ANNALES
POLONICI MATHEMATICI
XLIV (1984)

Representation generated by a finite number of Hilbert space operators

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Abstract. In the present paper we extend part of results on pairs of commuting Hilbert space operators contained in [3] and [6] to a multidimensional case.

The purpose of the present paper is to extend part of the results of [3] and [6] on pairs of commuting Hilbert space operators to the case of any finite number of operators.

The two-dimensional case is based on Cole's decomposition theorem for orthogonal measures (or rather on its generalization due to Bekken [2]). The main reason that Cole's decomposition holds true is that the algebra $R(K)$ has no completely singular orthogonal measures when K is a compact subset of the complex plane C. But this property does not hold for $K \subset C^n$ where $n \geqslant 2$. Therefore we cannot extend Cole's decomposition to n-dimensional case for $n \geqslant 3$ automatically.

1. Throughout this paper, $K_1, \ldots, K_n$ will denote compact subsets of the complex plane C, $C(K_1 \times \ldots \times K_n)$ will denote the algebra of all complex continuous functions, $R(K_1 \times \ldots \times K_n)$ the uniform closure in $C(K_1 \times \ldots \times K_n)$ of the algebra of all rational functions with singularities off $K_1 \times \ldots \times K_n$, and Q_i $(i = 1, \ldots, n)$ is the set of all non-peak points of $R(K_i)$.

For a set $E \subset C^n$ by ∂E we denote its topological boundary and by $M(E)$ the set of all complex Borel measures on E. We introduce the following notation:

$$K_{j_1,\ldots,j_m} \overset{\text{df}}{=} K_{j_1} \times \ldots \times K_{j_m}, \quad 1 \leqslant j_1 \leqslant \ldots \leqslant j_m \leqslant n,$$

$$\Gamma_{j_1,\ldots,j_m} \overset{\text{df}}{=} \partial K_{j_1} \times \ldots \times \partial K_{j_m},$$

$$R_{j_1,\ldots,j_m} \overset{\text{df}}{=} R(K_{j_1} \times \ldots \times K_{j_m}),$$

$$Q_{j_1,\ldots,j_m} \overset{\text{df}}{=} Q_{j_1} \times \ldots \times Q_{j_m}.$$

For details and terminology concerning function algebras we refer to [4].

A measure μ is said to be *orthogonal* to a function algebra R, if $\int u d\mu = 0$ for all $u \in R$. The set of all measures orthogonal to R will be denoted by $R^{\perp}$.

By $B(Q_{1,\ldots,n}, R_{1,\ldots,n})$ we will denote the algebra of all functions u on $K_{1,\ldots,n}$ such that there exists a bounded sequence u_k in $R_{1,\ldots,n}$ converging to u pointwise on $Q_{1,\ldots,n}$ (see [1], [2]), $\|u\| \overset{\text{df}}{=} \inf\{\sup_k \|u_k\|\colon u_k \in R_{1,\ldots,n},\ u_k \to u$ pointwise on $Q_{1,\ldots,n}\}$.

The letter H will stand for a Hilbert space, $L(H)$ for the algebra of all linear bounded operators an H. The algebra homomorphism $S\colon R_{1,\ldots,n} \to L(H)$ is a representation if $\|S(u)\| \leqslant \|u\| \overset{\text{df}}{=} \sup\{|u(x)|\colon x \in K_{1,\ldots,n}\}$ for all $u \in R_{1,\ldots,n}$ ($\|T\|$ denotes the norm of $T \in L(H)$).

It is well known that for every $f, g \in H$ there is a complex measure $\mu_{f,g}$ on $K_{1,\ldots,n}$ such that

$$(S(u)f, g) = \int u d\mu_{f,g}, \quad u \in R_{1,\ldots,n}, \quad \|\mu_{f,g}\| \leqslant \|f\|\,\|g\|,$$

with (f, g) denoting the scalar product of $f, g \in H$, and $\|\mu\|$ the total variation of a measure μ.

A collection $\{\mu_{f,g}\}_{f,g \in H}$ is called a *system of elementary measures* of S.

If for an n-tuple $T_1, \ldots, T_n$ of operators in $L(H)$ there is a representation $S\colon R_{1,\ldots,n} \to L(H)$ such that $S(e_i) = T_i$, where $e_i(z_1, \ldots, z_n) = z_i$, then we say that $K_{1,\ldots,n}$ is a spectral set for $T_1, \ldots, T_n$ and S is a representation of $R_{1,\ldots,n}$ generated by $T_1, \ldots, T_n$.

DEFINITION. An m-tuple of operators $T_1, \ldots, T_m$ on H has the *property F'* if the only subspace reducing $T_1, \ldots, T_m$ to normal operators with common spectral measure on $\Gamma_{1,\ldots,m}$ singular to $R^{\perp}_{1,\ldots,m}$ is the null space.

An n-tuple of operators $T_1, \ldots, T_n$ on H has the *property F* if every m-tuple $T_{j_1}, \ldots, T_{j_m}$ $(1 \leqslant j_1 \leqslant \ldots \leqslant j_m \leqslant n,\ m = 1, \ldots, n)$ has the property F'.

Our results are as follows:

THEOREM 1. *Let $T_1, \ldots, T_n$ be an n-tuple of commuting operators in $L(H)$, and $K_1 \times \ldots \times K_n$ a spectral set for $T_1, \ldots, T_n$.*

If $T_1, \ldots, T_n$ has property F, then the representation $S\colon R_{1,\ldots,n} \to L(H)$ generated by $T_1, \ldots, T_n$ has a system of elementary measures belonging to the band of measures on $\Gamma_{1,\ldots,n}$ generated by representing measures for points in $Q_{1,\ldots,n}$.

THEOREM 2. *Assume that an n-tuple $T_1, \ldots, T_n$ of commuting operators in $L(H)$ has property F, and $K_1 \times \ldots \times K_n$ is a spectral set for $T_1, \ldots, T_n$.*

Then there is an algebra homomorphism

$$B(Q_{1,\ldots,n}, R_{1,\ldots,n}) \ni u \to u(T_1, \ldots, T_n) \in L(H)$$

such that

(1) $e_i(T_1, \ldots, T_n) = T_i$, *where* $e_i(z_1, \ldots, z_n) = z_i$ $(i = 1, \ldots, n)$,

(2) $\|u(T_1, \ldots, T_n)\| \leqslant \|u\|$, $u \in B(Q_{1,\ldots,n}, R_{1,\ldots,n})$,

(3) *if* $\sup_k \|u_k\| < \infty$, *and* $u_k \to u$ *pointwise on* $Q_{1,\ldots,n}$, *then* $u_k(T_1, \ldots, T_n) \to u(T_1, \ldots, T_n)$ *in the weak operator topology,*

(4) $u(T_1, \ldots, T_n)^* = \tilde{u}(T_1^*, \ldots, T_n^*)$, *where* $\tilde{u}(z_1, \ldots, z_n) \stackrel{\mathrm{df}}{=} \overline{u(\bar{z}_1, \ldots, \bar{z}_n)}$.

2. We will employ a Cole-type decomposition of orthogonal measures, due to Bekken [1], [2], and the decomposition of operator representation induced by it.

Let E be an arbitrary set of complex measures on a compact set K. By E^s we will denote the set of all measures on K which are singular to all measures in E. A set B of measures on K is called a *band* (see [3], [5]) if $B^{ss} = B$. Every complex measure μ has a unique decomposition $\mu = \mu_B + \mu_s$ such that $\mu_B \in B$ and $\mu_s \in B^s$. It is easy to see that E^s is a band for every $E \subset M(K)$ and that E^{ss} is the smallest band which includes E. We call E^{ss} the *band generated by E*.

Let μ be a measure on $\Gamma_{1,\ldots,n}$. We define $\pi_{j_1,\ldots,j_m}\mu$ as follows:

$$\pi_{j_1,\ldots,j_m}\mu(E) \stackrel{\mathrm{df}}{=} \mu(\{(z_1, \ldots, z_n) \in \Gamma_{1,\ldots,n}\colon (z_{j_1}, \ldots, z_{j_m}) \in E\})$$

for every Borel $E \subset \Gamma_{j_1,\ldots,j_m}$, $1 \leqslant j_1 \leqslant \ldots \leqslant j_m \leqslant n$, $m = 1, \ldots, n$.

On the set $\Gamma_{1,\ldots,n}$ we introduce the following bands of measures:

B_0 – band generated by representing measures for points in $Q_{1,\ldots,n}$,

$$B_{j_1,\ldots,j_m} \stackrel{\mathrm{df}}{=} \pi_{j_1,\ldots,j_m}^{-1}((R_{j_1,\ldots,j_m}^{\perp})^s), \quad 1 \leqslant j_1 \leqslant \ldots \leqslant j_m \leqslant n, \ m = 1, \ldots, n.$$

Using properties of peak interpolation sets in the same way as in Lemma 4.1 of [2] and the decomposition theorem for orthogonal measures [4], II.7.11, we can prove the following

LEMMA 1. *The bands* B_0 *and* $B_{j_1,\ldots,j_m}$ $(1 \leqslant j_1 \leqslant \ldots \leqslant j_m \leqslant n)$ *are reducing, i.e., if* μ *is orthogonal to* $R_{1,\ldots,n}$, *so is* μ_{B_0} *and so are all* $\mu_{B_{j_1,\ldots,j_n}}$.

A measure μ on $\Gamma_{1,\ldots,m}$ in called an *A-measure* (for $R_{1,\ldots,m}$) if $u_k \to 0$ weak-star in $L^\infty(|\mu|)$ whenever u_k is a bounded sequence in $R_{1,\ldots,m}$, and $u_k \to 0$ pointwise on $Q_{1,\ldots,m}$, $|\mu|$ denoting the variation measure of μ.

The main parts of Theorems 1 and 2 are the following propositions:

PROPOSITION 1. *If a complex measure* μ *on* $\Gamma_{1,\ldots,n}$ *is singular to all bands* $B_{j_1,\ldots,j_m}$ $(1 \leqslant j_1 \leqslant \ldots \leqslant j_m \leqslant n, \ m = 1, \ldots, n)$, *then* μ *is an A-measure.*

PROPOSITION 2. *Every A-measure on* $\Gamma_{1,\ldots,n}$ *belongs to* B_0.

The proof of Proposition 2 is to be found in [2]. Proposition 1 will be proved by induction. We will use Bekken's result taken from [2] which does not depend on dimension:

LEMMA 2. *Suppose μ is a measure on $\Gamma_{1,\ldots,n}$ orthogonal to $R_{1,\ldots,n}$. If for every sequence $\{u_k\}_{k=1}^{\infty}$ in $R_{1,\ldots,n}$ converging pointwise boundedly to 0 on $Q_{1,\ldots,n}$ and for every $a_i \in Q_i$ $(i = 1, \ldots, n)$ we have $u_k(\cdot, a_i, \cdot) \to 0$ weak-star in $L^{\infty}(|\mu|)$, then μ is an A-measure.*

For $n = 1$ and 2 Proposition 1 is due to Bekken [2]. Assume that it is valid for $n-1$.

Let μ be a measure satisfying the assumption of Proposition 1. It is enough to check that μ satisfies the assumption of Lemma 2 for $i = n$, that is, it is enough to prove that for every $a \in Q_n$ we have $u_k(\cdot, a) \to 0$ weak-star in $L^{\infty}(|\mu|)$ whenever u_k is a bounded sequence in $R_{1,\ldots,n}$ which converges pointwise to 0 on $Q_{1,\ldots,n}$.

The measure μ is by the assumption singular to all bands $B_{j_1,\ldots,j_m}$, where $1 \leqslant j_1 \leqslant \ldots \leqslant j_m \leqslant n$; it is in particular singular to $(R_{1,\ldots,n}^{\perp})^s$. Therefore, by the same argument as in the proof of [4], II. 7, 5, there is a measure ν in $R_{1,\ldots,n}^{\perp}$ singular to all bands $B_{j_1,\ldots,j_m}$ such that μ is absolutely continuous with respect to $|\nu|$.

Let $\{u_k\}_{k=1}^{\infty}$ be a bounded sequence in $R_{1,\ldots,n}$ such that $u_k \to 0$ on $Q_{1,\ldots,n}$. Fix $a \in Q_n$, and put

$$L \stackrel{\text{df}}{=} \{(z_1, \ldots, z_n) \in \Gamma_{1,\ldots,n}\colon u_k(z_1, \ldots, z_{n-1}, a) \nrightarrow 0\}.$$

It is easy to see that $L = L' \times \partial K_n$ where

$$L' \stackrel{\text{df}}{=} \{(z_1, \ldots, z_{n-1}) \in \Gamma_{1,\ldots,n-1}\colon u_k(z_1, \ldots, z_{n-1}, a) \nrightarrow 0\}.$$

The sequence $\{u_k(\cdot, a)\}_{k=1}^{\infty}$ is bounded in $R_{1,\ldots,n-1}$ and converges pointwise to 0 on $Q_{1,\ldots,n-1}$. By the induction assumption $\pi_{1,\ldots,n-1}|\nu|$ is an A-measure, which implies $|\nu|(L) = (\pi_{1,\ldots,n-1}|\nu|)(L') = 0$. Then also $|\mu|(L) = 0$ which means that μ is an A-measure.

3. Proof of Theorem 1. Let S be the representation of $R_{1,\ldots,n}$ in $L(H)$ generated by $T_1, \ldots, T_n$. If B is a band of measures on $\Gamma_{1,\ldots,n}$, then we obtain a unique orthogonal decomposition of S into two representations ([7], Sec. 3):

$$S = S_B \oplus S_s,$$

where $S_B(S_s)$ has a system of elementary measures belonging singularly to B. We call S_B the *B-part* of S.

We will decompose S with respect to the bands $B_{j_1,\ldots,j_m}$ $(1 \leqslant j_1 \leqslant \ldots \leqslant j_m \leqslant n)$. Let $\{B_1, B_2, \ldots, B_l\}$ be the ordered set of these bands (the way of ordering is arbitrary). Then

$$S = S_0 \oplus S_1 \oplus \ldots \oplus S_l,$$

where S_0 is the B_0-part of S and S_i $(i = 1, \ldots, l)$ is the B_i-part of $S \ominus (S_0 \oplus S_1 \oplus \ldots \oplus S_{i-1})$.

The decomposition of S induces a decomposition of the space H into an orthogonal sum of closed subspaces

$$H = H_0 \oplus H_1 \oplus \ldots \oplus H_l,$$

where $S|_H = S_i$ $(i = 0, \ldots, l)$, $T|_K$ denoting the restriction of an operator (or an operator representation) T to a subspace K.

Fix a sequence $j_1, \ldots, j_m$ $(1 \leqslant j_1 \leqslant \ldots \leqslant j_m \leqslant n)$ and consider the subalgebra of all functions in $R_{1,\ldots,n}$ which depend only on variables $z_{j_1,\ldots,j_m}$. We can identify this subalgebra with the algebra $R_{j_1,\ldots,j_m}$ and restrict representations S_i $(i = 1, \ldots, n)$ to this algebra (see [6]). Let us denote those restrictions by S_i'.

There is a number $p \in \{1, \ldots, l\}$ such that $B_{j_1,\ldots,j_m} = B_p$. The representation S_p has a system of elementary measures $\{\mu_{f,g}^p\}_{f,g \in H} \subset B_p$. Then for every $v \in R_{j_1,\ldots,j_m}$ we have

$$(S_p'(v)f, g) = (S_p(v)f, g) = \int_{\Gamma_{1,\ldots,n}} v d\mu_{f,g}^p = \int_{\Gamma_{j_1,\ldots,j_m}} v d\pi_{j_1,\ldots,j_m} \mu_{f,g}^p.$$

Therefore $\{\pi_{j_1,\ldots,j_m} \mu_{f,g}^p\}_{f,g \in H}$ is a system of elementary measures of S_p' and, by definition of B_p, these all measures are singular to $R_{j_1,\ldots,j_m}^{\perp}$. Hence by [7], Theorem 4.2, S_p' can be extended to a $*$-representation of $C(\Gamma_{j_1,\ldots,j_m})$ with spectral measure singular to $R_{j_1,\ldots,j_m}^{\perp}$. It implies that $T_{j_q}|_{H_p} = S_p(e_{j_q}) = S_p'(e_{j_q})$ are normal operators with common spectral measure singular to $R_{j_1,\ldots,j_n}^{\perp}$. Since $T_1, \ldots, T_n$ have property F then $H_p = \{0\}$. The same is true for every $p \in \{1, \ldots, l\}$. Hence the proof is complete.

4. Proof of Theorem 2. Let S: $R_{1,\ldots,n} \to L(H)$ be the representation generated by $T_1, \ldots, T_n$. We construct its extension into $B(Q_{1,\ldots,n}, R_{1,\ldots,n})$ in a following way:

If $u \in B(Q_{1,\ldots,n}, R_{1,\ldots,n})$, then there exists a bounded sequence u_k in $R_{1,\ldots,n}$ converging to u pointwise on $Q_{1,\ldots,n}$. Since by Proposition 1 every elementary measure $\mu_{f,g}$ $(f, g \in H)$ of S is an A-measure,

$$\int u_k d\mu_{f,g} \to \int u d\mu_{f,g}.$$

Therefore the value $\int u d\mu_{f,g}$ does not depend on the choice of elementary measures of f, g, and we can define the operator $u(T_1, \ldots, T_n)$ as follows:

$$(u(T_1, \ldots, T_n)f, g) \overset{\mathrm{df}}{=} \int u d\mu_{f,g}, \quad f, g \in H.$$

It is easy to see that $u(T_1, \ldots, T_n)$ is indeed a linear bounded operator on H. One can also show, following [3], Theorem 4.4, that the extension $B(Q_{1,\ldots,n}, R_{1,\ldots,n}) \ni u \to u(T_1, \ldots, T_n)$ of S is multiplicative and satisfies (1)–(4) in Theorem 2.

5. In Theorems 1 and 2 we made the assumption that $T_1, \ldots, T_n$ had property F. This assumption is necessary, as the following example shows:

EXAMPLE. Denote by D the open unit disc on the complex plane. On $(\partial D)^3$ we define a measure $\mu \overset{\text{df}}{=} m_* \times \nu$, where m_* is the Lebesgue measure on the set $\Gamma_* = \{(z_1, z_2) \in (\partial D)^2 \colon z_1 = \bar{z}_2\}$, and $d\nu = e_3 \, dm$ with $e_3(z_1, z_2, z_3) = z_3$ and m denoting the Lebesgue measure on ∂D (Davie, unpublished).

LEMMA 3. *The measure μ is orthogonal to $A(D^3)$ (algebra of all complex continuous functions on $\bar{D}^3$ which are analytic on D^3) and singular to B_0.*

Proof. Since for $u \in A(D^3)$

$$\begin{aligned}\int_{(\partial D)^3} u \, d\mu &= \int_{\Gamma_*} \Big(\int_{\partial D} u(z_1, z_2, z_3) \, d\nu(z_3) \Big) dm_*(z_1, z_2) \\ &= \int_{\Gamma_*} \Big(\int_{\partial D} z_3 u(z_1, z_2, z_3) \, dm(z_3) \Big) dm_*(z_1, z_2) \\ &= \int_{\Gamma_*} 0 \, dm_*(z_1, z_2) = 0,\end{aligned}$$

the measure μ is orthogonal to $A(D^3)$.

The set $\Gamma_* \times \partial D$ is a peak set for $A(D^3)$; consider the function $g(z_1, z_2, z_3) = e^{z_1 z_2 - 1}$. Hence, by Lemma 4 below, every measure in B_0 vanishes on it, that means μ is singular to B_0.

LEMMA 4. *If P is a peak set for a function algebra R, $x \notin P$ and μ_x is a representing measure for x, then $\mu_x|_P \equiv 0$.*

Proof. Let g be a function in R such that $g = 1$ on P and $|g(x)| < 1$ for $x \notin P$. Then

$$\mu_x(P) = \int_P d\mu_x = \lim_{k \to \infty} \int g^k \, d\mu_x = \lim_{k \to \infty} g^k(x) = 0.$$

Since μ_x is positive, $\mu_x|_P \equiv 0$.

We define a system of commuting operators T_1, T_2, T_3 on $L^2(|\mu|)$, where T_i $(i = 1, 2, 3)$ is the operator of multiplication by the function e_i.

It is obvious that the representation generated by T_1, T_2, T_3 has a system of elementary measures which are singular to B_0, because they are absolutely continuous with respect to $|\mu|$.

We can easily find the reason of this fact:

The measure m_* is singular to $A(D^3)^\perp$ which implies that (T_1, T_2) does not have the property F' and then (T_1, T_2, T_3) does not have the property F.

Remark. The measure μ in the Example shows that Theorem of my paper *A property of measures orthogonal to a tensor product of function algebras*, Bull. Acad. Polon. Sci., Ser. Sci. Math. 27 (1979), 571–575, as stated there is not true. (The same refers to one of Bekken's results published in [1] [see VIII. 1. 10].) It is necessary to assume additionally in Theorems 1 and 3 of [6] that the pair of operators has property F'.

The Example implies also that it may exist a representation of algebra $R(K_1 \times K_2)$ in $L(H)$ such that its restrictions to single algebras $R(K_i)$ ($i = 1, 2$) may be absolutely continuous. Hence part 4° of Theorem 2 [5] needs correction.

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Reçu par la Rédaction le 25. 06. 1980