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Approximation by transcendental polynomials

by JÓZEF SICIAK (Kraków)

To the memory of my teacher Franciszek Leja

Abstract. The main object of this paper is the following result. Let h be a transcendental entire function of a complex variable. Let E be a compact subset of C such that $C-E$ is connected and let f be any function holomorphic in a neighbourhood of E.

Then there exists a sequence of polynomials of two variables $\{P_k(z, w)\}$ such that $\deg P_k \leqslant k$ and

$$(+)\qquad \lim_{k\to\infty} \big(\sup_{z\in E} |f(z)-P_k(z, h(z))|\big)^{1/k} = 0.$$

If, moreover, E is infinite, we show that polynomials P_k satisfying $(+)$ may be found by interpolation. Namely, we prove that for every $k \geqslant 1$ there exist $m_k := \binom{k+2}{2}$ points $z_{k1}, \ldots, z_{km_k}$ in E with the following properties:

1° for every k there exists a polynomial $L_k(z, w)$ of degree $\leqslant k$ uniquely determined by the equations

$$L_k(z_{kj}, h(z_{kj})) = f(z_{kj}), \quad j = 1, \ldots, m_k;$$

2° the polynomials $P_k := L_k$ satisfy $(+)$.

The result is obtained by the method of the extremal function Φ_K (see [7]–[11]) defined for every compact subset K of C^n by the formula

$$(++)\qquad \Phi_K(z) := \sup_{k\geqslant 1}\big(\sup\{|P(z)|;\ P(z) = \sum_{|\alpha|\leqslant k} c_\alpha z^\alpha,\ \|P\|_K \leqslant 1\}\big)^{1/k}, \quad z\in C^n.$$

If $n = 1$, then $\log\Phi_K(z) = G(z)$ for $z \in C-\hat{K}$, where G is the Green function with pole at infinity and $\hat{K}$ denotes the polynomially convex hull of K.

1. Introduction. Let Ω be a domain (open connected set) in C^n. We say that a holomorphic map $g = (g_1, \ldots, g_m)$: $\Omega \to C^m$ is *quasientire*, if

$$(1)\qquad \underline{\varrho}(g, K) := \liminf_{\nu\to\infty} \sqrt[\nu]{\varrho_\nu(g, K)} = 0$$

for some compact subset K of Ω with $\operatorname{int} K \neq \emptyset$, where

$$(2)\qquad \varrho_\nu(g, K) := \inf\{|g-P|_K\}, \quad |g-P|_K := \max_{1\leqslant j\leqslant m} \|g_j-P_j\|_K,$$

the infimum being taken over all polynomial mappings $P\colon C^n \to C^m$ of degree $\leqslant \nu$ and $\|g_j - P_j\|_K$ denoting the supremum norm of $g_j - P_j$ on K.

It is known [1] (see also Theorem 10.1 in [10]) that g is quasientire in Ω if and only if (1) is satisfied for all compact subsets of Ω. It is obvious that every entire mapping $g\colon C^n \to C^m$ is quasientire.

We say that a holomorphic map $g\colon \Omega \to C^m$ is *transcendental*, if there is no nonzero polynomial $P\colon C^n \times C^m \ni (z, w) \to P(z, w) \in C$ with $P(z, g(z)) \equiv 0$ in Ω.

If $g(z)$ is quasientire and $P(z, w)$ is a polynomial, then the function $P(z, g(z))$ is called a *transcendental polynomial*.

The main object of this paper are the following two theorems.

THEOREM 1. *Let $g\colon \Omega \to C^m$ be a quasientire transcendental map such that the set*

$$A := \{z \in \Omega;\ g(z) = 0\} \tag{3}$$

is a 1-dimensional complex submanifold of Ω. If K is a polynomially convex compact subset of A such that $A - K$ is connected, then

$$\varrho(f, K) := \limsup_{\nu \to \infty} \sqrt[\nu]{\varrho_\nu(f, K)} = 0 \tag{4}$$

for every function $f \in O(K)$ (i.e. f holomorphic in a neighbourhood of K).

COROLLARY. *If $h\colon D \to C$ is a transcendental quasientire function in a simple connected domain $D \subset C$, then for every polynomially convex compact subset E of D and for every function $f \in O(E)$ there exists a sequence of polynomials $\{P_\nu(z, w)\}$ of two complex variables such that $\deg P_\nu \leqslant \nu$ and*

$$\lim_{\nu \to \infty} \big(\sup_{z \in E} |f(z) - P_\nu(z, h(z))|\big)^{1/\nu} = 0. \tag{5}$$

In order to get the Corollary it is sufficient to apply Theorem 1 with $m = 1$, $n = 2$, $\Omega := D \times C$, $g_1(z, w) := w - h(z)$, $A :=$ the graph of h, $K := \{(z, h(z));\ z \in E\}$ and with $f(z)$ treated as a function of two variables z, w that does not depend on w.

If $E \subset C$ is of positive logarithmic capacity, $C - E$ is connected and f is holomorphic in a neighbourhood of E but not on the whole plane C, then by Bernstein–Walsh theorem [14] the best approximation of f by polynomials satisfies

$$0 < \varrho(f, E) := \limsup_{\nu \to \infty} \sqrt[\nu]{\varrho_\nu(f, E)} < 1, \tag{6}$$

while the behaviour of the best approximation of f by the transcendental polynomials $P_\nu(z, h(z))$ is given by (5).

In the Corollary we may take in particular $D = C$ and $h(z) = e^{iz}$. Then

$$P_\nu(z, h(z)) = \sum_{j=0}^{\nu} c_j(z) e^{ijz},$$

where c_j is a polynomial in z of degree $\leqslant \nu - j$.

Transcendental polynomials $P_\nu(z, h(z))$ satisfying (5) may be found by interpolation. Namely, let e_1, $e_2, \ldots$ be a sequence composed of all monomials $z^k w^l$ with $k+l \geqslant 0$ of two complex variables $(z, w) \in C^2$ such that $\deg e_\nu \leqslant \deg e_{\nu+1}$, e.g.

$$\{e_\nu\} = \{1, z, w, z^2, zw, w^2, \ldots, z^k, z^{k-1} w, \ldots, zw^{k-1}, w^k, \ldots\}.$$

Given any system of ν points $x_1, \ldots, x_\nu$ in C^2, we define the *generalized Vandermondian V of order ν* by

$$V(x_1, \ldots, x_\nu) := \det [e_i(x_j)]_{i,j=1,\ldots,\nu}. \tag{7}$$

A system of points $\{\xi_{\nu 1}, \ldots, \xi_{\nu\nu}\} \subset K$ is called a *system of the extremal points of K* with respect to the generalized Vandermondian V of order ν, if

$$|V(\xi_{\nu 1}, \ldots, \xi_{\nu\nu})| = \max \{|V(x_1, \ldots, x_\nu)|;\ \{x_1, \ldots, x_\nu\} \subset K\}. \tag{8}$$

The definition and properties of systems of the extremal points of compact subsets of C^n may be found in [7]–[11]; see also Section 4 of this paper.

We can now state

THEOREM 2. *Let h: $D \to C$ be a transcendental quasientire function in a simple connected plane domain D. Let E be an infinite polynomially convex compact subset of D. Put $K := \{(z, h(z));\ z \in E\}$ and let $m_\nu := \binom{\nu+2}{2}$ be the number of monomials of two variables of degree $\leqslant \nu$. Then*

1° $V(\xi_1, \ldots, \xi_\nu) \neq 0$ *for every* $\nu \geqslant 1$;

2° *If $f \in O(E)$, then the interpolating polynomials*

$$L_\nu(z, w) := \sum_{j=1}^{m_\nu} f(\pi(\xi_j)) \frac{V(\xi_1, \ldots, \xi_{j-1}, x, \xi_{j+1}, \ldots, \xi_{m_\nu})}{V(\xi_1, \ldots, \xi_{m_\nu})} \tag{9}$$

with $x := (z, w)$ satisfy (5). *In the formula* (9) *π denotes the canonical projection of C^2 onto the first coordinate plane, and $\xi_1, \ldots, \xi_\nu$ is a system of the extremal points of K with respect to the generalized Vandermondian V of order ν.*

Assertion 1° will be proved in Section 4 (Proposition 4.1). Assertion 2° follows from Theorem 1 owing to the following inequality

$$|f(z) - L_\nu(z, h(z))| \leqslant (m_\nu + 1)\varrho_\nu(\tilde{f}, K), \qquad z \in E, \tag{10}$$

where $\tilde{f}(z, w) := f(z)$ and $K := \{(z, h(z));\ z \in E\}$.

Inequality (10) is a simple consequence of the definition of the extremal points of K with respect to V (see [10], p. 199).

Remark. It would be interesting to find the behaviour (as $\nu \to \infty$) of the transcendental polynomials $P_\nu(z, h(z))$ of the best approximation (or the behaviour of the interpolating polynomials (9)) of functions $f \in \mathscr{C}^k[-1, 1]$, $k \geqslant 0$, especially in the case $h(z) = e^{iz}$.

The results of this paper are obtained by the method of the extremal function Φ_K (see [7]–[11]).

It is a good opportunity to recall at this place that Franciszek Leja, to whose memory this volume of Annales Polonici Mathematici is dedicated at the occasion of his centenary, was the creator of the method of extremal points and extremal functions on the complex plane. The method appeared to be a very useful tool in dealing with many problems of the complex analysis (for the references see e.g. [5], [8]). He has also initiated extending his method to the case of several variables by introducing extremal points and extremal function of a compact subset of C^2 with respect to homogeneous polynomials of two variables ([2]–[5]). His papers and ideas were fundamental for the authors extending and applying his method to the case of C^n, $n \geq 2$. The present author had a privilege to be a student of F. Leja and he has gained a lot from the personal contacts with his Teacher during a period of more than 25 years.

In the sequel we shall need the following known

PROPOSITION 1 (Lemma 8.4 in [10]). *Let p_i $(i = 1, \ldots, m)$ be polynomials of n complex variables of degree $\leq d$. Given $R > 1$ and $t > 0$, define*

$$D_t := \{z \in C^n;\ \varphi(z) < R^t\}, \quad \text{where } \varphi(z) := \max_{1 \leq j \leq m} |p_j(z)|^{1/d}.$$

Assume D_t is bounded.

If f is holomorphic in D_t, then

$$\varrho(f, \bar{D}_s) \leq R^{s-t} \quad \text{for all } s \text{ with } 0 < s < t.$$

Proposition 1 will be used to prove the following

LEMMA 1. *Given a compact subset K of C^n the following conditions are equivalent*:

(a) $\Phi_K(z) = +\infty$ *in* $C^n - K$;

(b) $\Phi_K(z) = +\infty$ *in* $\Omega - K$, Ω *being a neighbourhood of* K;

(c) $K = \hat{K}$ *and* $\varrho(f, K) = 0$ *for every* $f \in O(K)$;

(d) $\varrho(h_a, K \cup \{a\}) = 0$ *for every point* $a \in C^n - K$, *where* h_a *is the function defined by* $h_a(z) := 0$ *on* K *and* $h_a(a) := 1$.

Remark. If $n = 1$ and if $\varrho(f, K) = 0$ for every $f \in O(K)$, then $\Phi_K(z) = +\infty$ in $C - K$. We do not know whether the same implication is true in C^n, $n \geq 2$.

By means of Lemma 1 and of Sadullaev's [13] criterion for an analytic set to be algebraic, we shall get

LEMMA 2. *Let $g\colon \Omega \to C^m$ be a quasientire map of a domain $\Omega \subset C^n$ and let A be given by* (3).

Then

(i) $\Phi_K(z) = +\infty$ *in* $\Omega - A$ *for every compact subset* K *of* A.

If, moreover, Ω is polynomially convex, g is transcendental and A is a 1-dimensional connected complex submanifold of Ω, then

(ii) $\Phi_K(z) = +\infty$ *in* $C^n - \hat{K}$ *for every compact subset* K *of* A *with connected* $A - \hat{K}$.

Theorem 1 is a direct consequence of Lemmas 2 and 1.

2. Proof of Lemma 1. The implications (a)$\Rightarrow$(b) and (c)$\Rightarrow$(d) are obvious.

(b)$\Rightarrow$(c). One may assume Ω is bounded and $\Phi_K(z) = +\infty$ on $\Omega - K$. Let $r > 0$ be a fixed positive real number such that $\Omega \subset P(0, r) := \{z \in C^n;\ |z| := \max_{1 \leqslant j \leqslant n} |z_j| < r\}$. Let G be a neighbourhood of K such that $\bar{G}$ is a compact subset of Ω. Given any $R > 4r$, by the definition of Φ_K and by assumption (b), we can find a finite system of polynomials q_j $(j = 1, \ldots, m)$ such that $\|q_j\|_K \leqslant 1$ and $\max_{1 \leqslant j \leqslant m} |q_j(z)|^{1/d_j} > R/r$ on $\bar{\Omega} - G$, where $d_j := \deg q_j$. Put $d := d_1 \cdot \ldots \cdot d_m$, $p_j := q_j^{d/d_j}$ $(j = 1, \ldots, m)$, $p_{m+j}(z) := (z_j/r)^d$ $(j = 1, \ldots, n)$, $\varphi(z) := \max_{1 \leqslant j \leqslant m+n} |p_j(z)|^{1/d}$. Then the polynomial polyhedron

$$D_s := \{z \in C^n;\ \varphi(z) < (R/r)^s\}, \quad 0 < s \leqslant 1, \tag{2.1}$$

is contained in the polydisk $P(0, R)$ and $\hat{K} \subset D_s$ $(0 < s \leqslant 1)$. Put $\Delta_0 := D_1 \cap \Omega$ and $\Delta_1 := D_1 - \Delta_0$. Then $\bar{\Delta}_0 \cap \bar{\Delta}_1 = \emptyset$, because $\bar{\Delta}_0 \subset G$ and $\Delta_1 \subset P(0, R) - \Omega$. It is also obvious that $K \subset \Delta_0$.

We are now ready to show (c). At first we claim that $\hat{K} \subset \Delta_0$. If $\hat{K} \not\subset \Delta_0$, then $K \subset \Delta_0$ and $\hat{K} \cap \Delta_1 \neq \emptyset$. The function h defined by $h(z) := 0$ on Δ_0 and $h(z) := 1$ on Δ_1 is holomorphic in D_1. Thus by Proposition 1

$$\varrho(h, \hat{K}) \leqslant \varrho(h, \bar{D}_s) \leqslant (r/R)^{1-s}, \quad 0 < s < 1.$$

Hence

$$\varrho(h, \hat{K}) \leqslant r/R < 1/4.$$

Therefore there exist a polynomial P and an integer $\nu \geqslant 1$ with $\deg P \leqslant \nu$ and with $|h(z) - P(z)| < 4^{-\nu}$ for all z in $\hat{K}$. Then $|P(z)| < 1/4$ on K and $|P(z)| > 3/4$ on $\hat{K} \cap \Delta_1$. Hence $\hat{K} \cap \Delta_1 = \emptyset$. We have obtained a contradiction which shows that $\hat{K} \subset \Delta_0$.

Since $\Phi_K(z) = +\infty$ in $\Omega - K$, $\Phi_K(z) = 1$ on $\hat{K}$ and $\hat{K} \subset \Omega$, we conclude that $K = \hat{K}$.

Now we shall prove the second statement of (c). Given any $f \in O(K)$, we may choose a bounded neighbourhood Ω of K with $f \in O(\Omega)$ and with $\Phi_K(z) = +\infty$ in $\Omega - K$. Since $\bar{\Delta}_0 \cap \bar{\Delta}_1 = \emptyset$ and $K \subset \Delta_0$, we may assume $f(z) = 1$ on Δ_1. Then by Próposition 1

$$\varrho(f, K) \leqslant \varrho(f, \bar{D}_s) \leqslant (r/R)^{1-s}, \quad 0 < s < 1,$$

which implies $\varrho(f, K) \leqslant r/R$. By the arbitrariness of $R > 4r$ we get $\varrho(f, K) = 0$.

(c) $\Rightarrow$ (a) (The proof due to T. Winiarski). Let a be a fixed point of $C^n - K$ and let P be a polynomial with $P(a) = 1 > \|P\|_K$. By (c), if $f(z) := 1/(P(z)-P(a))$, we have $\varrho(f, K) = 0$, i.e. there exists a sequence of polynomials $\{P_\nu\}$ such that $\deg P_\nu \leqslant \nu$ and $\lim_{\nu\to\infty} \|f - P_\nu\|_K^{1/\nu} = 0$. Put $\varepsilon_\nu := \left(\sup_{z\in K} |1-(P(z)-P(a))P_\nu(z)|\right)^{1/\nu}$ and $Q_\nu(z) := [1-(P(z)-P(a))P_\nu(z)]\varepsilon_\nu^{-\nu}$. Then by the definition of Φ_K

$$|Q_\nu(z)|^{1/\nu} \leqslant \Phi_K(z) \quad \text{in } C^n \text{ for all } \nu \geqslant 1.$$

In particular $1/\varepsilon_\nu = |Q_\nu(a)|^{1/\nu} \leqslant \Phi_K(a)$, $\nu \geqslant 1$, which implies that $\Phi_K(a) = +\infty$, because $\varepsilon_\nu \to 0$ as $\nu \to \infty$.

3. Proof of Lemma 2. (i) Given any compact subset K of A and any point a in $\Omega - A$, put $E := K \cup \{a\}$ and let $P: C^n \to C^m$ be a polynomial map of degree $\leqslant \nu$ such that $|g-P|_E = \varrho_\nu(g, E)$. Since $g = 0$ on A and g is quasientire, we have

$$\liminf_{\nu\to\infty} |P_\nu|_K^{1/\nu} \leqslant \liminf_{\nu\to\infty} |g-P_\nu|_E^{1/\nu} = 0 = \lim_{j\to\infty} |g-P_{\nu_j}|_E^{1/\nu_j}$$

for a subsequent $\{\nu_j\}$ of $\{\nu\}$. Since $g(a) \neq 0$, we get $\lim_{j\to\infty} |P_{\nu_j}(a)|^{1/\nu_j} = 1$. Thus we may assume $P_{\nu_j}(a) \neq 0$ for all j. If there exists j with $P_{\nu_j} \equiv 0$ on K, then

$$m|P_{\nu_j}(a)|^{1/\nu_j} \leqslant \Phi_K(a) \quad \text{for all } m \geqslant 1,$$

so $\Phi_K(a) = +\infty$. If $|P_{\nu_j}|_K \neq 0$ for all j, then

$$(|P_{\nu_j}(a)|/|P_{\nu_j}|_K)^{1/\nu_j} \leqslant \Phi_K(a), \quad j \geqslant 1,$$

which gives $\Phi_K(a) = +\infty$. The proof of (i) is concluded.

(ii) By (i) $\hat{K} \cap (\Omega - A) = \emptyset$. If Ω is polynomially convex, then $\hat{K} \subset \Omega$ and consequently $\hat{K} \subset A$ for every compact subset K of A. Since $\Phi_K(z) = +\infty$ in $\Omega - A$ and A is a 1-dimensional complex submanifold, the extremal function $\log\Phi_K$ is on $A - \hat{K}$ locally a limit of an increasing sequence of harmonic functions (see the next Proposition). By the Harnack principle either $\Phi_K = +\infty$ in $A - \hat{K}$ or $\log\Phi_K$ is harmonic in $A - \hat{K}$. The second possibility is excluded, because by Sadullaev [13] the extremal function Φ_K is not locally bounded on $A - K$, if A is transcendental. Therefore $\Phi_K(z) = +\infty$ in $\Omega - \hat{K}$, which by Lemma 1 gives (ii).

In the proof of the next Proposition we shall need the following Zaharjuta's theorem [15] (see also [10], [12]):

Let L be the class of all plurisubharmonic functions u in C^n with $\sup\{u(z) - \log(1+|z|);\ z \in C^n\} < +\infty$. Then

$$\log\Phi_K(z) = \sup\{u(z);\ u \in L,\ u \leqslant 0 \text{ on } K\}, \quad z \in C^n. \tag{3.1}$$

PROPOSITION 3.1. *Let* $g\colon \Omega \to C^m$ *be a quasientire map of a domain* $\Omega \subset C^n$ *into* C^m *such that*

$$A := \{z \in \Omega;\ g(z) = 0\}$$

is a 1-*dimensional complex submanifold of* Ω.

Then for every compact subset K *of* A *the function* $\log \Phi_K$ *is on* $A - \hat{K}$ *locally a limit of an increasing sequence of harmonic functions.*

Proof. For the simplicity of notation we shall assume (without loss of generality) that $n = 2$. Let p be a fixed point of $A - \hat{K}$. The set A is locally a graph of a holomorphic function. More precisely, we can find a coordinate system in C^2 and a holomorphic function h on a disc $D(a, R) := \{z \in C;\ |z-a| < R\}$ such that $A_R := \{(z, h(z));\ z \in D(a, R)\}$ is contained in A and $p = (a, h(a))$. We may assume that $\bar{A}_R \cap K = \emptyset$.

Let r and ϱ be fixed real numbers with $0 < r < \varrho < R$. Let u be a fixed nonnegative continuous function of the class L. Let us define a subharmonic function v on A by the formulas

$$v(z, w) := u(z, w) \quad \text{on } A - A_r, \qquad v(z, w) := v_0(z) \quad \text{on } A_r,$$

where v_0 is the harmonic function in $D_r := D(a, r)$, continuous on $\bar{D}_r$ with $v_0(z) = u(z, h(z))$ on ∂D_r. The function v is continuous and subharmonic on A, harmonic in A_r, $v \geqslant u$ on A and $v = u$ on $A - A_r$.

We shall first prove the following

Claim. Given $\varepsilon > 0$, there exists a function $f \in L$ with $f = u$ on K and $f = \max\{v - \varepsilon, u\}$ on A_R.

In order to show the Claim let G be a neighbourhood of A_R such that $G \cap (A - A_R) = \emptyset$, $G \subset \Omega$, $\{z \in C;\ (z, w) \in G\} = D(a, R)$ and $v(z, h(z)) - \varepsilon = u(z, h(z)) - \varepsilon < u(z, w)$ for all $(z, w) \in G$ with $|z-a| = \varrho$. Put $E := K \cup \bar{A}_R$. It is known [10] that there exists an increasing sequence $\{F_j\}$ of continuous functions of the class L with $\log \Phi_E(z) = \sup_j F_j$. It is obvious that $\Phi_E(z) = 1$ on E and by Lemma 2 (i) $\Phi_E(z) = +\infty$ in $\Omega - E$. Take j so large that $F_j(z, w) > v(z, h(z))$ for $(z, w) \in \partial G$ with $|z-a| \leqslant \varrho$. Then the required function f is given by

$$f(z, w) := \begin{cases} \max\{F_j(z, w), u(z, w), v(z, h(z)) - \varepsilon\}, & \text{if } (z, w) \in G,\ |z-a| < \varrho, \\ \max\{F_j(z, w), u(z, w)\}, & \text{if } (z, w) \notin G \text{ or } |z-a| \geqslant \varrho. \end{cases}$$

We are now ready to accomplish the proof of Proposition 3.1. Let $\{u_j\}$ be an increasing sequence of continuous functions of the class L with $\log \Phi_K = \sup\{u_j;\ j \geqslant 1\}$. Apply the Claim with $u = u_j$ and $\varepsilon = 1/j$. Then for every j there exists $f_j \in L$ such that $f_j = \max\{v_j - 1/j,\ u_j\}$ on A_R and $f_j = u_j$ on K, where $v_j \in L$ and v_j is harmonic on A_r. Therefore

$$f_j(z, w) \leqslant \log \Phi_K(z, w) \quad \text{in } C^2$$

and

(§) $$u_j(z, w)-1/j \leqslant v_j(z, w)-1/j \leqslant \log \Phi_K(z, w) \quad \text{on } A_r \ (j \geqslant 1).$$

The sequence $\{u_j\}$ being increasing in C^2, the sequence $\{v_j\}$ is increasing on A_r and thus $\{v_j-1/j\}$ is increasing on A_r. By (§) we get $\log \Phi_K = \sup\{v_j-1/j;\ j \geqslant 1\}$ on A_r. The proof is concluded, because v_j-1/j are harmonic on A_r.

Remark. The above proof is a modification of the Sadullaev's proof of Corollary 3.1 in [13]. Here a modification is necessary, because in [13] the set A is given by $A = \{z \in C^n;\ g_1(z) = 0, \ldots, g_m(z) = 0\}$, where g_j are polynomials.

4. Unisolvent subsets of C^n and the Lagrange interpolation formula for polynomials of n complex variables. Let α: $N \ni \nu \to \alpha(\nu) \in Z_+^n$ be a bijection such that $|\alpha(\nu)| \leqslant |\alpha(\nu+1)|$, where $|\alpha| := \alpha_1 + \ldots + \alpha_n$ for any multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in Z_+^n$. Let

$$e_\nu(z) := z^{\alpha(\nu)} = z_1^{\alpha_1(\nu)} \ldots z_n^{\alpha_n(\nu)}, \quad \nu \geqslant 1,$$

be the sequence of all monomials of n variables corresponding to the bijection α. Then every polynomial P of n variables of degree $\leqslant \nu$ can be uniquely written in the form

$$P(z) = \sum_{k=1}^{m_\nu} c_k e_k(z) \quad \text{with } m_\nu := \binom{\nu+n}{n}.$$

We say that a subset E of C^n is *unisolvent of order* ν, if there exist points $x_1, \ldots, x_\nu$ in E such that

(4.1) $$V(x_1, \ldots, x_\nu) := \det [e_j(x_l)]_{l,j=1,\ldots,\nu} \neq 0.$$

We say that E is *unisolvent*, if it is unisolvent of order ν for every $\nu \geqslant 1$. The mapping

$$(C^n)^\nu \ni (x_1, \ldots, x_\nu) \to V(x_1, \ldots, x_\nu) \in C$$

will be called *generalized Vandermondian of order* ν.

By the elementary theory of systems of linear equations, if $V(x_1, \ldots, x_\nu) \neq 0$ and $P_\nu(z) = \sum_{k=1}^{\nu} c_k e_k(z)$, then the following *Lagrange interpolation formula* is true

(4.2) $$P_\nu(z) = \sum_{k=1}^{\nu} P_\nu(x_k) \frac{V(x_1, \ldots, x_{k-1}, z, x_{k+1}, \ldots, x_\nu)}{V(x_1, \ldots, x_\nu)},$$

i.e. the polynomial P is uniquely determined by its values at the points $x_1, \ldots, x_\nu$. In particular, if P is any polynomial on C^n of degree $\leqslant \nu$ and if $x_1, \ldots, x_{m_\nu}$ is a system of m_ν points of C^n for which the determinant V of order m_ν is different from zero, then

$$(4.3)\qquad P(z) = \sum_{k=1}^{m_\nu} P(x_k)\frac{V(x_1,\ldots,x_{k-1},z,x_{k+1},\ldots,x_{m_\nu})}{V(x_1,\ldots,x_{m_\nu})},\qquad z\in C^n.$$

PROPOSITION 4.1. *Given a holomorphic map $h\colon D\to C^m$ of a plane domain D into C^m, the following conditions are equivalent:*

(i) *h is transcendental, i.e. there is no nonzero polynomial $P\colon C\times\times C^m\ni(z,w)\to P(z,w)\in C$ with $P(z,h(z))\equiv 0$ in D;*

(ii) *for every infinite subset F of D with $F'\cap D\neq\emptyset$, F' denoting the set of the limit points of F, there exists a sequence of points z_k in F such that, if $x_k := (z_k, h(z_k))$, then (4.1) is true for every $\nu\geqslant 1$;*

(iii) *for every infinite subset F of D with $F'\cap D\neq\emptyset$ the subset $E := \{(z,h(z));\ z\in F\}$ of $C\times C^m$ is unisolvent.*

Proof. (i)$\Rightarrow$(ii). Put $C^n := C\times C^m$. If $\nu=1$, then $V(x_1)=e_1(x_1)=1$ for every x_1 in F. Given $\nu\geqslant 1$, assume the points $x_1,\ldots,x_\nu$ of F are already chosen in such a way that $V(x_1,\ldots,x_\nu)$ is different from zero. Then $P_{\nu+1}(x) := V(x_1,\ldots,x_\nu,x) = V(x_1,\ldots,x_\nu)e_{\nu+1}(x)+\sum_{k=1}^{\nu} c_k e_k(x)$ is a nonzero polynomial of $x:=(z,w)\in C\times C^m$. Since h is transcendental, $P_{\nu+1}(z,h(z))\not\equiv 0$ on D and consequently, by the identity property of holomorphic functions, $P_{\nu+1}(z,h(z))\not\equiv 0$ on F. Thus there exists a point $x_{\nu+1}=(z_{\nu+1},h(z_{\nu+1}))$ in F with $P_{\nu+1}(x_{\nu+1}) = V(x_1,\ldots,x_{\nu+1})\neq 0$. By the mathematical induction the proof of (ii) is concluded.

(ii)$\Rightarrow$(iii) is obvious.

(iii)$\Rightarrow$(i). Suppose h is not transcendental. Then there exists a nonzero polynomial P with $P(z,h(z))\equiv 0$ on D. In particular $P(z,w)=0$ on the unisolvent set $E:=\{(z,h(z));\ z\in F\}$. Thus $P=0$. This is a contradiction, which concludes the proof.

5. A necessary condition for the equation: $\Phi_K(z) = +\infty$ in $C^n - K$.

PROPOSITION 5.1. *If $\Phi_K(z) = +\infty$ in C^n-K, then K satisfies the following condition:*

(e) *For every irreducible algebraic subset X of C^n with $\dim X\geqslant 1$ the extremal function $\Phi_{K\cap X}$ is not locally bounded on X.*

Proof. If (e) does not hold, then there exists an irreducible algebraic set X in C^n with $\dim X\geqslant 1$ such that $\Phi_{K\cap X}$ is locally bounded on X. Let $R>0$ be so large that $K\cap X\subset P(0,R)$. By [12] there exists a function f holomorphic in $P(0,R)$ such that if S is any irreducible component of the analytic set $X\cap P(0,R)$, then the set S is the maximal domain of existence of $f|S$ on X. By (c) of Lemma 1 there exists a sequence of polynomials $\{P_\nu\}$ such that $\deg P_\nu\leqslant\nu$ and $\lim_{\nu\to\infty}\|f-P\|_K^{1/\nu}=0$. Hence $\lim_{\nu\to\infty}\|P_\nu-P_{\nu-1}\|_K^{1/\nu}=0$. By the obvious inequalities

$$|P_\nu(z)-P_{\nu-1}(z)| \leqslant \|P_\nu - P_{\nu-1}\|_K \Phi^\nu_{K\cap X}(z), \qquad z\in X,\ \nu \geqslant 1,$$

the series $P_1+\sum_{\nu=2}^{\infty}(P_\nu - P_{\nu-1})$ is locally uniformly convergent on X. Its sum f is a holomorphic function at all regular points of X and $\tilde{f}=f$ on $K\cap X$.

The extremal function $\Phi_{K\cap X}$ being locally bounded on X, the set $K\cap X$ is not pluripolar on X (see [13]). Since a countable union of pluripolar sets on X is pluripolar on X, and the set of irregular points of X is pluripolar on X, it follows that there exists a regular point a of X such that for every $r>0$ the set $K\cap X\cap P(a, r)$ is not pluripolar on X. Let S be the irreducible component of the analytic set $X\cap P(0, R)$ such that $a\in S$. Then $\tilde{f}=f$ on S, which implies that $\tilde{f}$ is an analytic continuation of $f|S$ on X beyond S. We have got a contradiction which shows that (e) is true.

Problem. By Proposition 5.1 condition (a) of Lemma 1 implies condition (e). Is the implication (e)$\Rightarrow$(a) true?

Proposition 5.2. *Given a compact subset K of C^n, let D be a bounded domain in C^n with $\bar{D}\cap K=\emptyset$. If $\Phi_K(z)=+\infty$ on ∂D, then $\Phi_K(z)=+\infty$ in D. In particular the set*

$$\{a\in C^n-K;\ \Phi_K(z)<+\infty\}$$

contains no isolated points.

Proof. Given $\varepsilon>0$ and a point $a\in\partial D$, there exists a polynomial P_a of degree d_a such that $\|P_a\|_K\leqslant 1$ and $|P_a(z)|^{1/d_a}>1/\varepsilon$ in a neighbourhood U_a of a. By the compactness argument there is a finite system of polynomials P_j $(j=1,\ldots,s)$ such that $\|P_j\|_K\leqslant 1$ and $\max_{1\leqslant j\leqslant s}|P_j(z)|^{1/d_j}>1/\varepsilon$ on a neighbourhood of ∂D. Put

$$u(z):=\max\left\{\frac{1}{d_j}\log|P_j(z)|;\ j=1,\ldots,s\right\} \quad \text{in } C^n$$

and

$$u_\varepsilon(z):=u(z) \quad \text{on } C^n-D, \qquad u_\varepsilon(z):=\max\left\{\log\frac{1}{\varepsilon},\ u(z)\right\} \quad \text{on } D.$$

Then $u_\varepsilon\in L$, $u_\varepsilon(z)\leqslant 0$ on K. Therefore $u_\varepsilon(z)\leqslant\log\Phi_K(z)$ in C^n, and in particular $\log(1/\varepsilon)\leqslant u_\varepsilon(z)\leqslant\log\Phi_K(z)$ in D. Q.E.D.

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INSTYTUT MATEMATYKI UNIWERSYTETU JAGIELLOŃSKIEGO
KRAKÓW

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