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On sufficiency of decomposable jets in $J^r(2,1)$

by SHIH-HUNG CHANG (University Park, Pa.)

Abstract. This paper deals with the determination of the smallest integer r so that the r-jet $j^{(r)}(f)$, from a given function f, is C^0-sufficient. Formulas have been obtained by using the orders of the branches in the decomposition of f. The proof is based on the construction of certain semi-algebraic set and the selection of an analytic arc in such a set. The result in this paper simplifies Kuo's method for the determination of sufficiency in $J^r(2,1)$.

1. Introduction. The jet space $J^r(n,1)$ consists of all real polynomials $Z\colon R^n \to R$ of degree $\leqslant r$ with $Z(0) = 0$. Let $f\colon R^n \to R$ be a C^{r+1}-function with $f(0) = 0$ and let $j^{(r)}(f) \in J^r(n,1)$ denote the Taylor expansion of f up to and including the terms of degree $\leqslant r$. We call f a *realization* of the r-jet $j^{(r)}(f)$.

DEFINITION. An r-jet $Z \in J^r(n,1)$ is called *C^0-sufficient* if for any two realizations f, g of Z, there exists a local homeomorphism $h\colon R^n \to R^n$, $h(0) = 0$, such that $f\circ h = g$ in a neighborhood of $0 \in R^n$.

The following theorem (due to Kuiper [2] and Kuo [3], [4] for (a) ⇒ (b), and Bochnak and Łojasiewicz [1] for (b) ⇒ (a)) is essential to our discussion:

THEOREM A. *Let $Z \in J^r(n,1)$. Then the following two conditions are equivalent:*

(a) *There exist $\varepsilon > 0$ and $\delta > 0$ such that*

$$|\operatorname{grad} Z(x)| \geqslant \varepsilon |x|^{r-\delta}$$

in a neighborhood of $0 \in R^n$;

(b) *Z is C^0-sufficient.*

Here we consider the space $J^r(2,1)$ and we shall refer to the notion of C^0-sufficiency simply as sufficiency from now on.

This paper deals with the actual finding of the smallest integer r so that $j^{(r)}(f)$ is sufficient for a given polynomial f of two variables. According to Lu [8], if the initial form of f can be factored as $P_{a_1}P_{a_2}\dots P_{a_q}$, where each P_{a_i} is homogeneous of degree a_i and $P_{a_1}, \dots, P_{a_q}$ are pairwise relatively prime, then one can always find formal power series f_i with initial form P_{a_i} such that

$$f(x, y) = f_1(x, y) \dots f_q(x, y).$$

The main result in [8] asserts that *if* $j^{(k_i)}(f_i)$ *is sufficient for each* i, *then* $j^{(m)}(f)$ *is sufficient, where*

$$m = \sum_{i=1}^{q} a_i + \max_{i=1,\dots,q} \{k_i - a_i\}.$$

And m is the smallest such integer if each k_i is respectively the smallest. In this paper we shall present some simple formulas for determining the smallest such integer k_i.

For this purpose one need only consider (using a local C^∞-change of coordinates if necessary, see [8]) the f_i's of the form

$$(1) \qquad Z(x, y) = x^t + H_{t+1}(x, y) + H_{t+2}(x, y) + \dots,$$

where the homogeneous forms H_j do not have terms involving any power x^i for $i > t-1$. Then by applying Puiseux's theorem and with the help of Newton polygons (Walker [10], p. 98–105), one can decompose Z into factors as follows:

$$Z(x, y) = (x - h_1(y)) \dots (x - h_t(y)),$$

where each $h_i(y)$ is a fractional power series of y with order

$$O(h_i(y)) > 1, \quad i = 1, \dots, t.$$

Then, using the orders $O(h_i(y))$, we shall give formulas for the smallest integer r so that $j^{(r)}(Z)$ is sufficient. The results are stated in Section 2 together with some illustrative examples, and proved in Section 3. Kuo [4] has also established a process to determine the smallest such integer r by using the orders of the branches from the decompositions of Z_x and Z_y. So the main progress here is in simplifying the determination of the sufficiency via a more direct approach. When the jets are already in decomposed form, the determination can be made in most cases even by inspection as this is evident from the examples. It is also hoped that our results will be useful in the qualitative study of algebraic curves.

2. Results and examples.

THEOREM 1. *Let* $Z = Z(x, y)$ *be a polynomial with a decomposition of the form*

$$(2) \qquad Z(x, y) = x(x + c_1 y^{s_1} + \dots)(x + c_2 y^{s_2} + \dots) \dots (x + c_k y^{s_k} + \dots),$$

where $1 \leqslant s_1 < s_2 < \dots < s_k$, $c_i \in R$ *and* $c_i \neq 0$, $i = 1, \dots, k$. *Then* $m = s_1 + \dots + s_{k-1} + 2s_k$ *is the smallest integer such that* $j^{(m)}(Z)$ *is a sufficient* m*-jet.*

Remark. The author understands that Y. C. Lu has also obtained the following result: Let $Z(x,y) = xg(x,y)$ and $g(x,y)$ be a polynomial of the form

$$g(x,y) = x+H_2(x,y)+H_3(x,y)+\ldots,$$

where $H_i(x,y)$ is a homogeneous form of degree i, for each i, and $O(g(0,y)) = s$. Then $2s$ is the smallest integer such that $j^{(2s)}(Z)$ is sufficient.

COROLLARY. *Let $Z = Z(x,y)$ be a polynomial with a decomposition of the form*

$$(3)\qquad Z(x,y) = (x+c_1y^{s_1}+\ldots)\ldots(x+c_ky^{s_k}+\ldots)(x+c_{k+1}y^{s_{k+1}}+\ldots),$$

where $1 \leqslant s_1 < \ldots < s_k \leqslant s_{k+1}$, $c_i \in R$ and $c_i \neq 0$, $i = 1, \ldots, k+1$.

(i) *When $c_ky^{s_k} \neq c_{k+1}y^{s_{k+1}}$, let $m = s_1+\ldots+s_{k-1}+2s_k$.*

(ii) *When $c_ky^{s_k} = c_{k+1}y^{s_{k+1}}$, consider that*

$$Z(x,y) = (x+c_1y^{s_1}+\ldots)\ldots(x+c_{k-1}y^{s_{k-1}}+\ldots)(x+h(y)+dy^r+\ldots)\times$$
$$\times(x+h(y)+ey^l+\ldots),$$

where $h(y)$ is a polynomial with $O(h(y)) = s_k$ and degree $< r \leqslant l$, $d, e \in R$, and $dy^r \neq ey^l$. Let $m = s_1+\ldots+s_{k-1}+2r$. Also use the same m when $ey^l+\ldots \equiv 0$ in the last factor of Z.

Then m is the smallest integer such that $j^{(m)}(Z)$ is a sufficient m-jet.

Note that in the decomposition of Z, if any two factors are the same, i.e. Z is divisible by $(x+c_iy^{s_i}+\ldots)^2$ for some i, then $j^{(r)}(Z)$ is not sufficient for any finite r (Kuo [5], Theorem 7.2, p. 175). However, if in the decompositions (2) and (3) all factors are distinct but not all of the numbers $s_1, \ldots, s_k$, then these numbers need not all be integers, and the coefficients $c_1, \ldots, c_k$ may be complex. In the next two theorems we shall give a complete discussion for polynomials of the form (1) with $t = 3$. Theorem 2 concerns the case where all coefficients in the decomposition are real and Theorem 3 the case with complex coefficients. Similar results may be obtained for $t \geqslant 4$ if needed.

THEOREM 2. *Let $Z = Z(x,y)$ be a polynomial with a decomposition of the form*

$$(4)\qquad Z(x,y) = x(x+c_1y^{s_1}+\ldots)(x+c_2y^{s_2}+\ldots),$$

where $1 \leqslant s_1 = s_2$, $c_i \in R$ and $c_i \neq 0$, $i = 1, 2$.

(i) *When $c_1 \neq c_2$, let $m = [3s_1]$, i.e. the largest integer $\leqslant 3s_1$.*

(ii) *When $c_1 = c_2$, consider that*

$$Z(x,y) = x(x+h(y)+dy^r+\ldots)(x+h(y)+ey^l+\ldots),$$

where $h(y)$ is a polynomial with $O(h(y)) = s_1$ and degree $< r \leqslant l$, $d, e \in R$, and $dy^r \neq ey^l$. Let $m = s_1+2r$. Also use the same m when $ey^l+\ldots \equiv 0$ in the last factor of Z.

Then m is the smallest integer such that $j^{(m)}(Z)$ is a sufficient m-jet.

EXAMPLE 1 (Kuo [4]). $Z(x, y) = x^3-3xy^k$, $k \geqslant 3$. Here we have

$$Z(x, y) = x(x+\sqrt{3}y^{k/2})(x-\sqrt{3}y^{k/2}),$$

and hence $m = [3k/2]$.

Remarks. 1. It is clear from (ii) of the above theorem that in general one cannot expect to get an upper bound for m merely from s_1 and s_2. A similar remark can be made for the decompositions (2) and (3).

2. With simple changes of variables one can reduce all possible cases concerning

$$(5) \qquad Z(x, y) = (x+c_1y^{s_1}+\ldots)(x+c_2y^{s_2}+\ldots)(x+c_3y^{s_3}+\ldots),$$

where $1 \leqslant s_1 \leqslant s_2 \leqslant s_3$ and c_1, c_2, c_3 are real, to the previous results.

EXAMPLE 2. $Z(x, y) = (x-y^2-y^3-y^4)(x-y^2-y^3-y^5)(x-y^2-y^5)$. The change of variables $X = x-y^2-y^5$ and $Y = y$ reduces Z to

$$W(X, Y) = X(X-Y^3)(X-Y^3-Y^4+Y^5),$$

and hence $m = 11$.

If in the decomposition (4) of Z there are complex coefficients, then Z is in general of the form

$$(6) \qquad Z(x, y) = x\big(x+h(y)+ig(y)\big)\big(x+h(y)-ig(y)\big),$$

where $h(y)$ is a power series in y with real coefficients and $O\big(h(y)\big) \geqslant 1$, and $g(y)$ is a power series in $y^{1/2}$ with real coefficients and $O\big(g(y)\big) \geqslant 1$. When $h(y) \equiv 0$, we use the convention $O\big(h(y)\big) = +\infty$. The following theorem takes care of this case.

THEOREM 3. *Let $Z = Z(x, y)$ be a polynomial with a decomposition of the form* (6).

(i) *When $g(y) = y^{l/2}g_1(y)$, where $l \geqslant 3$ is an odd integer and $g_1(y)$ is a power series in y with $O\big(g_1(y)\big) = 0$, let $m = l+O\big(h(y)\big)$ if $O\big(h(y)\big) < l/2$ and $m = [3l/2]$ if $O\big(h(y)\big) > l/2$.*

(ii) *When $g(y)$ is a power series in y with $O\big(g(y)\big) = k$, a positive integer, let $m = 2k+O\big(h(y)\big)$ if $O\big(h(y)\big) < k$ and $m = 2k+1$ if $O\big(h(y)\big) > k$.*

(iii) *When $O\big(h(y)\big) = k$ in* (ii), *consider that*

$$Z(x, y) = x\big(x+a_1y^k+\ldots+i(b_1y^k+\ldots)\big)\big(x+a_1y^k+\ldots-i(b_1y^k+\ldots)\big),$$

where a_1 and b_1 are non-zero real constants.

(A) *Let $m = 2k+1$ if $a_1^2-3b_1^2 < 0$ and $m = 3k$ if $a_1^2-3b_1^2 > 0$.*

(B) *When $a_1^2-3b_1^2 = 0$, consider further that*

$$Z(x, y) = x\big(x+a_1y^k+a_2y^r+\ldots+i(b_1y^k+b_2y^t+\ldots)\big)\big(x+a_1y^k+ \\ +a_2y^r+\ldots-i(b_1y^k+b_2y^t+\ldots)\big),$$

where a_2, b_2 are real constants, $k < r$, and $k < t$.

(a) $r > t$: *If k and t are both even or both odd and if $b_1 b_2 > 0$, $m = \min\{k+t+1, 3k\}$. Otherwise, $m = 3k$.*

(b) $r < t$: *If k and r are both even or both odd and if $a_1 a_2 < 0$, $m = \min\{k+r+1, 3k\}$. Otherwise, $m = 3k$.*

(c) $r = t$: *If k and r are both even or both odd and if $a_1 a_2 - 3b_1 b_2 < 0$, $m = \min\{k+r+1, 3k\}$. Otherwise, $m = 3k$ except possibly when $r < 2k-1$ and $a_1 a_2 - 3b_1 b_2 = 0$ with $a_2 \neq 0$ and $b_2 \neq 0$. In the latter case, $k+r+1 < m \leqslant 3k$ (more higher order terms in the decomposition of Z needed for a precise decision).*

Then m is the smallest integer such that $j^{(m)}(Z)$ is a sufficient m-jet.

EXAMPLE 3 (Kuo [4]). $Z(x, y) = x^3 + 3xy^{2k}$, where $k \geqslant 1$ is an integer. Here we have

$$Z(x, y) = x(x + i\sqrt{3}\, y^k)(x - i\sqrt{3}\, y^k),$$

and hence $m = 2k+1$.

Remark. If in the decomposition (5) of Z there are complex coefficients, then Z is in general of the form

$$Z(x, y) = \big(x + h(y) + ig(y)\big)\big(x + h(y) - ig(y)\big)\big(x + q(y)\big),$$

where $h(y)$, $q(y)$ are power series in y with real coefficients and orders $\geqslant 1$, and $g(y)$ is a power series in $y^{1/2}$ with real coefficients and order $\geqslant 1$. Then the change of variables $X = x + q(y)$ and $Y = y$ reduces it to Theorem 3.

3. The proofs.

Proof of Theorem 1. We have

$$(7) \qquad Z(x, y) = x^{k+1} + x^k(c_1 y^{s_1} + \ldots) + x^{k-1}(c_1 c_2 y^{s_1+s_2} + \ldots) + \ldots \\ + x^2(c_1 c_2 \ldots c_{k-1} y^{s_1+s_2+\ldots+s_{k-1}} + \ldots) + x(c_1 c_2 \ldots c_k y^{s_1+s_2+\ldots+s_k} + \ldots).$$

Let

$$A = \{(u, v) \in R^2 \mid |\mathrm{grad}\ Z(u, v)| = \min_{|(x,y)| = |(u,v)|} |\mathrm{grad}\ Z(x, y)|\}.$$

Then by Seidenberg–Tarski theorem ([6], p. 17), A is a semi-algebraic set. Hence, by the Bruhat–Cartan–Wallace curve selection lemma ([7], p. 103; [9], § 3, p. 25), there exists in A an analytic arc $a(t) = \big(x(t), y(t)\big)$, $0 \leqslant t < \eta$ for some $\eta > 0$, with $a(0) = (0, 0)$ and $a(t) \neq (0, 0)$ for $t > 0$. Such a construction has been previously used in [1], and we call such an analytic arc in A a *Łojasiewicz arc* for Z. Let

$$x(t) = a_1 t^{n_1} + a_2 t^{n_2} + \ldots,$$
$$y(t) = b_1 t^{m_1} + b_2 t^{m_2} + \ldots,$$

where both $x(t)$ and $y(t)$ are convergent power series for $0 \leqslant t < \eta$, and a_i, b_i, $i = 1, 2, \ldots$, are all real coefficients.

Now, for all (x, y) with $|(x, y)| = |(x(t), y(t))|$, we have

$$|\mathrm{grad}\, Z(x, y)| \geqslant |\mathrm{grad}\, Z(x(t), y(t))|$$

since $a(t) = (x(t), y(t)) \in A$. From (7) we have

$$\begin{aligned}(8)\quad \frac{\partial Z}{\partial x}(x(t), y(t)) &= (k+1)(a_1 t^{n_1}+\ldots)^k + k(a_1 t^{n_1}+\ldots)^{k-1}\big(c_1(b_1 t^{m_1}+\\ &+\ldots)^{s_1}+\ldots\big) + (k-1)(a_1 t^{n_1}+\ldots)^{k-2}\big(c_1 c_2 (b_1 t^{m_1}+\ldots)^{s_1+s_2}+\ldots\big)+\ldots+\\ &+2(a_1 t^{n_1}+\ldots)\big(c_1 c_2 \ldots c_{k-1}(b_1 t^{m_1}+\ldots)^{s_1+s_2+\ldots+s_{k-1}}+\ldots\big)+\\ &+\big(c_1 c_2 \ldots c_k (b_1 t^{m_1}+\ldots)^{s_1+s_2+\ldots+s_k}+\ldots\big),\end{aligned}$$

and

$$\begin{aligned}(9)\quad \frac{\partial Z}{\partial y}(x(t), y(t)) &= (a_1 t^{n_1}+\ldots)^k\big(s_1 c_1 (b_1 t^{m_1}+\ldots)^{s_1-1}+\ldots\big)+\\ &+(a_1 t^{n_1}+\ldots)^{k-1}\big((s_1+s_2) c_1 c_2 (b_1 t^{m_1}+\ldots)^{s_1+s_2-1}+\ldots\big)+\ldots+\\ &+(a_1 t^{n_1}+\ldots)^2\big((s_1+\ldots+s_{k-1}) c_1 \ldots c_{k-1}(b_1 t^{m_1}+\ldots)^{s_1+\ldots+s_{k-1}-1}+\ldots\big)+\\ &+(a_1 t^{n_1}+\ldots)\big((s_1+\ldots+s_k) c_1 \ldots c_k (b_1 t^{m_1}+\ldots)^{s_1+\ldots+s_k-1}+\ldots\big).\end{aligned}$$

Note that

$$(10)\qquad |(x(t), y(t))| \sim \begin{cases} t^{n_1}, & \text{if } n_1 \leqslant m_1, \\ t^{m_1}, & \text{if } m_1 < n_1, \end{cases}$$

where $E(t) \sim F(t)$ means that there exist two numbers ε_1 and ε_2 such that

$$0 < \varepsilon_1 < \frac{E(t)}{F(t)} < \varepsilon_2$$

for all sufficiently small $t > 0$. And clearly, $E(t) \sim F(t)$ and $F(t) \sim G(t)$ imply that $E(t) \sim G(t)$.

Now we consider the following possibilities:

1. $n_1 < m_1$. From (8) and (9) we see that

$$|\mathrm{grad}\, Z(x(t), y(t))| \sim t^{kn_1}.$$

Hence, by (10) we have

$$|\mathrm{grad}\, Z(x(t), y(t))| \sim |(x(t), y(t))|^k,$$

and then

$$(11)\qquad |\mathrm{grad}\, Z(x(t), y(t))| \geqslant \varepsilon |(x(t), y(t))|^k$$

for some $\varepsilon > 0$ and for all sufficiently small $t \geqslant 0$.

2. $n_1 = m_1$.

(i) $s_1 > 1$. Similar to case 1 and we get (11) again.

(ii) $s_1 = 1$. At least one of the coefficients of the leading terms in (8) and (9) is non-zero. Hence, we still have

$$|\operatorname{grad} Z(x(t), y(t))| \sim t^{kn_1}$$

and inequality (11).

3. $m_1 < n_1 < s_1 m_1$ $(s_1 > 1)$. Here we get

$$|\operatorname{grad} Z(x(t), y(t))| \sim t^{kn_1},$$

and hence

$$|\operatorname{grad} Z(x(t), y(t))| \sim |(x(t), y(t))|^{kn_1/m_1}$$

by (10). Thus

$$|\operatorname{grad} Z(x(t), y(t))| \geqslant \varepsilon |(x(t), y(t))|^{kn_1/m_1} \geqslant \varepsilon |(x(t), y(t))|^{ks_1}$$

for some $\varepsilon > 0$ and for all sufficiently small $t \geqslant 0$.

4. $n_1 = s_1 m_1$ $(s_1 > 1)$. Since at least one of the coefficients of the leading terms in (8) and (9) is non-zero, we have either

$$|\operatorname{grad} Z(x(t), y(t))| \sim t^{kn_1}$$

or

$$|\operatorname{grad} Z(x(t), y(t))| \sim t^{kn_1+(s_1-1)m_1}.$$

Therefore we have at least

$$|\operatorname{grad} Z(x(t), y(t))| \geqslant \varepsilon |(x(t), y(t))|^{(k+1)s_1-1}.$$

5. $s_1 m_1 < n_1 < s_2 m_1$. Here we have

$$|\operatorname{grad} Z(x(t), y(t))| \geqslant \varepsilon |(x(t), y(t))|^{s_1+(k-1)s_2}.$$

6. $n_1 = s_2 m_1$. We have either

$$|\operatorname{grad} Z(x(t), y(t))| \sim t^{(k-1)n_1+s_1 m_1}$$

or

$$|\operatorname{grad} Z(x(t), y(t))| \sim t^{kn_1+(s_1-1)m_1},$$

and hence at least

$$|\operatorname{grad} Z(x(t), y(t))| \geqslant \varepsilon |(x(t), y(t))|^{s_1+ks_2-1}.$$

$2k+1$. $s_{k-1} m_1 < n_1 < s_k m_1$. We have

$$|\operatorname{grad} Z(x(t), y(t))| \geqslant \varepsilon |(x(t), y(t))|^{s_1+\dots+s_k}.$$

$2k+2$. $n_1 = s_k m_1$. Here we have either

$$|\operatorname{grad} Z(x(t), y(t))| \sim t^{(s_1+\dots+s_k)m_1}$$

or

$$\left|\operatorname{grad} Z(x(t), y(t))\right| \sim t^{(s_1+\dots+s_{k-1}+2s_k-1)m_1},$$

and hence at least

$$\left|\operatorname{grad} Z(x(t), y(t))\right| \geqslant \varepsilon\left|(x(t), y(t))\right|^{s_1+\dots+s_{k-1}+2s_k-1}.$$

$2k+3$. $s_k m_1 < n_1$. We have

$$\left|\operatorname{grad} Z(x(t), y(t))\right| \geqslant \varepsilon\left|(x(t), y(t))\right|^{s_1+\dots s_k}.$$

Therefore, taking account of all possibilities, we have at least (case $2k+2$) that

$$\left|\operatorname{grad} Z(x(t), y(t))\right| \geqslant \varepsilon\left|(x(t), y(t))\right|^{s_1+\dots+s_{k-1}+2s_k-1}$$

for some $\varepsilon > 0$ and for all sufficiently small $t \geqslant 0$. Hence, for all (x, y) with $|(x, y)| = \left|(x(t), y(t))\right|$, we have

$$|\operatorname{grad} Z(x, y)| \geqslant \left|\operatorname{grad} Z(x(t), y(t))\right| \geqslant \varepsilon|(x, y)|^{s_1+\dots+s_{k-1}+2s_k-1}.$$

By Theorem A, $j^{(m)}(Z)$ is sufficient with $m = s_1+\dots+s_{k-1}+2s_k$.

To show that m is the smallest such integer, we compute Z_x and decompose it (using Puiseux's theorem with the help of Newton polygons [10], p. 98–105) as follows:

$$Z_x(x, y) = (k+1)\left(x+\frac{k}{k+1}c_1y^{s_1}+\dots\right)\left(x+\frac{k-1}{k}c_2y^{s_2}+\dots\right)\dots$$
$$\dots(x+\tfrac{2}{3}c_{k-1}y^{s_{k-1}}+\dots)(x+\tfrac{1}{2}c_ky^{s_k}+\dots).$$

Let $p_1(y) = -\frac{1}{2}c_ky^{s_k}-\dots$, i.e. $x-p_1(y)$ is the last factor in the above decomposition of Z_x. Then we express Z as

$$Z(x, y) = Z(p_1, y)+(x-p_1)Z_x(p_1, y)+(x-p_1)^2R(x-p_1, y)$$

or

$$Z(x, y) = Z(p_1, y)+(x-p_1)^2R(x-p_1, y)$$

since $Z_x(p_1, y) = 0$. Now $O\big(Z(p_1, y)\big) = m$, and hence $Z(x, y)-Z(p_1, y) = (x-p_1)^2R(x-p_1, y)$ is a realization of $j^{(m-1)}(Z)$. Therefore $0 \in R^2$ is not an isolated critical point of $j^{(m-1)}(Z)$, and by Theorem A ((b) ⇒ (a)), $j^{(m-1)}(Z)$ is not C^0-sufficient.

Proof of Corollary. The change of variables $X = x+c_{k+1}y^{s_{k+1}}+\dots$ and $Y = y$ reduces (i) to Theorem 1 when s_{k+1} is an integer. Otherwise, a method similar to the proof of Theorem 1 can be used to establish the result. (ii) is an immediate consequence of (i) above and Theorem 1 after using the change of variables $X = x+h(y)$ and $Y = y$.

Proof of Theorem 2. For (i) we use a similar argument to Theorem 1 to establish that

$$|\operatorname{grad} Z(x, y)| \geqslant \varepsilon |(x, y)|^{3s_1 - 1} = \varepsilon |(x, y)|^{[3s_1]-\delta}$$

for some δ, $0 < \delta \leqslant 1$. To show that $m = [3s_1]$ is the smallest such integer, we again decompose Z_x and obtain

$$Z_x(x, y) = 3(x - a_1 y^{s_1} - \ldots)(x - a_2 y^{s_1} - \ldots),$$

where

$$a_1 = \frac{-(c_1 + c_2) + (c_1^2 + c_2^2 - c_1 c_2)^{1/2}}{3} \quad \text{and} \quad a_2 = \frac{-(c_1 + c_2) - (c_1^2 + c_2^2 - c_1 c_2)^{1/2}}{3}.$$

Let $p_1(y) = a_2 y^{s_1} + \ldots$, i.e. $x - p_1(y)$ is the last factor in Z_x. Then since $O(Z(p_1, y)) = 3s_1$, $Z(x,y) - Z(p_1, y) = (x - p_1)^2 R(x - p_1, y)$ is a realization of $j^{([3s_1]-1)}(Z)$. (Note that a realization of an r-jet is a C^{r+1}-function.) Hence, $j^{([3s_1]-1)}(Z)$ is not C^0-sufficient.

The case (ii) becomes special cases of Theorem 1 and its corollary following the change of variables $X = x + h(y)$ and $Y = y$.

Proof of Theorem 3. The case (i) follows from Theorem 2 after one uses the change of variables $(x, y) \to (x, -y)$. Indeed,

$$Z(x, -y) = x\big(x + \hat{h}(y) - y^{l/2}\hat{g}_1(y)\big)\big(x + \hat{h}(y) + y^{l/2}\hat{g}_1(y)\big),$$

where $\hat{h}(y) = h(-y)$ and $\hat{g}_1(y) = g_1(-y)$, and Theorem 2 then implies the result.

For the cases (ii) and (iii), one may decompose Z_x and Z_y and then apply Theorem A. The computations are laborious but essentially elementary. We omit such computations here. (In some cases one may use an initial change of variables to simplify the computation.)

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DEPARTMENT OF MATHEMATICS
THE PENNSYLVANIA STATE UNIVERSITY
UNIVERSITY PARK

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