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The formal inverse and the Jacobian Conjecture

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Abstract. In the present note we write a simple formula for the formal inverse of a polynomial mapping of the form $I+H$, where H is a homogeneous polynomial of degree 3. Some of its applications to the Keller Jacobian Conjecture are given.

1. Introduction. Let $\boldsymbol{K}$ denote either the field of complex numbers or the field of real numbers. By a polynomial automorphism (or, briefly, automorphism) of $\boldsymbol{K}^n$ we mean a one-to-one polynomial mapping of $\boldsymbol{K}^n$ onto $\boldsymbol{K}^n$ with polynomial inverse. It is well known that if F is an automorphism of $\boldsymbol{K}^n$ with inverse F^{-1}, then $\deg F^{-1} \leqslant (\deg F)^{n-1}$ (see [2]).

Let $F = (F_1, \ldots, F_n)$: $\boldsymbol{K}^n \to \boldsymbol{K}^n$ be a polynomial mapping and let $J(F)$ denote, as usual, the Jacobian matrix $(\partial F_i/\partial x_j)_{i,j=1,\ldots,n}$. When F is an automorphism, the chain rule implies that $\operatorname{Jac} F = \det J(F)$ is a non-zero constant.

The Keller Jacobian Conjecture (KJC) asserts that the converse is true, i.e., that for every natural $n \geqslant 2$, every polynomial mapping F: $\boldsymbol{K}^n \to \boldsymbol{K}^n$ with $\operatorname{Jac} F = \text{const} \neq 0$ is an automorphism. A survey of results in this direction is contained, for example, in [2].

To explain the motives for this paper, we recall two recent reductions of KJC together with the relevant definitions.

Definition 1.1. A polynomial mapping F: $\boldsymbol{K}^n \to \boldsymbol{K}^n$ is called a *(J)-endomorphism of $\boldsymbol{K}^n$* if $F = I - H$, where I denotes the identity transformation of $\boldsymbol{K}^n$ and H: $\boldsymbol{K}^n \to \boldsymbol{K}^n$ is a homogeneous polynomial of degree 3.

The mapping F is called a *(D)-endomorphism* if $F = I - D$, where $D = (d_1^3, \ldots, d_n^3)$ and $d_1, \ldots, d_n$ are $\boldsymbol{K}$-linear forms on $\boldsymbol{K}^n$ (so, (D)-endomorphisms are a special kind of (J)-endomorphisms).

A (J)-endomorphism ((D)-endomorphism) which is an automorphism is called a *(J)-automorphism* (*(D)-automorphism*).

Let us note that for every (J)-endomorphism F of $\boldsymbol{K}^n$ we have $\operatorname{Jac} F = 1 + W$, where $W \in \boldsymbol{K}[x_1, \ldots, x_n]$, $W(0) = 0$ (in consequence, $\operatorname{Jac} F = \text{const} \neq 0$

iff $\operatorname{Jac} F = 1$). Therefore, every (J)-endomorphism $F = (F_1, \ldots, F_n)$ has a formal inverse $G \in (\boldsymbol{K}[[x_1, \ldots, x_n]])^n$, i.e., there exist uniquely determined formal power series $G_1, \ldots, G_n \in \boldsymbol{K}[[x_1, \ldots, x_n]]$ without constant terms such that $F_i(G_1, \ldots, G_n) = x_i$ $(i = 1, \ldots, n)$ and $G = (G_1, \ldots, G_n)$.

Now the above mentioned reductions of KJC are summarized by the following theorem:

THEOREM 1.2. *The following statements are equivalent*:

(i) KJC *is true*.

(ii) (Jagžev's reduction; see [4] or [2]). *For every* $n \in \boldsymbol{N}$ *and any* (J)-*endomorphism* F *of* $\boldsymbol{K}^n$ *such that* $\operatorname{Jac} F = 1$, F *is an automorphism.*

(iii) (cubic linear reduction; see [3]). *For every* $n \in \boldsymbol{N}$ *and any* (D)-*endomorphism* F *of* $\boldsymbol{K}^n$ *such that* $\operatorname{Jac} F = 1$, F *is an automorphism.*

An explicit formula for the formal inverse of a (J)-endomorphism seems to be a useful tool for proving KJC in its versions (ii) or (iii) (as we already know the inverse of a (J)-automorphism of $\boldsymbol{K}^n$ is a polynomial of degree not exceeding 3^{n-1}).

Numerous such formulas are known (see, for example, [1], [2]). In the present note we fix our attention on a simple recurrence formula for the formal inverse of any (J)-endomorphism (Theorem 2.1). It has the property of being differentiation-free and permits us to calculate quickly several successive initial terms of the expansion. To show its usefulness we prove the global invertibility of some (J) and (D)-endomorphisms with constant jacobians and give the formulas for their inverses. This improves and generalizes some results of Bass, Connell, Wright [2] and Drużkowski [3].

2. The inversion formula. In the sequel we propose the following convenient notation: if $H\colon E_1 \to E_2$ is a homogeneous polynomial mapping between vector spaces E_1 and E_2 of degree k, by H we also denote the unique symmetric k-linear mapping of E_1^k into E_2 such that $H(x, \ldots, x) = H(x)$ for $x \in E_1$.

We have

THEOREM 2.1. *Let* $F = I - H$ *be a* (J)-*endomorphism of* $\boldsymbol{K}^n$ *and let* $G = \sum_{j=1}^{\infty} G_j$ *be the formal inverse of* F $(G_j\colon \boldsymbol{K}^n \to \boldsymbol{K}^n$ *is a homogeneous polynomial of degree* j). *Then*

$$G_1 = I,$$

$$G_{2k+1} = \sum_{p+q+r=k-1} H(G_{2p+1}, G_{2q+1}, G_{2r+1}), \qquad k = 1, 2, 3, \ldots,$$

$$G_{2k} = 0, \qquad k = 1, 2, 3, \ldots$$

In particular

$$G_3 = H,$$
$$G_5 = 3H(I, I, H),$$
$$G_7 = 9H(I, I, H(I, I, H)) + 3H(I, H, H),$$
$$G_9 = 27H(I, I, H(I, I, H(I, I, H))) + 9H(I, I, H(I, H, H)) + \\ + 18H(I, H, H(I, I, H)) + H(H, H, H).$$

Proof. Since $F(0) = I$, the formal inverse $G = \sum_{j=1}^{\infty} G_j$ is convergent in a neighbourhood of $0 \in \boldsymbol{K}^n$.

Moreover, the mapping F is odd (i.e., $F(x) = -F(-x)$ for $x \in \boldsymbol{K}^n$), and hence, using the identity principle, we can easily see that $G_j = 0$ for even j, i.e., $G = I + \sum_{k=1}^{\infty} G_{2k+1}$.

Let V be a neighbourhood of zero such that G is analytic in V and $F \circ G = \mathrm{id}_v$. Then

$$\begin{aligned} x &= F(G(x)) \\ &= x + \sum_{k=1}^{\infty} G_{2k+1}(x) - H\Big(\sum_{p \geq 0} G_{2p+1}(x), \sum_{q \geq 0} G_{2q+1}(x), \sum_{r \geq 0} G_{2r+1}(x)\Big) \\ &= x + \sum_{k=1}^{\infty} G_{2k+1}(x) - \sum_{p,q,r \geq 0} H(G_{2p+1}(x), G_{2q+1}(x), G_{2r+1}(x)) \end{aligned}$$

for $x \in V$.

Using the identity principle we obtain the equality of power series:

$$\sum_{k=1}^{\infty} G_{2k+1} = \sum_{p+q+r \geq 1} H(G_{2p+1}, G_{2q+1}, G_{2r+1}),$$

and hence, comparing the degrees of corresponding terms, we obtain the desired formulas:

$$G_{2k+1} = \sum_{p+q+r=k-1} H(G_{2p+1}, G_{2q+1}, G_{2r+1})$$

for $k = 1, 2, 3, \ldots$ QED.

Let us formulate the following immediate corollary, which will be very useful in applications.

COROLLARY 2.2. *Let* $F = I - H$ *be a* (J)*-endomorphism of* $\boldsymbol{K}^n$ *and let* $G = \sum_{j=1}^{\infty} G_j$ *be the formal inverse of* F. *Assume that there exists* $k \in N$ *such that*

$$G_{3^k+2} = \ldots = G_{3^{k+1}} = 0.$$

Then F is a polynomial automorphism of K^n and $\deg F^{-1} \leqslant 3^k$.

3. Applications to KJC. For every nilpotent matrix A with entries in some integral domain we define the index of A by putting $\operatorname{ind} A = \min\{p \in N\colon A^p = 0\}$. Let us recall that for every (J)-endomorphism $F = I - H$, $\operatorname{Jac} F = 1$ if and only if the matrix $J(H)$ is nilpotent (see, for example, [3], p. 308).

In view of the above it seems to be natural to try to prove KJC by induction with respect to $\operatorname{ind} J(H)$ (compare [2], Section III).

The first inductive step ($\operatorname{ind} J(H) = 2$) is given in [2]. We propose here an immediate elementary proof based on our inversion formula.

PROPOSITION 3.1 ([2], Corollary 5.4). *If $F = I - H$ is a (J)-endomorphism of K^n such that $J(H)$ is nilpotent with* $\operatorname{ind} J(H) = 2$, *then F is an automorphism with the inverse*

$$F^{-1} = I + H.$$

Proof. The condition $J(H)^2 = 0$ is equivalent to the identity

$$H(x, x, H(x, x, y)) = 0 \quad \text{for every } x, y \in K^n. \tag{1}$$

Differentiating (1) twice with respect to x, we obtain

$$H(t, u, H(x, x, y)) + 2H(x, u, H(t, x, y)) + \\ + 2H(t, x, H(u, x, y)) + H(x, x, H(t, u, y)) = 0 \tag{2}$$

for every $t, u, x, y \in K^n$.

Putting $x = y$ in (1), we have

$$H(I, I, H) = 0. \tag{3}$$

By putting $t = x = y$, $u = H(x)$ in (2), because of (3) we obtain

$$H(I, H, H) = 0. \tag{4}$$

Finally, the substitution $t = u = H(x)$, $y = x$ in (2) gives, *via* (3) and (4),

$$H(H, H, H) = 0. \tag{5}$$

Identities (3)–(5) permit us to apply Corollary 2.2, which ends the proof. QED.

In the case of (D)-endomorphisms one more step is possible.

PROPOSITION 3.2. *If $F = I - D$ is a (D)-endomorphism of K^n such that $J(D)$ is nilpotent with* $\operatorname{ind} J(D) = 3$, *then F is an automorphism with the inverse*

$$F^{-1} = I + D + 3D(I, I, D) + 3D(I, D, D) + D(D, D, D).$$

Proof. Let $D = (d_1^3, \ldots, d_n^3)$, where $d_1, \ldots, d_n$ are linear forms on K^n and let $L = (d_1, \ldots, d_n)$. It is easy to see that

$$\operatorname{Ker} L \subset \operatorname{Ker} D(u, v, \cdot) \quad \text{for every } u, v \in K^n. \tag{D.0}$$

The condition $J(D)^3 = 0$ is equivalent to the identity

$$D(x, x, D(x, x, D(x, x, y))) = 0 \quad \text{for every } x, y \in \boldsymbol{K}^n,$$

which implies

$$\text{(D.1)} \qquad L(D(x, x, D(x, x, y))) = 0 \quad \text{for every } x, y \in \boldsymbol{K}^n.$$

In particular

$$\text{(D.2)} \qquad L(D(x, x, D(x))) = 0 \quad \text{for } x \in \boldsymbol{K}^n.$$

Differentiating (D.2) with respect to x and using (D.1) we obtain

$$\text{(D.3)} \qquad L(D(x, y, D(x))) = 0 \quad \text{for every } x, y \in \boldsymbol{K}^n.$$

By (D.0) and (D.3) we have

$$\text{(D.4)} \qquad D(u, v, D(x, y, D(x))) = 0 \quad \text{for every } u, v, x, y \in \boldsymbol{K}^n.$$

Let $G = \sum_{j=1}^{\infty} G_j$ be the formal inverse of F. As a consequence of (D.4) and Theorem 2.1 we obtain

$$D(G_{2p+1}, G_{2q+1}, G_{2r+1}) = 0$$

for any non-negative integers p, q, r such that $4 \leqslant p+q+r \leqslant 13$.

Therefore

$$G_{11} = G_{3^2+2} = \ldots = G_{33} = G_{27} = 0$$

and, by Corollary 2.2, the proof is concluded. QED.

Remark 3.3. It is easy to see that, in fact, the above considerations prove that if KJC is true for (J)-endomorphisms with $\operatorname{ind} J(H) = p$ it is true for (D)-endomorphisms with $\operatorname{ind} J(D) = p+1$.

Remark 3.4. Since for every nilpotent matrix A we have the relation $\operatorname{ind} A \leqslant 1 + \operatorname{rank} A$, the statement of Proposition 3.2 is true for (D)-endomorphisms $F = I - D$ of $\boldsymbol{K}^n$ such that $\operatorname{Jac} F = 1$ and $\operatorname{rank} J(D) = 2$. Therefore Proposition 3.2 is the generalization of the result from Theorem 5 in [3] with $\operatorname{rank} J(D) = 2$.

The formulas for inverses in Proposition 3.1 and 3.2 suggest the following

CONJECTURE. For every (J)-automorphism $F = I - H$ with $\operatorname{ind} J(H) = p$ we have $\deg F^{-1} \leqslant 3^{p-1}$.

Let us note that the general upper bound for the degree of the inverse gives the validity of the Conjecture for (J)-automorphisms of $\boldsymbol{K}^n$ with $\operatorname{ind} J(H) = n$.

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