (see e.g. [5], p. 185).

5. Compact-valued mappings and partitions into compact subsets. In this section X denotes an arbitrary metric separable space and $\mathscr{C}(X)$ the space of all compact subsets of X, endowed with the Vietoris topology.

First, let us note that, $\boldsymbol{L}$ being supposed a σ-algebra and F compact-valued, conditions (2) and (2′) are equivalent ([6], p. 274, Corollary 5′) and, moreover (ibidem Theorem 4), they are equivalent to the condition of F being an $\boldsymbol{L}$-mapping (that means that $F^{-1}(\boldsymbol{G})\in\boldsymbol{L}$ for each $\boldsymbol{G}$ open in $\mathscr{C}(Y)$).

Consequently, in what follows we may write briefly $\boldsymbol{L}$-mapping instead of writing $\boldsymbol{L}^+$ (or $\boldsymbol{L}^-$) mapping.

The following theorem has been deduced from Theorem A (see [7], p. 400).

THEOREM C. *For each Polish space X there is a choice-function $f\colon 2^X\to X$ of the first Baire class (which means that $f(K)\in K$ for each closed $K \subset X$ and the set $f^{-1}(\boldsymbol{G})$ is an F_σ-set for each open $\boldsymbol{G} \subset 2^X$).*

Here we are going to give a very simple proof of that statement for the case where 2^X is replaced by $\mathscr{C}(X)$ (the space X being supposed arbitrary metric separable).

Since each metric separable space can be embedded in the Hilbert cube, H, the proof reduces to the case $X = H$.

So let $g\colon I \to H$ be a continuous mapping of the closed interval I onto H. Then the inverse mapping $g^{-1}\colon H \to 2^I$ is upper semi-continuous (see e.g. [5], p. 57), hence of the first class ([5], p. 70).

Next denote by $h(A)$ the first point of A for $A \in 2^I$. Obviously the function $h\colon 2^I \to I$ is continuous (see [5], p. 49).

It remains to put

$$f = g \circ h \circ g^{-1}.$$

Now we shall deduce from Theorem C, thus modified, both Theorems A and B, always under the assumptions that Y is metric separable, $\boldsymbol{L}$ a σ-algebra, $F\colon X \to \mathscr{C}(Y)$ an $\boldsymbol{L}$-mapping, and $\boldsymbol{Q}$ an $\boldsymbol{L}$-partition of the metric separable space X into compact sets.

Proof of Theorem A. Let, the mapping $e\colon \mathscr{C}(Y) \to Y$, be a choice-function (i.e. $e(A) \in A$ for $A \in \mathscr{C}(Y)$) of the first class. Then $f = e \circ F$ is a *selector for F* and f is an *$\boldsymbol{L}$-mapping.*

The fact that $f(x) \in F(x)$ is obvious. It remains to show (3). Let G be open in Y. We have

$$f^{-1}(G) = F^{-1}[e^{-1}(G)];$$

since e is of the first class, the set $e^{-1}(G)$ is F_σ in $\mathscr{C}(Y)$. So let

$$e^{-1}(G) = \boldsymbol{K}_1 \cup \boldsymbol{K}_2 \cup \ldots, \quad \text{where } \boldsymbol{K}_i \text{ is closed in } \mathscr{C}(Y).$$

Since F is an $\boldsymbol{L}$-mapping, the set $F^{-1}(\boldsymbol{H})$ belongs to $\boldsymbol{L}$ for each $\boldsymbol{H}$ open in $\mathscr{C}(Y)$, and so does $F^{-1}(\boldsymbol{K}_i)$ (because $\boldsymbol{L}$ is a σ-algebra). Consequently

$$F^{-1}(\boldsymbol{K}_1 \cup \boldsymbol{K}_2 \cup \ldots) = [F^{-1}(\boldsymbol{K}_1) \cup F^{-1}(\boldsymbol{K}_2) \cup \ldots] \in \boldsymbol{L}.$$

Proof of Theorem B. We have to show the Propositions 1 and 2.

1. As shown, the mapping $f = e \circ P\colon X \to X$ is a selector for P and is an $\boldsymbol{L}$-mapping. Moreover, f is constant on each $E \in \boldsymbol{Q}$, because

$$x' \in P(x) \Rightarrow P(x') = P(x).$$

2. The proof of Proposition 2 given in Section 3 remains unchanged.

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