ANNALES
POLONICI MATHEMATICI
XXXVIII (1980)

Radial limits of the Poisson kernel on the classical Cartan domains

by MANFRED STOLL (Columbia)

Abstract. Let D be a classical Cartan domain of type I–IV, B the Bergman–Šilov boundary of D, and P the Poisson kernel on $D\times B$. In the paper we prove that for each domain D, there exists a constant p_D depending only on D, such that the following holds: for each $u\in B$ and every neighborhood $\mathcal{U}$ of u, there exists a finite constant M such that

$$\limsup_{r\to 1}(1-r)^{p_D}P(rt,u)\leqslant M \qquad \text{for all } t\in B-\mathcal{U},$$

and for each $a<p_D$, there exists $t\neq u$ such that $\limsup_{r\to 1}(1-r)^{a}P(rt,u)=+\infty$. The constants p are computed for each of the four types of domains.

Let D denote the unit disc in C, T the boundary of D, and $P(z,t)$ the Poisson kernel on $D\times T$. Then P satisfies

$$\frac{1-r}{1+r}\leqslant P(rt,u)\leqslant\frac{1+r}{1-r} \tag{1}$$

for all $t,u\in T$. Furthermore, given any neighborhood $\mathcal{U}$ of $u\in T$, there exists a constant M depending only on $\mathcal{U}$, such that for all $t\in T-\mathcal{U}$

$$\limsup_{r\to 1}(1-r)^{-1}P(rt,u)\leqslant M<\infty. \tag{2}$$

Harnack's inequality (1) has been extended to the classical Cartan domains by Tung in [5], and recently to irreducible bounded symmetric domains by Koranyi in [3]. In this note we prove the following generalization of (2) for the classical Cartan domains.

THEOREM. *Let D be a classical Cartan domain of type* I–IV, *B the Bergman–Šilov boundary of D, and P the Poisson kernel on $D\times B$. For each domain D, there exists a constant p_D, depending only on D, such that the following hold:*

(a) *for each $u\in B$ and every neighborhood $\mathcal{U}$ of u, there exists a constant $M<\infty$ such that*

$$\limsup_{r\to 1}(1-r)^{p_D}P(rt,u)\leqslant M \tag{3}$$

for all $t \in B - \mathcal{U}$, *and*

(b) *for every* $a < p_D$, *there exists* $t \neq u$ *such that* $\limsup_{r \to 1} (1-r)^a P(rt, u) = +\infty$.

For the domain $D_1(m, n)$ $(m \leqslant n)$, $p_1(m, n) = -n(2-m)$; for $D_2(n)$, $p_2(n) = (n-2)(n+1)/2$; for $D_3(n)$, $p_3(n) = (n-1)(n-4)/2$ for even n and $p_3(n) = n(n-5)/2$ for odd n; for $D_4(n)$, $p_4 = 0$.

1. Notation. The classical Cartan domains of type I–IV, which we will denote by D_k, $k = 1, \ldots, 4$, may be defined as spaces of matrices [2]. For a matrix z of complex entries, $z^* = \bar{z}'$ denotes the complex conjugate of the transposed matrix z', and $I^{(n)}$ denotes the identity matrix of order n. Also, for a hermitian matrix H, $H > 0$ means that H is positive definite.

The first three Cartan domains are defined by

$$D_k = \{z\colon\ I - zz^* > 0\},$$

$k = 1, 2, 3$, where, for $D_1 = D_1(m, n)$, z is a (m, n) matrix. Since the condition $I - zz^* > 0$ and $I - z^*z > 0$ are equivalent, we assume that $m \leqslant n$. For $D_2 = D_2(n)$, z is a symmetric matrix of order n, and for $D_3 = D_3(n)$, z is a skew-symmetric matrix of order n. The fourth domain, $D_4 = D_4(1, n)$ is the set of all $(1, n)$ matrices or n-dimensional vectors $(n > 2)$ of complex numbers satisfying

$$1 + |zz'|^2 - 2zz^* > 0, \qquad |zz'| < 1.$$

The (complex) dimensions of these four domains are mn, $n(n+1)/2$, $n(n-1)/2$ and n respectively.

The Bergman–Šilov boundaries B_k of the domains D_k, $k = 1, \ldots, 4$ are as follows: $B_1 = B_1(m, n)$ consists of all (m, n) matrices u satisfying $uu^* = I^{(m)}$ and $B_2 = B_2(n)$ consists of all symmetric unitary matrices of order n. $B_3 = B_3(n)$, for even n consists of all skew-symmetric unitary matrices of order n. For odd n, B_3 consists of all matrices of the form $u'Du$, where u is an arbitrary unitary matrix and

$$D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \dotplus \ldots \dotplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \dotplus 0.$$

$B_4 = B_4(1, n)$ consists of all $(1, n)$ matrices u of the form

$$u = e^{i\theta} x, \qquad xx' = 1, \qquad 0 \leqslant \theta \leqslant \pi,$$

where x is a real vector.

For each domain D, let G denote the connected component of the identity of the group of holomorphic automorphisms of D and let K denote the isotropy subgroup of G at the origin. Since K acts transitively on B and $P(rt, k\cdot u) = P(rk^{-1}\cdot t, u)$, it suffices to prove the result for a conveniently chosen $u \in B$, which we will denote by e.

Throughout the paper we will denote the unitary group of order m by $\mathscr{U}(m)$. Furthermore, the topology on B will always be the Euclidean topology in C^n restricted to B.

2. Proof of the theorem for D_1 and D_2. The Poisson kernel P_1 on $D_1 \times \times B_1$ is given by

$$P_1(z, u) = \frac{1}{V_1} \frac{[\det(I - zz^*)]^n}{|\det(I - zu^*)|^{2n}}, \tag{4}$$

where $z \in D_1(m, n)$, $u \in B_1(m, n)$ and V_1 is the Euclidean volume of B_1. For the domain D_1 we take $e = [I^{(m)}, 0]$, where 0 is the zero matrix of order $(m, n-m)$. Furthermore, for each $u \in B_1(m, n)$, we write $u = [u_1, u_2]$, where u_1 is an (m, m) matrix and u_2 is an $(m, n-m)$ matrix. Since $u \in B_1$, we have $u_1 u_1^* + u_2 u_2^* = I^{(m)}$. By (4),

$$P_1(ru, e) = \frac{1}{V_1} \frac{(1-r^2)^{mn}}{|\det(I - ru_1)|^{2n}}. \tag{5}$$

Since any (m, m) matrix is similar to an upper triangular matrix, we can choose $s \in \mathscr{U}(m)$ such that $s^* u_1 s = [t_{ij}]$, $i, j = 1, \ldots, m$, and $t_{ij} = 0$ for $j < i$. Consequently,

$$|\det(I - ru_1)| = \prod_{j=1}^{m} |1 - rt_{jj}|, \tag{6}$$

where the t_{jj} are the eigenvalues of u_1.

We now proceed to show that for any neighborhood $\mathscr{U}$ of e, there exists $\varrho > 0$, depending only on $\mathscr{U}$, such that for any $u = [u_1, u_2] \in B_1 - -\mathscr{U}$ (the complement of $\mathscr{U}$ in B_1), there is at least one eigenvalue t_{kk} of u_1 such that $|1 - t_{kk}| \geqslant \varrho$.

We define a mapping τ of $\mathscr{U}(m) \times B_1(m, n)$ onto $B_1(m, n)$ by

$$\tau(s, u) = [su_1 s^*, su_2],$$

where $s \in \mathscr{U}(m)$, $u = [u_1, u_2] \in B_1(m, n)$. Clearly $\tau(s, u) \in B_1$ for all $s \in \mathscr{U}(m)$ and all $u \in B_1$. Also, $\tau(s, e) = e$ for all $s \in \mathscr{U}(m)$ and $\tau(I, u) = u$ for all $u \in B_1$. Since matrix multiplication is continuous, τ is a continuous mapping of $\mathscr{U}(m) \times B_1$ onto B_1.

Let $\mathscr{U}$ be an arbitrary neighborhood of e in B_1. By continuity, for each $s \in \mathscr{U}(m)$ there exist neighborhoods $\mathscr{V}_s$ and $\mathscr{O}_s$ of s and e respectively such that $\tau(t, u) \in \mathscr{U}$ for all $(t, u) \in \mathscr{V}_s \times \mathscr{O}_s$. Since the unitary group $\mathscr{U}(m)$ is compact, a finite number of the $\mathscr{V}_s$, say $\mathscr{V}_{s_1}, \ldots, \mathscr{V}_{s_n}$ covers $\mathscr{U}(m)$. Let

$$\mathscr{O} = \bigcap_{j=1}^{n} \mathscr{O}_{s_j}.$$

Then $\mathcal{O}$ is an open neighborhood of e and

$$(7) \qquad [su_1s^*, su_2] \in \mathcal{U}$$

for all $s \in \mathcal{U}(m)$ and all $u = [u_1, u_2] \in \mathcal{O}$.

For each $\varepsilon > 0$, let

$$(8) \qquad \mathcal{N}_\varepsilon(e) = \Big\{u \in B_1\colon |1-u_{ii}|^2 + \sum_{j\neq i} |u_{ij}|^2 < \varepsilon^2,\ i = 1, \ldots, m\Big\}.$$

Since the collection $\{\mathcal{N}_\varepsilon(e)\}$ is a neighborhood base at e, choose $\varepsilon > 0$ such that $\mathcal{N}_\varepsilon(e) \subset \mathcal{O}$.

Let $u \in B - \mathcal{U}$ be arbitrary. Then for all $s \in \mathcal{U}(m)$, $[s^*u_1s, s^*u_2] \notin \mathcal{N}_\varepsilon(e)$. If not, then by (7) it would follow that $[u_1, u_2] \in \mathcal{U}$. Choose $s \in \mathcal{U}(m)$ such that $s^*u_1s = [t_{ij}]$, $i, j = 1, \ldots, m$, with $t_{ij} = 0$ for $j < i$. For convenience, we denote the elements of s^*u_2 by t_{ij}, $i = 1, \ldots, m$, $j = m+1, \ldots, n$. Since $[s^*u_1s, s^*u_2] = [t_{ij}] \notin \mathcal{N}_\varepsilon(e)$, there exists a k, $1 \leqslant k \leqslant m$, such that

$$(9) \qquad |1-t_{kk}|^2 + \sum_{j=k+1}^{n} |t_{kj}|^2 \geqslant \varepsilon^2.$$

Since $[t_{ij}] \in B_1$, $\sum_{j=i}^{n} |t_{ij}|^2 = 1$ for all $i = 1, \ldots, m$ and $|t_{ij}| \leqslant 1$ for all i, j. Therefore $\sum_{j=k+1}^{n} |t_{kj}|^2 = 1 - |t_{kk}|^2$. Furthermore, since $|1-t|^2 + 1 - |t|^2 \leqslant 4|1-t|$ for all t, $|t| \leqslant 1$, by (9) we obtain $|1-t_{kk}| \geqslant \frac{1}{4}\varepsilon^2$. Therefore by (6), $|\det(I-ru_1)| \geqslant (1-r)^{m-1}|1-rt_{kk}|$, and by (5),

$$P_1(ru, e) \leqslant \frac{2^{mn}}{V_1} \frac{(1-r)^{n(2-m)}}{|1-rt_{kk}|^{2n}}.$$

Let $p_1 = p_1(m, n) = -n(2-m)$. Then

$$\limsup_{r\to 1} (1-r)^{p_1} P_1(ru, e) \leqslant \frac{2^{mn}}{V_1}\left(\frac{2}{\varepsilon}\right)^{4n}$$

for all $u \in B_1 - \mathcal{U}$. Since we can always choose u such that $u_1 = \operatorname{diag}[\lambda, 1, \ldots, 1]$, $|\lambda| = 1$, $\lambda \neq 1$, $u \neq e$ and for any $a < p_1$, $\limsup_{r\to 1}(1-r)^a P(ru, e) = +\infty$, which proves the result for D_1.

For the domain D_2, the Poisson kernel P_2 is given by

$$(10) \qquad P_2(z, u) = \frac{1}{V_2} \frac{[\det(I-z\bar{z})]^{(n+1)/2}}{|\det(I-z\bar{u})|^{n+1}},$$

where $z \in D_2(n)$, $u \in B_2(n)$ and V_2 is the Euclidean volume of B_2. For D_2, we take $e = I^{(n)}$. Since B_2 consists of all symmetric unitary matrices of order n, for each $u \in B_2$, we can choose $s \in U(n)$ such that $s^*us = \operatorname{diag}[\lambda_1, \ldots, \lambda_n]$ (diagonal matrix), where the λ_i are the eigenvalues of u.

Hence by (10),

$$P_2(ru, e) = \frac{1}{V_2} \frac{(1-r^2)^{n(n+1)/2}}{[\prod_{i=1}^{n} |1-r\lambda_i|]^{n+1}}.$$

Let $\mathscr{U}$ be an arbitrary open neighborhood of e in B_2. As in the proof for D_1, there exists an open neighborhood $\mathscr{O}$ of e in B_2 such that $sus^* \in \mathscr{U}$ for all $u \in \mathscr{O}$ and all $s \in \mathscr{U}(n)$. Choose $\varepsilon > 0$ such that $\mathscr{N}_\varepsilon(e) \subset \mathscr{O}$. Then for all $u \in \mathscr{B}_2 - \mathscr{U}$, $s^*us \notin \mathscr{N}_\varepsilon(e)$ for all $s \in \mathscr{U}(n)$. Let $u \in B_2 - \mathscr{U}$ be arbitrary. Choose $s \in \mathscr{U}(n)$ such that $s^*us = \operatorname{diag}[\lambda_1, \ldots, \lambda_n]$. Hence, since $s^*us \notin \mathscr{N}_\varepsilon(e)$, there exists k such that $|1-\lambda_k| \geqslant \varepsilon$. Therefore,

$$|\det(I - rue^*)| \geqslant (1-r)^{n-1}|1-r\lambda_k|$$

and

$$P_2(ru, e) \leqslant \frac{2^{n(n+1)/2}}{V_2} \frac{(1-r)^{(n+1)(2-n)/2}}{|1-r\lambda_k|^{(n+1)}}.$$

Therefore

$$\limsup_{r\to 1} (1-r)^{p_2} P_2(ru, e) \leqslant \frac{2^{n(n+1)/2}}{V_2} \left(\frac{1}{\varepsilon}\right)^{n+1}$$

for all $u \in B_2 - \mathscr{U}$, where $p_2 = p_2(n) = (n-2)(n+1)/2$.

3. Proof of the theorem for D_3. The Poisson kernel P_3 on $D_3 \times B_3$ is given by

$$P_3(z, u) = \frac{1}{V_3} \frac{[\det(1-zz^*)]^a}{|\det(I-zu^*)|^{2a}}, \tag{11}$$

where $z \in D_3(n)$, $u \in B_3(n)$, V_3 is the Euclidean volume of B_3, and $a = (n-1)/2$ for even n and $a = n/2$ for odd n.

For n even, we let $e_n = \operatorname{diag}[E_1, \ldots, E_{n/2}]$, where

$$E_j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and for n odd, we take

$$e_n = \operatorname{diag}[E_1, \ldots, E_{(n-1)/2}, 0] = \begin{bmatrix} e_{(n-1)} & 0 \\ 0 & 0 \end{bmatrix}. \tag{12}$$

For even n, e_n is a skew-symmetric unitary matrix and for odd n, e_n satisfies,

$$e_n' = e_n^* = -e_n, \quad e_n e_n' = \begin{bmatrix} I^{(n-1)} & 0 \\ 0 & 0 \end{bmatrix}.$$

In both cases

$$[\det(I-r^2ee^*)]^a = (1-r^2)^{n(n-1)/2}. \tag{13}$$

We first consider the case where n is even. In this case, $B_3(n)$ consists of all skew-symmetric unitary matrices of order n. Consider the mapping τ of $B_3(n)$ into $\mathscr{U}(n)$ given by

$$\tau(u) = ue'.$$

Clearly τ is continuous, $\tau(u) \in \mathscr{U}(n)$ for all n, and $\tau(u) = I^{(n)}$ if and only if $u = e_n$.

Let $\mathscr{U}$ be an arbitrary open neighborhood of e_n in $B_3(n)$. Since τ is continuous and $\tau(u) = I^{(n)}$ if and only if $u = e_n$, there exists a neighborhood $\mathscr{V}$ of $I^{(n)}$ in $\mathscr{U}(n)$ such that $\mathscr{U} \cap \tau(B_3 - \mathscr{U}) = \emptyset$. Hence for all $u \in B_3 - \mathscr{U}$, $ue_n' \notin \mathscr{V}$. As in Section 2, we can choose $\mathscr{O}$, an open neighborhood of $I^{(n)}$ such that $s\mathscr{O}s^* \subset \mathscr{V}$ for all $s \in \mathscr{U}(n)$. Now choose $\varepsilon > 0$ such that $\mathscr{N}_\varepsilon(I^{(n)}) \subset \mathscr{O}$. Combining the above, we obtain that if $u \in B_3 - \mathscr{U}$, then $s^*(ue_n')s \notin \mathscr{N}_\varepsilon(I^{(n)})$. Since ue_n' is unitary, choose $s \in \mathscr{U}(n)$ such that

$$s^*(ue_n')s = \operatorname{diag}[\lambda_1, \ldots, \lambda_n].$$

Consequently, $|\det(I - rue_n')| = \prod_{j=1}^{n} |1-r\lambda_j|$, where λ_j are the eigenvalues of ue_n', and for each $u \in B_3 - \mathscr{U}$ there exists k, $1 \leqslant k \leqslant m$, such that $|1-\lambda_k| \geqslant \varepsilon$. Furthermore, as in [4], since u is skew-symmetric, $xe_n - u$ (x a variable) is skew-symmetric, and by [1], p. 481, $\det(xI - ue_n') = \det(xe_n - u)$ is a perfect square. Consequently the eigenvalues of ue_n' occur in pairs. Therefore

$$|\det(I - rue_n')| \geqslant (1-r)^{n-2}|1-r\lambda_k|^2,$$

where k is such that $|1-\lambda_k| \geqslant \varepsilon$. Hence for even n,

$$P_3(ru, e_n) \leqslant \frac{2^{n(n-1)/2}}{V_3} \frac{(1-r)^{(n-1)(4-n)/2}}{|1-r\lambda_k|^{2(n-1)}}$$

and

$$\limsup_{r\to 1} (1-r)^{p_3} P_3(ru, e_n) \leqslant \frac{2^{n(n-1)/2}}{V_3} \left(\frac{1}{\varepsilon}\right)^{2(n-1)}$$

for all $u \in B_3 - \mathscr{U}$, where $p_3(n) = (n-1)(n-4)/2$. Part (b) of the theorem is obvious.

For odd n, $B_3(n) = \{W'e_nW\colon W \text{ is unitary of order } n\}$, and e_n is given (12). Suppose $U = W'e_nW \in B_3(n)$. Let

$$W = \begin{bmatrix} w & h \\ d & w_{nn} \end{bmatrix}, \quad \text{where } w = [w_{ij}]_{i,j=1}^{n-1},\ h = [w_{1,n}, \ldots, w_{n-1,n}]',$$
$$d = [w_{n,1}, \ldots, w_{n,n-1}].$$

Since

$$e_n = \begin{bmatrix} e_{(n-1)} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad h' e_{(n-1)} h = 0, \qquad U = \begin{bmatrix} u & b \\ -b' & 0 \end{bmatrix},$$

where $u = w' e_{(n-1)} w$, $b = w' e_{(n-1)} h$. Consider

$$UU^* = \begin{bmatrix} uu^* + bb^* & -u\bar{b} \\ -(u\bar{b})^* & b'\bar{b} \end{bmatrix}.$$

Since W is unitary, $ww^* + hh^* = w^* w + d^* d = I^{(n-1)}$, and therefore

$$uu^* + bb^* = w' e_{(n-1)} (ww^* + hh^*) e'_{(n-1)} \overline{w} = w' \overline{w}.$$

Since $w'\overline{w} + d'\overline{d} = I^{(n-1)}$, each element on the diagonal of $w'\overline{w}\,(\sum_{i=1}^{n-1} |w_{ij}|^2$, $j = 1, \ldots, n-1)$ is less than or equal to one. Hence the same is true for $uu^* + bb^*$, and consequently also for uu^*. Therefore if

$$U = \begin{bmatrix} u & b \\ -b' & 0 \end{bmatrix} \in B_3(n),$$

and if $u = [u_{ij}]_{i,j=1}^{n-1}$, then

$$\sum_{j=1}^{n-1} |u_{ij}|^2 \leqslant 1, \qquad i = 1, \ldots, n-1. \tag{12}$$

Next we consider the mapping τ of $B_3(n)$ into the space of $(n-1, n-1)$ matrices (with the euclidean topology) given by

$$\tau\colon\; U = \begin{bmatrix} u & b \\ -b' & 0 \end{bmatrix} \to u e'_{(n-1)}.$$

Clearly τ is continuous. Furthermore $\tau(U) = I^{(n-1)}$ if and only if $U = e_n$. To see this, we note that $\tau(U) = I^{(n-1)}$ if and only if

$$U = \begin{bmatrix} e_{n-1} & b \\ -b' & 0 \end{bmatrix}.$$

However, by the above, every diagonal element of $e_{(n-1)} e^*_{(n-1)} + bb^* = I^{(n-1)} + bb^*$ must be less than or equal to one. Hence every element on the diagonal of bb^* is zero. Consequently $b = 0$ and $U = e_n$.

Let $\mathscr{U}$ be an arbitrary open neighborhood of e_n in B_3. Since B_3 is compact and τ is continuous $\tau(B_3 - \mathscr{U})$ is a closed set not containing $I^{(n-1)}$. Hence there exists an open neighborhood $\mathscr{V}$ of $I^{(n-1)}$ such that $\mathscr{V} \cap \tau(B_3 - \mathscr{U}) = \emptyset$. As in Section 2, since $\mathscr{U}(n-1)$ is compact and the mapping $(s, v) \to s^* v s$ is continuous, there exists an open neighborhood $\mathscr{O}$ of $I^{(n-1)}$ such that $s^* \mathscr{O} s \subset \mathscr{V}$ for all $s \in \mathscr{U}(n-1)$. As in (8) we let

$$\mathscr{N}_\varepsilon(I^{(n-1)}) = \Big\{v\colon\; |1 - v_{ii}|^2 + \sum_{j \neq i} |v_{ij}|^2 < \varepsilon^2,\; i = 1, \ldots, n-1\Big\}.$$

Choose $\varepsilon > 0$ such that $\mathcal{N}_\varepsilon(I^{(n-1)}) \subset \mathcal{O}$. Then for all

$$U = \begin{bmatrix} u & b \\ -b' & 0 \end{bmatrix} \in B_3 - \mathcal{U}, \qquad s(ue'_{n-1})s^* \notin \mathcal{N}_\varepsilon(I^{(n-1)}).$$

Let

$$U = \begin{bmatrix} u & b \\ -b' & 0 \end{bmatrix} \in B_3 - \mathcal{U},$$

and choose $s \in \mathcal{U}(n-1)$ such that $s(ue'_{n-1})s^* = t = [t_{ij}]$ is upper triangular. Then

$$|\det(I^n - rUe'_n)| = |\det(I^{(n-1)} - rue'_{n-1})| = \prod_{j=1}^{n-1} |1 - rt_{jj}|,$$

where the t_{jj} are the eigenvalues of ue'_{n-1}. Since $t \notin \mathcal{N}_\varepsilon(I^{(n-1)})$, there exists k, $1 \leqslant k \leqslant n-1$, such that

$$|1 - t_{kk}|^2 + \sum_{j=k+1}^{n-1} |t_{kj}|^2 \geqslant \varepsilon^2. \tag{15}$$

Let

$$S = \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $S \in \mathcal{U}(n)$ and therefore $SUS' \in B_3(n)$. Since

$$SUS' = \begin{bmatrix} sus' & sb \\ -b's' & 0 \end{bmatrix}$$

and $(sus')(sus')^* = suu^*s^* = (sue'_{n-1}s^*)(se_{n-1}u^*s^*) = tt^*$, by (14)

$$\sum_{j=i}^{n-1} |t_{ij}|^2 \leqslant 1, \quad i = 1, \ldots, n-1. \tag{16}$$

Therefore, by (15) and (16), if $1 \leqslant k \leqslant n-2$

$$|1 - t_{kk}|^2 + 1 - |t_{kk}|^2 \geqslant \varepsilon^2$$

and consequently $|1 - t_{kk}| \geqslant \frac{1}{4}\varepsilon^2$. If $k = n-1$, then we obviously have $|1 - t_{kk}| \geqslant \varepsilon \geqslant \frac{1}{4}\varepsilon^2$. Therefore, for all

$$U = \begin{bmatrix} u & b \\ -b' & 0 \end{bmatrix} \in B_3 - \mathcal{U},$$

there exists at least one eigenvalue t_{kk} of ue'_{n-1} such that $|1 - t_{kk}| \geqslant \frac{1}{4}\varepsilon^2$. However, as for the even case, since u is skew-symmetric, the eigenvalues of ue'_{n-1} occur in pairs. Therefore,

$$|\det(I^{(n)} - rUe'_n)| \geqslant (1-r)^{n-3}|1 - rt_{kk}|^2.$$

Therefore by (11) and (13)

$$P_3(rU, e_n) \leqslant \frac{2^{n(n+1)/2}}{V_3} \frac{(1-r)^{n(5-n)/2}}{|1-rt_{kk}|^{2n}},$$

and hence for all $U \in B_3 - \mathcal{U}$,

$$\limsup_{r\to 1} (1-r)^{p_3} P_3(rU, e_n) \leqslant \frac{2^{n(n+1)/2}}{V_3} \left(\frac{4}{\varepsilon^2}\right)^{2n},$$

where $p_3(n) = n(n-5)/2$ (n odd).

Using diagonal matrices for W, then if $U = W' e_n W$,

$$U = \operatorname{diag}[D_1, \ldots, D_{(n-1)/2}, 0],$$

where

$$D_j = \begin{bmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{bmatrix}, \qquad |\lambda_j| = 1,$$

and for such a U, $Ue_n' = \operatorname{diag}[\lambda_1, \lambda_1, \ldots, \lambda_a, \lambda_a, 0]$, $a = (n-1)/2$, and

$$P_3(rU, e_n) = \frac{1}{V_3} \frac{(1-r^2)^{n(n-1)/2}}{\prod_{j=1}^{(n-1)/2} |1-r\lambda_j|^{2n}}.$$

With the appropriate choice for λ_j, part (b) of the theorem follows.

4. Proof of the theorem for D_4. On $D_4(n)$, the Poisson kernel is given by

$$P_4(z, u) = \frac{1}{V_4} \frac{(1+|zz'|^2 - 2\bar{z}z')^{n/2}}{|1+zz'\bar{u}\bar{u}' - 2zu^*|^n},$$

where $z \in D_4(n)$, $u \in B_4(n)$, and V_4 is the volume of B_4. Here we take $e = (1, 0, \ldots, 0)$. Then for $u = e^{i\theta}x$, $x = (x_1, \ldots, x_n)$, $xx' = 1$, $0 \leqslant \theta \leqslant \pi$,

$$P_4(ru, e) = \frac{1}{V_4} \left[\frac{(1-r^2)^2}{|1+r^2 e^{2i\theta} - 2re^{i\theta}x_1|^2}\right]^{n/2}.$$

For $x_1 \neq \cos\theta$,

$$\lim_{r\to 1} (1-r)^{-n} P_4(ru, e) = \frac{1}{V_4} \frac{1}{|\cos\theta - x_1|^n}. \tag{17}$$

Consider the case where $x_1 = \cos\theta \neq 1$. Since

$$|1+r^2 e^{2i\theta} - 2re^{i\theta}x_1|^2 = (1-r^2)^2 \sin^2\theta + \big((1+r^2)\cos\theta - 2rx_1\big)^2,$$

$$\lim_{r\to 1} P_4(ru, e) = \frac{1}{V_4} (1-x_1^2)^{-n/2}. \tag{18}$$

For $x_1 = \cos\theta = 1$ $(u = e)$, $\lim\limits_{r\to 1} P(ru, e) = +\infty$. Combining (17) and (18), we obtain that for all $u \neq e$,

$$\lim_{r\to 1} P_4(ru, e) = \begin{cases} 0, & x_1 \neq \cos\theta, \\ \dfrac{1}{V_4}(1-x_1^2)^{-n/2}, & x_1 = \cos\theta. \end{cases} \tag{19}$$

For $\varepsilon > 0$, define $\mathcal{N}_\varepsilon(e)$ by

$$N_\varepsilon(e) = \Big\{u = e^{i\theta}x \colon |e^{i\theta}x_1 - 1|^2 + \sum_{j=2}^{n} |x_j|^2 < \varepsilon^2\Big\}.$$

Then for any $u \in B_4$, $u \notin \mathcal{N}_\varepsilon(e)$, $u = e^{i\theta}x$,

$$\varepsilon^2 \leqslant |e^{i\theta}x_1 - 1|^2 + \sum_{j=2}^{n} |x_j|^2 = |e^{i\theta}x_1 - 1|^2 + 1 - x_1^2 = 2(1 - x_1\cos\theta).$$

Consequently if $u \notin \mathcal{N}_\varepsilon(e)$,

$$\limsup_{r\to 1} P_4(ru, e) \leqslant \begin{cases} 0, & x_1 \neq \cos\theta, \\ \dfrac{1}{V_4}\left(\dfrac{2}{\varepsilon^2}\right)^{n/2}, & x_1 = \cos\theta. \end{cases}$$

Hence for D_4, $p_4 = 0$.

Taking $u = e^{i\theta}(\cos\theta, \sin\theta, 0, \ldots, 0)$, $\theta \neq 0$, shows that

$$\limsup_{r\to 1} (1-r)^a P_4(ru, e) = +\infty$$

for all $a < 0$, which proves the result.

References

[1] L. K. Hua, *On the theory of automorphic functions of a matrix variable*, I, *Geometrical basis*, Amer. J. Math. 66 (1944), p. 470–488.

[2] — *Harmonic analysis of functions of several complex variables in the classical domains*, Amer. Math. Soc., Rhode Island, 1963.

[3] A. Koranyi, *Inequalities for Poisson kernels on symmetric spaces*, Proc. Amer. Math. Soc. 43 (1974), p. 465–469.

[4] J. Mitchell and K. T. Hahn, *Representation of linear functionals on H^P spaces over bounded star-shaped circular domains in C^n*, lecture notes of talk given by J. Mitchell at Differential Geometry Summer Institute Stanford, 1973.

[5] S. H. Tung, *Harnack's inequalities on the classical Cartan domains*, Pacific J. Math. 16 (1966), p. 373–378.

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF SOUTH CAROLINA
COLUMBIA, U.S.A.

Reçu par la Rédaction le 10. 12. 1977