ANNALES
POLONICI MATHEMATICI
XXIII (1970)

A continuation theorem for holomorphic mapping into a Hilbert space

by M. SKWARCZYŃSKI (Warszawa)

Introduction. Let us consider a domain D in the space of n complex variables. Denote by $L^2H(D)$ the space of all holomorphic functions which are square integrable with respect to the Lebesque measure in D. Let $K_D(z, \bar{t})$ be the Bergman function of D (see [1]). An example of anti-holomorphic mapping into a Hilbert space is furnished by the imbedding $\mathscr{X}: D \to L^2H(D)$ given by $\mathscr{X}(t)(z) = K_D(z, \bar{t})$ (see [1] and [3]). Bremermann in [2] has studied the necessary and sufficient conditions for the mapping $\mathscr{X}$ to possess an analytic continuation into a larger domain $G, D \subset G$. The aim of the present note is to extend Bremermann's result to the case of an arbitrary holomorphic mapping.

1. Preliminary remarks.

1.1. A mapping $\mathscr{X}: D \to H$ of a domain $D \subset C^n$ into a Hilbert space H is called *holomorphic* if for every $f \in H$, $(\mathscr{X}(z), f)$ is a holomorphic function of z in the domain D.

1.2. If $\mathscr{X}: D \to H$ is holomorphic, then the function $k(z, \bar{t}) = (\mathscr{X}(z), \mathscr{X}(t))$ is holomorphic in $D \times D^*$, where $D^* = \{\bar{z}: z \in D\}$ and the function $k(z, \bar{z}) = \|\mathscr{X}(z)\|^2$ is real-analytic in D.

1.3. If $k(z, \bar{z})$ is a real-analytic function in a domain G, then there exists a neighborhood U of the "diagonal" $\{(z, \bar{t}) \in G \times G^*: z = t\}$ and the unique extension of $k(z, \bar{z})$ to the holomorphic in U function $k(z, \bar{t})$.

2. Continuation theorem. We state the following

THEOREM 2.1. *Consider a holomorphic mapping into a separable Hilbert space H*

$$\mathscr{X}: D \to H, \quad D \subset C^n.$$

Suppose that the function $\|\mathscr{X}(z)\|^2$ possesses a real-analytic extension to a larger domain $G, D \subset G$. Then the mapping $\mathscr{X}(z)$ can be continued analytically along an arbitrary path in G.

Proof. Denote by $k(z, \bar{z})$ the real-analytic extension of $\|\mathscr{X}(z)\|^2$

defined in G. According to 1.3, $k(z, \bar{z})$ extends to a holomorphic function $k(z, \bar{t})$ on U. Consider now an arbitary path

$$\gamma(s) \in G, \quad 0 \leqslant s \leqslant 1, \quad \gamma(0) = z^1 \in D.$$

We can cover γ by a sequence of polydiscs Q_i $(i = 1, 2, \ldots, m)$ in such a way that:

1° The radius r_i of each polydisc is smaller than the distance in the norm $|z| = \max_i |z_i|$ of the set

$$\Gamma = \{(z, \bar{t}) \in G \times G^*: z = \gamma(s), \ \bar{t} = \overline{\gamma(s)}, \ 0 \leqslant s \leqslant 1\}$$

to the boundary of U.

2° The center of polydisc Q_i is $z^i = \gamma(s^i)$, and $0 = s^1 < s^2 < \ldots < s^m = 1$.

3° $\gamma(s) \in Q_i$ for $s^i \leqslant s \leqslant s^{i+1}$, $i = 0, 1, \ldots, m-1$.

We shall show first that $\mathscr{X}(z)$ possesses a holomorphic extension to Q_1. By repetition of this argument the possibility of further extensions to polydiscs $Q_2, \ldots, Q_m$ will then follow. As a result we obtain a continuation of $\mathscr{X}(z)$ along γ.

Since $k(z, \bar{t})$ is holomorphic in a neighborhood of the polydisc

$$P_1 = \{(z, \bar{t}) \in G \times G^*: |z - z^1| \leqslant r_1, \ |\bar{t} - \bar{z}^1| \leqslant r_1\},$$

it can be developed into the power series convergent uniformly and absolutely in P_1

$$k(z, \bar{t}) = \sum_{p,q=0} \frac{1}{p!}\frac{1}{q!}(z - z^1)^p (\bar{t} - \bar{z}^1)^q \left.\frac{\partial^{p+q} k(z, \bar{t})}{\partial z^p \partial \bar{t}^q}\right|_{\substack{z=z^1 \\ \bar{t}=\bar{z}^1}}. \tag{2.2}$$

It follows that there exists a constant M such that for all p, q

$$\left|\frac{1}{p!}\frac{1}{q!} k^{p,q}\right| r_1^{p+q} < M, \tag{2.3}$$

where $k^{p,q}$ is an abbreviation for the corresponding partial derivative.

Let us now fix a complete orthonormal system φ_ν, $\nu = 1, 2, \ldots$, in H. The Fourier coefficients $a_\nu(z)$ in the norm-convergent power series

$$\mathscr{X}(z) = \sum_{\nu=1} a_\nu(z) \varphi_\nu \tag{2.4}$$

are holomorphic in D. Since $k(z, \bar{z})$ is continuous in D, the series

$$k(z, \bar{z}) = \sum_{\nu=1} |a_\nu(z)|^2 \tag{2.5}$$

converges locally uniformly in D.

We can now apply the Schwarz inequality to obtain the locally uniform convergence of the series

$$k(z,\bar{t}) = \sum_{\nu=1} a_\nu(z)\overline{a_\nu(t)} \tag{2.6}$$

in $D\times D^*$.

Differentiating (2.6) at $(z^1, \bar{z}^1)\in D\times D^*$ term by term and substituting the result into (2.3) we obtain

$$\Big|\sum_{\nu=1} a_\nu^p \bar{a}_\nu^q\Big| r_1^{p+q} < M,$$

where

$$a_\nu^p = \frac{1}{p!}\frac{\partial^p a_\nu(z)}{\partial z^p}\bigg|_{z=z^1}.$$

In particular, for $p = q$

$$\sum_{\nu=1} |a_\nu^p|^2 r_1^{2p} < M$$

holds. Using this estimate and the Schwarz inequality, we can prove exactly as in [2] that

1° The Taylor series for $a_\nu(z)$ at z^1 converges in Q_1, and therefore $a_\nu(z)$ possesses an extension to the polydisc Q_1.

2° Series (2.6) converges locally uniformly and absolutely in P_1.

We see that $\mathscr{X}(z)$ has an extension to Q_1 given by (2.4) for $z\in Q_1$. Now the whole argument can be repeated for Q_1, Q_2, z^2 instead of D, Q_1, z^1, and so on. After a finite number of steps we shall continue $\mathscr{X}(z)$ into a neighborhood of the point $\gamma(1)$.

The proof is completed.

COROLLARY 2.7. *Under the assumptions of Theorem* 2.1, *the function* $k(z,\bar{t})$ *possesses a holomorphic continuation along any path in the domain* $G\times G^*$.

Proof. This continuation is given by (2.6).

In general, however, the continuation of $\mathscr{X}(z)$ into the domain G may not be univalent. We end our considerations with the following

EXAMPLE. Let $D = \{z\colon |z|<1,\ \mathrm{Im}\, z>0\}$. Let $H = C^1$, $\mathscr{X}(z) = \sqrt{z}$, $\mathscr{X}(\frac{1}{2}e^{i\pi/2}) = \frac{1}{2}e^{i\pi/4}$. The real-analytic function $\|\mathscr{X}(z)\|^2 = |z|$ possesses an analytic extension to the domain $G = \{z\colon z\neq 0\}$. However, the continuation of $\sqrt{z}$ is not univalent in G.

References

[1] S. Bergman, *The kernel function and conformal mapping*, Math. Surveys No. 5, Amer. Math. Soc. (1950).

[2] H. Bremermann, *Holomorphic continuation of the kernel function and the Bergman metric in several complex variables*, Univ. of Michigan Press, Ann. Arbor (1955), p. 349–383.

[3] M. Skwarczyński, *The invariant distance in the theory of pseudoconformal transformations and the Lu Qi-keng conjecture*, Proc. Amer. Math. Soc. 22 (2) (1969), p. 305–310.

Reçu par la Rédaction le 10. 3. 1969