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Dilations of Banach space operator valued functions

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Abstract. The paper deals with existence and uniqueness theorems concerning dilations of positive definite functions with values in the space of all antilinear bounded operators from a complex Banach space to its dual. These dilations are representations of semigroups. Basic results concern the case of non-unital semi-groups.

1. Let E be a complex Banach space. By $\bar{L}(E)$ we denote the space of all bounded antilinear operators from E to the dual E^* of E. $L(E, F)$ stands for the space of all linear bounded operators from the Banach space E into the Banach space F. We write $L(E) = L(E, E)$. Let Z be a set.

DEFINITION ([9], [10], se also [2]). We say that the function $B(\cdot, \cdot)$: $Z \times Z \to \bar{L}(E)$ is *positive definite* and write then $B \gg 0$, if for every n, every n-tuples $z_1 \dots z_n \in Z$, $f_1 \dots f_n \in E$ the inequality

$$\sum_{i,k=1}^{n} \big(B(z_i, z_k)f_i\big)(f_k) \geqslant 0$$

holds true.

We have (see [10], Theorem 4.6, also [2]) the following

PROPOSITION 1. *Suppose that the function $B(\cdot, \cdot)$: $Z \times Z \to \bar{L}(E)$ is positive definite. Then there is a Hilbert space K and an operator function $X(\cdot)$: $Z \to L(E, K)$ such that*

$$B(u, v) = X(v)^* X(u) \quad \text{for } u, v \in Z. \tag{1}$$

Notice that if K in the above proposition is *minimal*, i.e.

$$K = \bigvee_{z \in Z} X(z)E, \tag{2}$$

then K and $X(\cdot)$ are *unique up to unitary equivalence*. This means that if K_1, K_2 are Hilbert spaces and $X_i(\cdot)$: $E \to K_i$ as well

$$K_i = \bigvee_{z \in Z} X_i(z)E \quad (i = 1, 2)$$

and

$$X_1(v)^* X_1(u) = X_2(v)^* X_2(u) \quad \text{for } u, v \in Z,$$

then there is an unitary map $U\colon K_1 \to K_2$ such that $UX_1(u) = X_2(u)$ for $u \in Z$.

If (2) holds true, then expresion (1) of B through $X(\cdot)$ is called *canonical.*

Let S be a multiplicative semi-group and K a Hilbert space. The operator-valued function $\pi(\cdot)\colon S \to L(K)$ is called a *representation* if $\pi(uv) = \pi(u)\pi(v)$ for $u, v \in S$. If S is unital, that is, S has a unit, say e, then π is called *unital* if $\pi(e) = I_K$ = identity operator in K.

It is plain that if E is a complex Banach space, K is a Hilbert space, $R \in L(E, K)$ and $\pi(\cdot)\colon S \to L(K)$ is a representation of the semi-group S, then the $\bar{L}(E)$ valued function B defined as

$$B(u, v) = R^* \pi(v)^* \pi(u) R \qquad (u, v \in S) \tag{3}$$

is positive definite.

If (3) holds true, then we say that $\pi(\cdot)$ is an *R-dilation* of B. Moreover, (3) implies that there is a finite function $\varrho\colon S \to \mathbf{R}^+$ such that

$$\sum_{i,k=1}^{n} \big(B(su_i, su_k)f_i\big)(f_k) \leqslant \varrho(s) \sum_{i,k=1}^{n} \big(B(u_i, u_k)f_i\big)(f_k) \tag{4}$$

for all $n, u_1 \dots u_n \in S$, $f_1 \dots f_n \in E$. One takes simply $\varrho(s) = \|\pi(s)\|^2$.

We say that $B \gg 0$ satisfies the ***boundedness condition*** if (4) holds true form some $\varrho(\cdot)$, all n and all $u_1 \dots u_n$, $f_1 \dots f_n$.

Suppose $B \gg 0$ and B satisfies the boundedness condition. Let $B(u, v) = X(v)^* X(u)$ be the canonical expression of B. Following [10] (the case where S is a group) and [2] (case of general semi-group) we define for $s \in S$ using (4) the operator $\hat{\pi}(s) \sum_j X(u_j) f_j = \sum_j X(su_j) f_j$ (finite sums) on a dense linear manifold of $K = \bigvee_{s \in S} X(s)E$. Condition (4) implies that $\hat{\pi}(s)$ extends in a unique way to $\pi(s) \in L(K)$. We then have $\|\pi(s)\|^2 \leqslant \varrho(s)$ $(s \in S)$. Since $\pi(s_1 s_2) X(u) f = X(s_1 s_2 u) f = \pi(s_1)\big(X(s_2 u) f\big) = \pi(s_1)\big(\pi(s_2) X(u) f\big)$ for $s_1, s_2, u \in S, f \in E$, thus $\pi(\cdot)$ is a representation of S into $L(K)$. Moreover,

$$B(su, rv) = X(v)^* \pi(r)^* \pi(s) X(u) \tag{5}$$

for $s, r, u, v \in S$. Now if S has the unit e we put $R = X(e)$ in (5) and derive therefore the following, probably the proper, formulation of a basic result of [2], which reads as follows.

PROPOSITION 2. *Let S be a unital semi-group and $B(\cdot, \cdot)\colon S \times S \to \bar{L}(E)$ a positive definite operator function. If B satisfies the boundedness condition* (4), *then B has an R-dilation, which is a unital representation π of S. The minimality condition $K = \bigvee_{s \in S} \pi(s) RE$ determines K and π up to unitary equivalence.*

Proof. Only uniqueness requires the proof. So, suppose that π_i $(i=1,2)$ is unital representation of S and $K_i = \bigvee_{s\in S} \pi_i(s)R_iE$, where $R_i\colon E\to K_i$ $(i=1,2)$ and $R_i^*\pi_i(v)^*\pi_i(u)R_i = B(u,v)$ for $i=1,2$, $u,v\in S$. Since for any n, $u_1\ldots u_n\in S$ and $f_1\ldots f_n\in E$

$$\Big\|\sum_i \pi_1(u_i)R_1f_i\Big\|_{K_1}^2 = \sum_{i,k}\big(B(u_i,u_k)f_i\big)(f_k) = \Big\|\sum_i \pi_2(u_i)R_2f_i\Big\|_{K_2}^2,$$

the map

$$\sum_i \pi_1(u_i)R_1f_i \to \sum_i \pi_2(u_i)R_2f_i$$

extends to the unitary operator $U\colon K_1\to K_2$. Since S has an unit, we have $UR_1 = R_2$ and obviously $U\pi_1(s) = \pi_2(s)U$ for $s\in S$.

COROLLARY 1. *Notice that since S has the unit e, we have $\pi(u)Rf = \pi(u)X(e)f = X(u)f$ for $u\in S, f\in E$, which implies that $\bigvee_{s\in S}\pi(s)RE = \bigvee_{s\in S}X(s)E = K$ in this case.*

COROLLARY 2. *If S is a $*$-semi-group* (see [8], where such semi-groups have been introduced) *and $B(u,v) = B(v^*u)$ ($v\to v^*$ is the involution in S), then Proposition 2 reduces to the extension of the Sz.-Nagy theorem* [8] *proved in* [1]. *In this case $\pi(\cdot)$ becomes a $*$-representation, i.e. it is a representation and $\pi(u^*) = \pi(u)^*$ for $u\in S$. For dilations of this type for Hilbert space case we refer also to* [3], [4] *and* [6], [7], *where the question of unitization as well the boundedness condition for non-unital S is discussed.*

Now we give some example of positive definite function which satisfy the boundedness condition (4). These examples are related to the probability theory. We consider only a complex valued functions, but we remark that the operator-valued function arise in the same way when one studies multivariate stochastic processes, see [10].

EXAMPLE. Let $(\Omega,\mathscr{B},P)$ be a probability space. By $L_0^2(\Omega,\mathscr{B},P)$ we denote the linear space of (equivalence classes of) complex-valued random variables ξ defined on the probability space for which $\boldsymbol{E}\xi = 0$ and $\boldsymbol{E}|\xi|^2<\infty$, where $\boldsymbol{E}$ denotes the mathematical expectation (the integral with respect to P).

By *second order process* we mean a mapping $x\colon \boldsymbol{R}\to L_0^2(\Omega,\mathscr{B},P)$. It is well know that the correlation function

$$r(s,t) = \boldsymbol{E}x(s)\overline{x(t)}$$

is positive definite. If we consider stationary stochastic processes, i.e. $r(s,t) = r(s-t,0)$ for each $s,r\in\boldsymbol{R}$, then $r(s,t)$ satisfies the boundedness condition with $\varrho(s)\equiv 1$.

Now we consider a class of second order stochastic processes x for which there exists a finite function $M\colon \boldsymbol{R}\to\boldsymbol{R}^+$ such that for all n, $t_1\ldots t_n$

$\in \boldsymbol{R}$ and $a_1 \ldots a_n \in \boldsymbol{C}$

$$(\alpha) \qquad \Big\| \sum_{k=1}^{n} a_k x(s+t_k) \Big\| \leqslant M(s) \Big\| \sum_{k=1}^{n} a_k x(t_k) \Big\|,$$

where $\|\cdot\|$ denotes the norm in $L_0^2(\Omega, \mathscr{B}, P)$. This class consists, in general, of non-stationary processes. If $M(s) = \mathrm{const}$, this class coincides with the so-called uniformly bounded linearly stationary stochastic processes which were ivestigated from the application oriented point of view in [5]. It was shown that every continuous such process $x(t)$ has a spectral representation

$$x(t) = \int_{-\infty}^{+\infty} e^{its} d\mu(s), \qquad t \in \boldsymbol{R},$$

where μ is bounded stochastic measure.

Let us notice that the correlation function of stochastic processes from the class (α) satisfies the boundedness condition with $\varrho(s) = M^2(s)$. Clearly, for all n, $t_1 \ldots t_n$, $s \in \boldsymbol{R}$ and $a_1 \ldots a_n \in \boldsymbol{C}$

$$\begin{aligned}\sum_{i,k=1}^{n} r(s+u_i, s+u_k) a_i \bar{a}_k &= \sum_{i,k=1}^{n} a_i \bar{a}_k \boldsymbol{E} x(s+u_i) \overline{x(s+u_k)} \\ &= \boldsymbol{E} \Big| \sum_{i=1}^{n} a_i x(s+u_i) \Big|^2 = \Big\| \sum_{i=1}^{n} a_i x(s+u_i) \Big\|^2 \\ &\leqslant M^2(s) \Big\| \sum_{i=1}^{n} a_i x(u_i) \Big\|^2 = M^2(s) \sum_{i,k=1}^{n} r(u_i, u_k) a_i \bar{a}_k.\end{aligned}$$

In the example we have a case: the semi-group $S = \boldsymbol{R}$, the Banach space $E = \boldsymbol{C}$, and the function $\boldsymbol{B} = \boldsymbol{r}$.

2. Suppose that the operator-valued function $B(\cdot, \cdot)\colon S \times S \to \bar{L}(E)$ is positive definite and S is a semi-group. If B satisfies the boundedness condition (4), then formula (5) holds true and $\pi(\cdot)$ is a representation of S, namely $\pi(\cdot)\colon S \to L(K)$, where $K = \bigvee_{s \in S} X(s)E$. The space K may be replaced by a smaller one and (5) will be still valid with suitably redefined factors. To see this we will briefly recall some facts related to von Neumann algebras.

Suppose, namely, that K is a Hilbert space and let $\mathscr{F} \subset L(K)$. Define $M = \{h \in K\colon Ah = 0, A^*h = 0 \text{ for all } A \in \mathscr{F}\}$ and let Q be the orthogonal projection on $K \ominus M$. Then

$$(6) \qquad AQ = QA = A \quad \text{for all } A \in \mathscr{F}.$$

Moreover,

(7) Q belongs to the von Neumann algebra generated by $\mathscr{F}$ (i.e., Q is in the strong operator closure of selfadjoint algebra generated by $\mathscr{F}$).

Now, if (5) holds true and if we take $\mathscr{F} = \{\pi(u): u \in S\}$, then by (6)

(8) $$B(su, rv) = X(v)^* Q^* \pi(r)^* \pi(s) Q\, X(u) \quad \text{for } s, r, u, v \in S.$$

Let $K_0 = QK$ $(K = \bigvee_{s \in S} X(s)E)$ and define $Y(\cdot): S \to L(E, K_0)$ as $Y(s)f = QX(s)f$ $(s \in S, f \in E)$. By (6), K_0 reduces all operators $\pi(s)$. We define $\varphi(s) = \pi(s) \mid K_0 \in L(K_0)$ and are then able to rewrite (8) as follows

(9) $$B(su, rv) = Y(v)^* \varphi(r)^* \varphi(s) Y(u); \quad r, s, u, v \in S.$$

Certainly $\varphi(\cdot): S \to L(K_0)$ is a representation of S.

We introduce now the following condition:

(A) There is a net $e_a \in S$ such that

(10) $$\sup_a \|B(e_a, e_a)\| < +\infty,$$

(11) $$\lim_a \big(B(ue_a, v)f\big)(g) = \big(B(u, v)f\big)(g) \quad \text{for } u, v \in S; f, g \in E,$$

(12) $$\lim_a \big(B(e_a, v)f\big)(g) \quad \text{exists for every } v \in S \text{ and every } f, g \in E.$$

Since

$$\|X(e_a)f\|^2 = \big(B(e_a, e_a)f\big)(f) \leqslant \|B(e_a, e_a)f\| \cdot \|f\| \leqslant \|B(e_a, e_a)\| \cdot \|f\|^2,$$

condition (10) yields that

(13) $$\sup_a \|X(e_a)\| < +\infty.$$

We write $R_a = QX(e_a)$.

Next by (11), since $\pi(u) = \pi(u)Q = Q\pi(u)$,

(14) $$\begin{aligned} \big(B(ue_a, v)f\big)(g) &= \big(X(ue_a)f, X(v)g\big) = \big(\pi(u)X(e_a)f, X(v)g\big) \\ &= \big(\varphi(u)R_a f, X(v)g\big) = \big(R_a f, \varphi(u)^* X(v)g\big) \\ &\to \big(X(u)f, X(v)g\big). \end{aligned}$$

Notice now that the set of vectors $\{\varphi(u)Y(s)g, \varphi(v)^* Y(t)f;\ u, v \in S;\ f, g \in E\}$ is linearly dense in K_0. Indeed, if $h \in K_0$ and $h \perp \varphi(u)Y(s)g = Q\pi(u)X(s)g$, then, for $s \in S, g \in E$, $\pi(u)^* Qh \perp X(s)g$, i.e., $\pi(u)^* Qh \perp K = \bigvee_{s \in S} X(s)E$ which implies that $\pi(u)^* Qh = 0$. By similar token the relation $h \perp \varphi(v)^* Y(t)f$ $(t \in S, f \in E)$ implies $\pi(v)Qh = 0$. Hence $h = Qh \in M \perp K_0$ which proves that $h = 0$, q.e.d.

It follows from (12) that

$$\lim_a (R_a f, \varphi(u) Y(s) g) = \lim_a (B(e_a) f, X(us) g) \tag{15}$$
$$= \lim_a (B(e_a, us) f)(g) \quad \text{exists.}$$

We conclude that $\lim_a (R_a f, h)$ exists for h running over a dense linear manifold of K_0. Since $\sup_a \|R_a\| \leqslant \sup_a \|X(e_a)\| < +\infty$ by (13), $\{R_a\}$ is a bounded net, which satisfies the Cauchy condition for the weak convergence. Hence $\lim_a R_a \overset{\mathrm{df}}{=} R$ exists and is a bounded linear operator from E into K_0.

It follows from (11) that

$$(\varphi(u) Rf, \varphi(v) R_a g) = \lim_\beta (\varphi(u) R_\beta f, \varphi(v) R_a g) = \lim_\beta (B(ue_\beta, ve_a) f)(g)$$
$$= (B(u, ve_a) f)(g).$$

On the other hand, since $\overline{(B(s, t) f)(g)} = (B(t, s) g)(f)$ for $s, t \in S$ and $f, g \in E$, we get from (11) once again that

$$\lim_a (B(u, ve_a) f)(g) = (B(u, v) f)(g)$$

and conclude finally that

$$(\varphi(u) Rf, \varphi(v) Rg) = \lim_a (B(u, ve_a) f)(g) = (B(u, v) f)(g).$$

Since u, v, f, g are arbitrary, we have

$$B(u, v) = R^* \varphi(v)^* \varphi(u) R, \qquad u, v \in S.$$

Summing up we proved the following theorem:

THEOREM 1. *Let S be a semi-group and $B(\cdot, \cdot)$: $S \times S \to \bar{L}(E)$ a positive definite operator function which satisfies the boundedness condition. Assume also that B satisfies condition* (A). *Then B has an R-dilation which is a representation of S.*

Using Theorem 1, one easily formulates an analogon of an existence theorem of [4] for Banach space valued operator functions in a case where the non-unital S is a $*$-semi-group and $B(u, v) = \tilde{B}(v^* u)$.

3. Our next topic concerns uniqueness of dilations. The uniqueness is that one up to unitary isomorphism. Our result given below is modelled after a uniqueness theorem of [3], [4].

To begin with we suppose that S is a semi-group, π_i: $S \to L(K_i)$ $(i = 1, 2)$. Assume that π_1, π_2 correspond to the same function $B(u, v)$

$\in \bar{L}(E)$, that is,

$$R_1^* \pi_1(v)^* \pi_1(u) R_1 = B(u, v) = R_2^* \pi_2(v)^* \pi_2(u) R_2$$

for all $u, v \in S$. Then, since for any n and $u_1 \ldots u_n \in S, f_1 \ldots f_n \in E$

$$\Big\| \sum_{j=1}^{n} \pi_1(u_j) R_1 f_j \Big\|_{K_1}^2 = \sum_{i,k=1}^{n} \big(B(u_i, u_k) f_i\big)(f_k)$$

$$= \Big\| \sum_{j=1}^{n} \pi_2(u_j) R_2 f_j \Big\|_{K_2}^2,$$

the minimality of K_1 and K_2 implies that there is a unitary map U: $K_1 \to K_2$ such that $U\pi_1(u) = \pi_2(u) U$ for $u \in S$. It follows that $U\pi_1(v)^* = \pi_2(v)^* U$ for $v \in S$.

Let now $\varphi_i(u_k) = \pi_i(u_k)$ or $\pi_i(u_k)^*$, the same choice being fixed for both $i = 1, 2$. Then

$$U \prod_{k=1}^{n} \varphi_1(u_k) R_1 f = \prod_{k=1}^{n} \varphi_2(u_k)\, U R_2 f.$$

Consequently

$$U \Big(\sum_{i=1}^{m} a_i \prod_{k=1}^{n} \varphi_i(u_{k,i}) \Big) R_1 f = \Big(\sum_{i=1}^{m} a_i \prod_{k=1}^{n} \varphi_2(u_{k,i}) \Big)\; U R_2 f$$

for all $m, n, u_{k,i} \in S$ and scalars a_i and $f \in E$. It follows that

$$UAR_1 f = UAU^{-1} R_2 f \tag{16}$$

for every A in the self-adjoint algebra $\mathscr{A}_1$ generated by $\mathscr{F}_1 = \{\pi_1(u): u \in S\}$.

We define the closed subspaces

$$M_i = \{h \in K_i:\ \pi_i(u) h = 0 = \pi_i(v)^* h \text{ for all } u, v \in S\}$$

and

$$N_i = \{h \in K_i:\ \pi_i(u)^* h = 0 \text{ for all } u \in S\}$$

for $i = 1, 2$. Certainly $M_i \subset N_i$ which implies that $N_i^\perp \subset M_i^\perp$. But $N_i^\perp = K_i$. Indeed, if $h \in N_i$, then $\pi_i(u)^* h = 0$ for $u \in S$ which implies that

$$\big(\pi_i(u)^* h, R_i f\big) = \big(h, \pi_i(u) R_i f\big) = 0 \quad \text{for } u \in S \text{ and } f \in E.$$

Since $K_i = \bigvee_{s \in S} \pi_i(s) R_i E$, we must have $h = 0$. q.e.d.

Let now C_i be an orthogonal projection on M_i and D_i an orthogonal projection on N_i. Obviously, $C_i \leqslant D_i$, which implies that $I_{K_i} - D_i \leqslant I_{K_i} - C_i \overset{\text{df}}{=} Q_i$. But $D_i = 0$. It follows that $Q_i = I_{K_i}$ (I_K stands as usually for the identity operator in K). Notice now that I_{K_i} is (by (7) of Section 2) in the strong operator closure of the self-adjoint operator algebra $\mathscr{A}_i$

generated by $\mathscr{F}_i = \{\pi_i(u): u \in S\}$. Hence, for $\varepsilon > 0$ and $f \in E$, there is an $A \in \mathscr{A}_1$ such that

$$\|AR_1 f - R_1 f\| = \|UAR_1 f - UR_1 f\| < \tfrac{1}{2}\varepsilon$$

and simultaneously

$$\|AU^{-1}R_2 f - U^{-1}R_2 f\| = \|UAU^{-1}R_2 f - R_2 f\| < \tfrac{1}{2}\varepsilon.$$

By (16) we have that $UAR_1 f = UAU^{-1}R_2 f$. It follows now from the above inequalities that

$$\|UR_1 f - R_2 f\| < \varepsilon.$$

Since f and ε are arbitrary, we conclude that $UR_1 = R_2$.

Summing all up we proved the following theorem:

THEOREM 2. *Let S be a semi-group and let $\pi_i(\cdot): S \to L(K_i)$ be a representation of S into $L(K_i)$, where K_i is a Hilbert space. Let E be a Banach space and $R_i \in L(E, K_i)$ $(i = 1, 2)$.*

If K_1, K_2 are minimal, i.e., $K_i = \bigvee_{s \in S} \pi_i(s) R_i E$ for $i = 1, 2$ and

$$R_1^* \pi_1(v)^* \pi_1(u) R_1 = R_2^* \pi_2(v)^* \pi_2(u) R_2 \quad \text{for } u, v \in S,$$

then π_1 and π_2 are unitary equivalent in the following sense: there is a unitary map $U: K_1 \to K_2$ such that $U\pi_1(u) = \pi_2(u) U$ for $u \in S$ and $UR_1 = R_2$.

Remark. If S is a $*$-semi-group and the things are going on with $*$-representations π_1, π_2 of S, then the above theorem, for Hilbert space E, reduces to uniqueness theorem of [3], [4]. The point is that $N_i = M_i$ in this case.

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