ANNALES
POLONICI MATHEMATICI
LI (1990)

Approximate fixed points for nonexpansive mappings in uniformly convex spaces

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Abstract. Let X be a uniformly convex Banach space and K a nonempty bounded closed and convex subset of X. For $\varepsilon > 0$, let $\mathfrak{T}_\varepsilon$ denote the family of all nonexpansive mappings defined on K and taking values in an ε-neighborhood of K, and let $\mathfrak{T}$ denote the family of all nonexpansive self-mappings of K. It is shown that if $T \in \mathfrak{T}_\varepsilon$ and if $d = \operatorname{diam}(K)$, then

$$\inf\{\|x-T(x)\|:\ x \in K\} \leqslant (d+2\varepsilon)\delta^{-1}(2\varepsilon/(d+2\varepsilon))+\varepsilon,$$

where δ denotes the modulus of convexity of X. It is also shown that if $n \in N$ is chosen so that $(1-\delta(2\varepsilon/d))^n \leqslant \varepsilon/d$, then for each $T \in \mathfrak{T}$ and $x \in K$, $\|S^n(x)-S^{n+1}(x)\| \leqslant \varepsilon$, where $S = (1/2)(I+T)$.

1. Notation and definitions. Throughout the paper, X will denote a Banach space. For $x \in X$ and $r > 0$, let $B(x; r)$ denote the closed ball centered at x with radius r. The *modulus of convexity* of X is the function δ: $[0,2] \to [0,1]$ defined by

$$\delta(\varepsilon) = \inf\{1-(1/2)\|x+y\|:\ x, y \in B(0; 1),\ \|x-y\| \geqslant \varepsilon\}.$$

The space X is said to be *uniformly convex* ([1]) if $\delta(\varepsilon) > 0$ for each $\varepsilon > 0$. For such spaces it is known that the function δ is continuous and strictly increasing on $[0,2)$ (see [4], [9]). Also:

(1.1) $t\delta(\varepsilon/t)$ increases as $t > 0$ decreases (with ε fixed; see [5]).

We shall use the following routine consequence of the definition of δ (cf. [10]).

(1.2) If $u, v \in X$ satisfy $\|u\| \leqslant \varrho$, $\|v\| \leqslant \varrho$ and $\|u-v\| \geqslant \varepsilon$, then

$$(1/2)\|u+v\| \leqslant \varrho(1-\delta(\varepsilon/\varrho)).$$

For $K \subset X$ and $x \in X$, let

$$\operatorname{dist}(x, K) = \inf\{\|x-u\|:\ u \in K\}; \quad \operatorname{diam}(K) = \sup\{\|u-v\|:\ u, v \in K\};$$

and for $\varepsilon > 0$, let $N_\varepsilon(K)$ denote the closed ε-neighborhood of K:

$$N_\varepsilon(K) = \{x \in X:\ \operatorname{dist}(x, K) \leqslant \varepsilon\}.$$

Recall that a mapping $T\colon K\to X$ is said to be *nonexpansive* if $\|T(u)-T(v)\|\leqslant\|u-v\|$ for each $u,v\in K$. (Much of the extensive fixed point theory for nonexpansive mappings is summarized in [6], [7].)

2. Approximate fixed points for approximate self-maps. Let K be a fixed bounded closed and convex subset of a Banach space ($K\neq\emptyset$) and for $\varepsilon>0$ let $\mathfrak{T}_\varepsilon$ denote the family of all nonexpansive mappings T for which $T\colon K\to N_\varepsilon(K)$. Let

$$\varrho(\varepsilon)=\sup_{T\in\mathfrak{T}_\varepsilon}\{\inf\{\|x-T(x)\|:\ x\in K\}\}.$$

In [8] it is shown that if K has a nonempty interior and if $\bar{r}=\sup\{r>0\colon B(x;r)\subset K$ for some $x\in K\}$, then

$$\varrho(\varepsilon)\leqslant\left[\frac{\operatorname{diam}(K)-\bar{r}+\varepsilon}{\bar{r}+\varepsilon}\right]\varepsilon. \tag{1}$$

(Note that this implies $\varrho(\varepsilon)=\varepsilon$ if K itself is a ball.) For arbitrary K, the following qualitative result is also proved in [8].

$$\varrho(\varepsilon)\to 0\quad\text{as } \varepsilon\to 0^+. \tag{2}$$

In this section we show that the above results may be further refined if X is assumed to be uniformly convex. Specifically, we have:

THEOREM 1. *Let X be a uniformly convex Banach space with modulus of convexity δ, let K a nonempty bounded closed and convex subset of X with* $\operatorname{diam}(K)=d$, *and let $\mathfrak{T}_\varepsilon$ denote the family of all nonexpansive mappings T for which $T\colon K\to N_\varepsilon(K)$. Then for $T\in\mathfrak{T}_\varepsilon$,*

$$\inf\{\|x-T(x)\|:\ x\in K\}\leqslant(d+2\varepsilon)\delta^{-1}\left[\frac{2\varepsilon}{d+2\varepsilon}\right]+\varepsilon.$$

Proof. Let $\mathfrak{K}$ denote the family of all nonempty closed and convex subsets of K with the property $C\in\mathfrak{K}$ if and only if $T(C)\subset N_\varepsilon(C)$. Partially order $\mathfrak{K}$ by set inclusion and let $\mathfrak{C}=\{C_\alpha\colon \alpha\in A\}$ be any descending chain in $\mathfrak{K}$. The set $C_0=\bigcap_\alpha C_\alpha$ is closed convex and nonempty (by weak compactness). Also, if $x\in C_0$, then for each α there exists $x_\alpha\in C_\alpha$ such that $\|x_\alpha-T(x)\|\leqslant\varepsilon$. Since K is weakly compact, the net $\{x_\alpha\colon \alpha\in A\}$ has a subnet which converges weakly, say to w, with $w\in C_0$. Since $\|w-T(x)\|\leqslant\sup_\alpha\|x_\alpha-T(x)\|\leqslant\varepsilon$, it follows that $T(C_0)\subset N_\varepsilon(C_0)$, i.e., $C_0\in\mathfrak{K}$. This proves that every chain in $\mathfrak{K}$ has a lower bound, so by Zorn's Lemma $\mathfrak{K}$ has a minimal element K_0. Also, since

$$\inf\{\|x-T(x)\|:\ x\in K\}\leqslant\inf\{\|x-T(x)\|:\ x\in K_0\},$$

we may assume at the outset that K itself is minimal (with $\operatorname{diam}(K)=d'\leqslant d$).

Now let z and r denote, respectively, the Chebyshev center and radius of K. (Thus $z \in K$ and $B(z;\, r)$ is the smallest ball centered at any point of K which contains K.) Since X is uniformly convex, $d/2 \leqslant r < d$. Set

$$H = B(T(z);\, r+\varepsilon) \cap K.$$

We claim $H = K$. To see this it suffices, by minimality of K, to prove $T(H) \subset N_\varepsilon(H)$. Let $x \in H$. By assumption there exists $y \in K$ such that $\|T(x) - y\| \leqslant \varepsilon$. Thus

$$\|T(z)-y\| \leqslant \|T(z)-T(x)\| + \|T(x)-y\| \leqslant \|z-x\| + \|T(x)-y\| \leqslant r+\varepsilon,$$

proving $y \in H$. Hence $T(x) \in N_\varepsilon(H)$, establishing the claim.

Since $K = H = B(T(z); r+\varepsilon) \cap K$, we conclude $K \subset B(T(z);\, r+\varepsilon)$. Also, by assumption, there exists $p \in K$ such that $\|p - T(z)\| \leqslant \varepsilon$, so it follows that $K \subset B(p;\, r+2\varepsilon)$. Also, since r is the Chebyshev radius of K there exists $x \in K$ such that

$$r \leqslant \left\| \frac{p+z}{2} - x \right\|.$$

We now have $\|x-p\| \leqslant r+2\varepsilon$ and $\|x-z\| \leqslant r+2\varepsilon$. By (1.2),

$$\left\| \frac{p+z}{2} - x \right\| \leqslant \left[1 - \delta\left[\frac{\|p-z\|}{r+2\varepsilon} \right] \right](r+2\varepsilon).$$

Therefore (assuming $\|z - T(z)\| \geqslant \varepsilon$),

$$\delta\left[\frac{\|z-T(z)\| - \varepsilon}{r+2\varepsilon} \right] \leqslant \delta\left[\frac{\|p-z\|}{r+2\varepsilon} \right] \leqslant 1 - \frac{r}{r+2\varepsilon} = \frac{2\varepsilon}{r+2\varepsilon}.$$

Consequently,

$$(3) \qquad (d+2\varepsilon)\,\delta\left[\frac{\|z-T(z)\| - \varepsilon}{d+2\varepsilon} \right] \leqslant (r+2\varepsilon)\delta\left[\frac{\|z-T(z)\| - \varepsilon}{r+2\varepsilon} \right] \leqslant 2\varepsilon.$$

It follows that

$$\|z - T(z)\| \leqslant (d+2\varepsilon)\delta^{-1}\left[\frac{2\varepsilon}{d+2\varepsilon} \right] + \varepsilon.$$

3. Uniform iteration. It is shown in [5] (cf. also [2]) that if K is any nonempty bounded closed and convex subset of a Banach space and if $\varepsilon > 0$, then there exists an integer N such that if $T\colon K \to K$ is nonexpansive, if $x_0 \in K$, and if $n \geqslant N$, then $\|S^n(x_0) - S^{n+1}(x_0)\| \leqslant \varepsilon$, where S denotes the mapping $\frac{1}{2}(I+T)$. The proof given in [5] is purely an existence proof offering no estimate on the magnitude of N. Indeed, it seems unlikely that such estimates would be easy to obtain in general settings. Although the problem appears to have received little attention, it would appear to be a tractable one in special settings.

(In a conversation with the first author, J. Alexander observed that it suffices to take $N \geqslant \varepsilon^{-1}-2$ if K is the interval $[0, 1]$ in $\boldsymbol{R}^1$.)

Here we give an estimate for N in terms of the modulus of convexity of a uniformly convex space.

THEOREM 2. *Let X be a uniformly convex Banach space with modulus of convexity δ, let K be a nonempty bounded close and convex subset of X with* $\operatorname{diam}(K) = d$, *and let $\varepsilon > 0$ ($\varepsilon \leqslant d/2$). If $T\colon K \to K$ is nonexpansive and if $S = (1/2)(I+T)$, then for any $x \in K$, $\|S^n(x)-S^{n+1}(x)\| \leqslant \varepsilon$ for all $n \in N$ satisfying*

$$(1-\delta(2\varepsilon/d))^n \leqslant \varepsilon/d. \tag{4}$$

Proof. Under the assumptions of the theorem it is well known that T (hence S) has at least one fixed point $p \in K$. Suppose n satisfies (4). Since $\{\|S^k(x)-S^{k+1}(x)\|\}$ is monotone decreasing, if $\|S^k(x)-S^{k+1}(x)\| \leqslant \varepsilon$ holds for some $k < n$ there is nothing to prove. So we may assume

$$\varepsilon < \|S^{n-1}(x)-S^n(x)\| \leqslant \|S^{n-2}(x)-S^{n-1}(x)\| \leqslant \ldots \leqslant \|x-S(x)\| \leqslant d. \tag{5}$$

Now, $\|S^{n-1}(x)-S^n(x)\| > \varepsilon$ implies $\|S^{n-1}(x)-T(S^{n-1}(x))\| > 2\varepsilon$. Also, we have $\|T(S^{n-1}(x))-p\| \leqslant \|S^{n-1}(x)-p\|$. Thus, since $S^n(x) = (1/2)(S^{n-1}(x)+T(S^{n-1}(x)))$,

$$\|S^n(x)-p\| \leqslant \left[1-\delta\left[\frac{2\varepsilon}{\|S^{n-1}(x)-p\|}\right]\right]\|S^{n-1}(x)-p\|. \tag{6}$$

In view of (6),

$$\|S^n(x)-p\| \leqslant \prod_{j=1}^{n}\left[1-\delta\left[\frac{2\varepsilon}{\|S^{n-j}(x)-p\|}\right]\right]\|x-p\|,$$

and by monotonicity of δ,

$$\|S^n(x)-p\| \leqslant [1-\delta[2\varepsilon/d]]^n d \leqslant (\varepsilon/d)d = \varepsilon.$$

Since p is a fixed point of T with T nonexpansive, $\|T(S^n(x))-p\| \leqslant \varepsilon$; hence $\|S^n(x)-T(S^n(x))\| \leqslant 2\varepsilon$ yielding

$$\|S^n(x)-S^{n+1}(x)\| = (1/2)\|S^n(x)-T(S^n(x))\| \leqslant \varepsilon.$$

4. Remarks.

(4.1) Since $r \leqslant d(1-\delta(1))$, implicit in (3) is a sharper estimate for Theorem 1, namely:

$$\inf\{\|x-T(x)\|\colon x \in K\} \leqslant [d(1-\delta(1))+2\varepsilon]\,\delta^{-1}\left[\frac{2\varepsilon}{d(1-\delta(1))+2\varepsilon}\right]+\varepsilon.$$

However, even this estimate is not precise since it does not yield the known (see [8]) fact that if X is a Hilbert space,

$$\inf\{\|x-T(x)\|\colon x \in K\} \leqslant \varepsilon.$$

(4.2) In cases where the modulus of convexity is explicitly known, the estimate of Theorem 2 can be improved. For example, if $X = l^p$, $2 \leqslant p < \infty$, then $\delta(\varepsilon) = 1-(1-(\varepsilon/2)^p)^{1/p}$. Computing directly from (6):

$$\|S(x)-p\| \leqslant (1-\delta(2\varepsilon/d))d = (1-(\varepsilon/d)^p)^{1/p}d = (d^p-\varepsilon^p)^{1/p};$$

$$\|S^2(x)-p\| \leqslant \left[1-\delta\left[\frac{2\varepsilon}{(d^p-\varepsilon^p)^{1/p}}\right]\right](d^p-\varepsilon^p)^{1/p} = (d^p-2\varepsilon^p)^{1/p};$$

and continuing:

$$\|S^n(x)-p\| \leqslant (d^p-n\varepsilon^p)^{1/p}, \qquad n = 1, 2, \ldots$$

Also, $(d^p-n\varepsilon^p)^{1/p} \leqslant \varepsilon$ if and only if $n \geqslant (d/\varepsilon)^p-1$. We note that this estimate for n is better than that given by (4).

Also, we note that for $X = \mathbf{R}^1$ and $K = [0, 1]$, $\delta(\varepsilon) = \varepsilon/2$, and a repetition of the above argument yields $|S^n(x)-p| \leqslant 1-n\varepsilon$; hence $|S^n(x)-p| \leqslant \varepsilon$ provided $n \geqslant \varepsilon^{-1}-1$.

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Reçu par la Rédaction le 25.04.1988