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On a class of extremal quasi-conformal mappings

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Abstract. In this paper I begin by defining two classes of functions as follows:

DEFINITION. A strictly increasing self-homeomorphism u of $[0, \infty)$ is said to be *ratio bounded* if there are numbers $L(u)$ and $M(u)$ for which

$$0 < L(u) < \frac{xu'(x)}{u(x)} < M(u) < \infty \quad \text{a.e. on } (0, \infty).$$

DEFINITION. A function U mapping $[0, \infty)$ onto itself is called *linear radial* if

$$U(x) = \log u(e^x), \quad x \in [0, \infty)$$

for some function u satisfying

(i) u is ratio bounded and $u(1) = 1$,

(ii) $\lim\limits_{x\to\infty} \dfrac{xu'(x)}{u(x)} = \max\{M(u), 1/L(u)\}$.

The major results of this paper are then

THEOREM. *Let h be ratio bounded on $[0, \infty)$ with $h(1) = 1$, and let G be the domain*

$$G = \{z = x+iy \mid -\infty < x < \infty, y > h(|x|)\}.$$

Furthermore, let

$$F(z) = F(x+iy) = (\operatorname{sign} x)\, U(|x|) + iy$$

in G, where U is linear radial. If $G' = F(G)$, then F is extremal in the class of all quasi-conformal mappings from G to G' that agree with F on the boundary ∂G. Moreover,

$$K(F) = \max\{M(u), 1/L(u)\},$$

where $K(F)$ represents the quasi-conformal dilatation of F on G.

COROLLARY. *If there is some $\varepsilon > 0$ and some interval $(a, b) \subset (0, \infty)$ on which*

$$\frac{1}{Q-\varepsilon} < U'(x) < Q-\varepsilon,$$

then F is extremal but not unique extremal.

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1. Introduction. Let G be the domain $G = \{z \mid y > |x|^a\}$ for $z = x+iy$ and $a > 1$. Further, let F be defined on G as $F(z) = Kx+iy$ for some $K > 1$ and let $G' = F(G)$.

It has been shown in both [3] and [4] that the function F defined above is extremal in the class of all quasi-conformal mappings of G onto G' that agree with F on the boundary ∂G of G. In this paper we will generalize both the function F and the domain G and show that the functions in this larger class are also extremal quasi-conformal for the boundary homeomorphisms they induce. We then consider the question of uniqueness and non-uniqueness for this class of extremal mappings.

2. Ratio bounded and linear radial functions. The following two classes of functions will be quite useful later in the construction of our extremal quasi-conformal mappings.

DEFINITION 1. Let u be a strictly increasing self-homeomorphism of $[0, \infty)$. Then u is said to be *ratio bounded* on $[0, \infty)$, or *RB*, if there are numbers L, M, $0 < L \leqslant M < \infty$, such that

$$L \leqslant xu'(x)/u(x) \leqslant M \quad \text{a.e. on } (0, \infty). \tag{1}$$

The lower (upper) ratio bound $L(u)$ ($M(u)$) of u on $[0, \infty)$ is defined as the supremum (infimum) of all numbers $L(M)$ satisfying (1).

DEFINITION 2. A function U mapping $[0, \infty)$ onto itself will be called *linear radial*, or *LR*, if

$$U(x) = \log u(e^x) \quad \text{for all } x \in [0, \infty), \tag{2}$$

where u is any function satisfying the conditions:

(i) u RB on $[0, \infty)$ with $u(1) = 1$,

(ii) $0 < L \leqslant xu'(x)/u(x) \leqslant M < \infty$ a.e. on $(1, \infty)$,

(iii) $\lim_{x\to\infty} [xu'(x)/u(x)] = Q = \max\{M, 1/L\}$.

By its definition U is a continuous, strictly increasing map of $[0, \infty)$ onto itself with

$$0 < L \leqslant U' \leqslant M < \infty \quad \text{a.e. on } (0, \infty)$$

and

$$\lim_{x\to\infty} U'(x) = Q = \max\{M, 1/L\}.$$

3. Preliminary results. In this section we develop some of the properties of ratio bounded functions and linear radial functions that will be needed in the proof of our main result.

THEOREM 1. *If u is an RB function, then so is u^{-1}, and*

$$L(u^{-1}) = 1/M(u), \quad M(u^{-1}) = 1/L(u).$$

Proof. Let $v = u^{-1}$. It is obvious that the continuity and monotonicity of u imply the same properties for v. Since v is strictly monotonic it is differentiable almost everywhere. Let y_0 be a point at which v is differentiable. Then also u must be differentiable at x_0, where $x_0 = v(y_0)$. Also $v'(y_0) = 1/u'(x_0)$ implies that

$$\frac{1}{M(u)} \leqslant \frac{y_0 v'(y_0)}{v(y_0)} = \frac{u(x_0)\dfrac{1}{u'(x_0)}}{x_0} = \frac{u(x_0)}{x_0 u'(x_0)} \leqslant \frac{1}{L(u)} \qquad \text{a.e. on } (0, \infty).$$

THEOREM 2. *Let u be RB on $[0, \infty)$ with ratio bounds L_0 and M_0. Then*

$$(b/a)^{L_0} \leqslant u(b)/u(a) \leqslant (b/a)^{M_0}$$

for all $a, b \in (0, \infty)$ with $a \leqslant b$.

Proof.

$$\begin{aligned} L_0 \leqslant xu'(x)/u(x) \leqslant M_0 &\Rightarrow L_0/x \leqslant u'(x)/u(x) \leqslant M_0/x \\ &\Rightarrow \log x^{L_0}]_a^b \leqslant \log u(x)]_a^b \leqslant \log x^{M_0}]_a^b \quad \text{(by integration)} \\ &\Rightarrow \log(b/a)^{L_0} \leqslant \log[u(b)/u(a)] \leqslant \log(b/a)^{M_0} \\ &\Rightarrow (b/a)^{L_0} \leqslant u(b)/u(a) \leqslant (b/a)^{M_0}. \end{aligned}$$

LEMMA 1. *Let U be LR on $[0, \infty)$. Then for each $\varepsilon > 0$ there exists an $N(\varepsilon) > 0$ such that*

$$(1-\varepsilon)Qx \leqslant U(x) \quad \textit{for all } x > N(\varepsilon).$$

Proof. Consider

$$\frac{U(x)}{Qx} = \frac{\int_0^x U'(s)\,ds}{Qx}.$$

Since $\lim_{x\to\infty} U'(x) = Q$, there is some $\hat{x} > 0$ such that $x \geqslant \hat{x}$ implies that $U'(x) \geqslant Q - \hat{\varepsilon}$, where $\hat{\varepsilon} = Q\varepsilon/2$. Therefore

$$\begin{aligned} \frac{U(x)}{Qx} &= \frac{\int_0^{\hat{x}} U'(s)\,ds}{Qx} + \frac{\int_{\hat{x}}^x U'(s)\,ds}{Qx} \\ &\geqslant \frac{\int_{\hat{x}}^x U'(s)\,ds}{Qx} \geqslant \frac{\int_{\hat{x}}^x (Q-\hat{\varepsilon})\,ds}{Qx} = \left(1-\frac{\hat{\varepsilon}}{Q}\right)\left(\frac{x-\hat{x}}{x}\right). \end{aligned}$$

Now since $\lim_{x\to\infty}\frac{x-\hat{x}}{x} = 1$, there exists an $N(\varepsilon)$ for which

$$\left(1-\frac{\hat{\varepsilon}}{Q}\right)\left(\frac{x-\hat{x}}{x}\right) \geqslant \left(1-\frac{\hat{\varepsilon}}{Q}\right)-\frac{\hat{\varepsilon}}{Q} = 1-\varepsilon \quad \text{for } x > N(\varepsilon).$$

Hence

$$\frac{U(x)}{Qx} \geqslant 1-\varepsilon \quad \text{for } x > N(\varepsilon),$$

and the lemma is proved.

4. The main result. We are now ready to state and prove the main result of this paper:

THEOREM 3. *Let h be an RB function on $[0, \infty)$ with ratio bounds L, M, where $1 < L \leqslant M < \infty$, and with $h(1) = 1$. Let G be the domain*

$$G = \{z = x+iy \mid -\infty < x < \infty,\ y > h(|x|)\}.$$

Finally, let

$$F(z) = F(x+iy) = (\operatorname{sign} x)\, U(|x|)+iy$$

in G, where U is LR. If $G' = F(G)$, then F is extremal in the class of all quasi-conformal maps from G to G' that agree with F on the boundary of G. Moreover,

$$K(F) = Q = \max\{M, 1/L\},$$

where $K(F)$ represents the quasi-conformal dilatation of F on G.

Proof. Clearly F is a homeomorphism so our first problem is to prove that F is quasi-conformal. To do this we will use the analytic definition of quasi-conformality as given on p. 24 of [1]. We must show that F is absolutely continuous on lines in G and that the maximal dilatation of F on G is finite. The proof will be in three parts.

(i) F is absolutely continuous on all vertical lines in G: Fix x_0 and let x_0+iy_1, x_0+iy_2 be any two points on the line $\operatorname{Re} z = x_0$ in G with $y_1 \leqslant y_2$. Then

$$|F(x_0+iy_2)-F(x_0+iy_1)| = |i(y_2-y_1)| = |y_2-y_1|,$$

so

$$\frac{|F(x_0+iy_2)-F(x_0+iy_1)|}{|(x_0+iy_2)-(x_0+iy_1)|} = \frac{|y_2-y_1|}{|y_2-y_1|} = 1.$$

Thus F is Lipschitz continuous, hence absolutely continuous, on each vertical line in G.

(ii) F is absolutely continuous on all horizontal lines in G: Fix y_0 and let x_1+iy_0, x_2+iy_0 be any two points on the line $\operatorname{Im} z = y_0$ in G with $0 \leqslant x_1 \leqslant x_2$. Then

$$\begin{aligned}|F(x_2+iy_0)-F(x_1+iy_0)| &= |U(x_2)-U(x_1)| = \log u(e^{x_2}) - \log u(e^{x_1}) \\ &= \log[u(e^{x_2})/u(e^{x_1})] \\ &\leqslant \log[e^{x_2}/e^{x_1}]^M \quad \text{(by Theorem 2)} \\ &= M(x_2-x_1).\end{aligned}$$

A similar result holds if $x_1 \leqslant x_2 \leqslant 0$. If $x_1 < 0 < x_2$, then

$$\begin{aligned}|F(x_2+iy_0)-F(x_1+iy_0)| &\leqslant |F(x_2+iy_0)-F(0+iy_0)| + \\ &\qquad + |F(0+iy_0)-F(x_1+iy_0)| \\ &\leqslant M(x_2-0)+M(0-x_1) = M(x_2-x_1).\end{aligned}$$

Hence

$$\frac{|F(x_2+iy_0)-F(x_1+iy_0)|}{|(x_2+iy_0)-(x_1+iy_0)|} \leqslant \frac{M(x_2-x_1)}{(x_2-x_1)} = M < \infty,$$

which shows that F is also Lipschitz continuous, and thus also absolutely continuous, on each horizontal line in G.

(iii) F has finite quasi-conformal dilatation on G: As we will show at the end of this proof,

$$K(F) \leqslant Q = \max\{M, 1/L\} < \infty.$$

In fact, $K(F) = Q$. For now, however, all we need is the upper bound.

We have thus shown that F is indeed a quasi-conformal map from G to G'. We must now show that it is an extremal quasi-conformal mapping as well. Hence let $f\colon G \to G'$ be any K-quasi-conformal map of G to G' that agrees with F on the boundary of G. Now choose any $\varepsilon > 0$. By Lemma 1 there exists an $N(\varepsilon) > 0$ such that $U(x) \geqslant (1-\varepsilon)Qx$ whenever $x > N(\varepsilon)$. Let $y_0 = h(N(\varepsilon))$. Then for any $\eta > y_0$, we have $h^{-1}(\eta) \geqslant h^{-1}(y_0) = N(\varepsilon)$, so that

$$2(1-\varepsilon)Qh^{-1}(\eta) \leqslant 2U(h^{-1}(\eta)) \leqslant L(\eta) = \int_{-h^{-1}(\eta)}^{h^{-1}(\eta)} |f_z+f_{\bar z}|\,d\xi, \tag{3}$$

for any $\eta > y_0$, where $L(\eta)$ is the length of the f-image of the segment

$$\gamma_\eta = \{z \,|\, \operatorname{Im} z = \eta,\ -h^{-1}(\eta) \leqslant \operatorname{Re} z \leqslant h^{-1}(\eta)\}.$$

Integrating (3) with respect to η from 0 to y for any $y > y_0$ gives

$$2(1-\varepsilon)Q \int_{y_0}^{y} h^{-1}(\eta)\,d\eta \leqslant \int_{y_0}^{y} L(\eta)\,d\eta \leqslant \int_{0}^{y} \int_{-h^{-1}(\eta)}^{h^{-1}(\eta)} |f_z+f_{\bar z}|\,d\xi.$$

Squaring and applying the Schwarz inequality gives

$$\Big(2(1-\varepsilon)Q\int_{y_0}^{y} h^{-1}(\eta)\,d\eta\Big)^2 \leqslant \Big(\int_0^y L(\eta)\,d\eta\Big)^2 \leqslant \iint_{G_y} J\,d\xi\,d\eta \iint_{G_y} \frac{|1+\chi|^2}{1-|\chi|^2}\,d\xi\,d\eta,$$

where $J(z) = |f_z|^2-|f_{\bar z}|^2$ and $\chi(z) = f_{\bar z}/f_z$ are the Jacobian and complex quasi-conformal dilatation, respectively, of f, and $G_y = G\cap\{z\,|\operatorname{Im} z < y\}$.

Clearly

$$U(x) = \int_0^x U'(s)\,ds \leqslant \int_0^x Qs\,ds = Qx \quad \text{for any } x \geqslant 0.$$

Hence, if $\delta(y) \geqslant 0$ is the maximal upper deviation of $f(\gamma_y)$ above the horizontal line $\operatorname{Im} z = y$, i.e. $\delta(y) = \sup\limits_{x\in\gamma_y}\{\operatorname{Im} f(x)-y\}$, then it is easy to see by considering the relevant areas that

$$\iint_{G_y} J\,d\xi\,d\eta \leqslant 2\int_0^{y+\delta(y)} U\big(h^{-1}(\eta)\big)\,d\eta \leqslant 2Q\int_0^{y+\delta(y)} h^{-1}(\eta)\,d\eta$$

and, using the same reasoning as on p. 354 of [3],

$$\iint_{G_y} \frac{|1+\chi|^2}{1-|\chi|^2}\,d\xi\,d\eta \leqslant 2K\int_0^y h^{-1}(\eta)\,d\eta,$$

where $(K-1)/(K+1) = \operatorname*{ess\,sup}\limits_{z\in G_y}|\chi(z)|$. Therefore

$$\Big(2(1-\varepsilon)Q\int_{y_0}^{y} h^{-1}(\eta)\,d\eta\Big)^2 \leqslant 4KQ\int_0^y h^{-1}(\eta)\,d\eta \int_0^{y+\delta(y)} h^{-1}(\eta)\,d\eta$$

or

$$Q \leqslant \frac{K}{(1-\varepsilon)^2}\,\frac{\int_0^y h^{-1}(\eta)\,d\eta \int_0^{y+\delta(y)} h^{-1}(\eta)\,d\eta}{\big(\int_{y_0}^{y} h^{-1}(\eta)\,d\eta\big)^2}. \tag{4}$$

If we can show that the term in brackets in (4) approaches 1 as y tends to ∞, then we will have $Q \leqslant K/(1-\varepsilon)$. After this we can let ε tend to 0 and achieve $Q \leqslant K$, from which it will follow that F is extremal for its boundary values. For convenience we will write

$$\frac{\int_0^y h^{-1}(\eta)\,d\eta \int_0^{y+\delta(y)} h^{-1}(\eta)\,d\eta}{\big(\int_{y_0}^{y} h^{-1}(\eta)\,d\eta\big)^2} = \left(\frac{\int_0^y h^{-1}(\eta)\,d\eta}{\int_{y_0}^{y} h^{-1}(\eta)\,d\eta}\right) + \left(\frac{\int_0^y h^{-1}(\eta)\,d\eta}{\int_{y_0}^{y} h^{-1}(\eta)\,d\eta}\right)\left(\frac{\int_y^{y+\delta(y)} h^{-1}(\eta)\,d\eta}{\int_{y_0}^{y} h^{-1}(\eta)\,d\eta}\right). \tag{5}$$

In order to simplify calculations, we will let $h^{-1}(\eta) = g(\eta)$ in the rest of this proof. Now by Theorem 1, since h is an RB function with

$$1 < L \leqslant \eta h'(\eta)/h(\eta) \leqslant M,$$

it follows that g is also RB with

$$1/M \leqslant \eta g'(\eta)/g(\eta) \leqslant 1/L < 1. \tag{6}$$

Clearly

$$\lim_{y\to\infty} \frac{\int_0^y g(\eta)\,d\eta}{\int_{y_0}^y g(\eta)\,d\eta} = 1 + \lim_{y\to\infty} \frac{\int_0^{y_0} g(\eta)\,d\eta}{\int_{y_0}^y g(\eta)\,d\eta} = 1$$

and, using (6) and integration by parts,

$$\int_y^{y+\delta(y)} g(\eta)\,d\eta \leqslant M \int_y^{y+\delta(y)} \eta g'(\eta)\,d\eta = M\eta g(\eta)\big]_y^{y+\delta(y)} - M \int_y^{y+\delta(y)} g(\eta)\,d\eta.$$

Thus

$$(M+1) \int_y^{y+\delta(y)} g(\eta)\,d\eta \leqslant M\eta g(\eta)\big]_y^{y+\delta(y)}$$

or

$$\int_y^{y+\delta(y)} g(\eta)\,d\eta \leqslant \frac{M}{M+1}\left[\big(y+\delta(y)\big)g\big(y+\delta(y)\big) - yg(y)\right].$$

Similarly,

$$\int_{y_0}^{y} g(\eta)\,d\eta \geqslant \frac{L}{L+1}\left[yg(y) - y_0 g(y_0)\right].$$

Therefore, if we let $C = M(L+1)/L(M+1)$, then

$$0 \leqslant \frac{\int_y^{y+\delta(y)} g(\eta)\,d\eta}{\int_{y_0}^{y} g(\eta)\,d\eta} \leqslant C\,\frac{\big(y+\delta(y)\big)g\big(y+\delta(y)\big) - yg(y)}{yg(y) - y_0 g(y_0)} \tag{7}$$

$$= \frac{C}{1 - \dfrac{y_0 g(y_0)}{yg(y)}}\left[\left(1 + \frac{\delta(y)}{y}\right)\frac{g\big(y+\delta(y)\big)}{g(y)} - 1\right].$$

But by Theorem 2

$$\frac{g\big(y+\delta(y)\big)}{g(y)} \leqslant \left(\frac{y+\delta(y)}{y}\right)^{1/L} = \left(l + \frac{\delta(y)}{y}\right)^{1/L}.$$

Hence (7) gives

$$(8)\qquad 0 \leqslant \frac{\int_{y}^{y+\delta(y)} g(\eta)\,d\eta}{\int_{y_0}^{y} g(\eta)\,d\eta} \leqslant \frac{C}{1-\big(y_0 g(y_0)/y g(y)\big)}\left[\left(1+\frac{\delta(y)}{y}\right)^{(L+1)/L}-1\right].$$

Now for any $\eta > 1$, Theorem 2 with $u = g$, $b = \eta$, $a = 1$ and $M_0 = 1/L$ shows that $g(\eta) \leqslant \eta^{1/L}$, $1/L < 1$. Hence, the proof on the top of p. 355 in [3] that $\lim\limits_{y\to\infty} \frac{\delta(y)}{y} = 0$ still goes through and so

$$\lim_{y\to\infty}\left(1+\frac{\delta(y)}{y}\right)^{1+1/L} - 1 = 0.$$

Moreover, since

$$\lim_{y\to\infty} \frac{C}{1-\big(y_0 g(y_0)/y g(y)\big)} = C,$$

we conclude from (8) that

$$\lim_{y\to\infty} \frac{\int_{y}^{y+\delta(y)} g(\eta)\,d\eta}{\int_{y_0}^{y} g(\eta)\,d\eta} = 0.$$

Hence, by (5),

$$\lim_{y\to\infty} \frac{\int_0^{y} h^{-1}(\eta)\,d\eta \int_0^{y+\delta(y)} h^{-1}(\eta)\,d\eta}{\left(\int_{y_0}^{y} h^{-1}(\eta)\,d\eta\right)^2} = 1,$$

so that $Q \leqslant K$.

Therefore any quasi-conformal mapping $f\colon G \to G'$ agreeing with F on the boundary of G must have quasi-conformal dilatation $K(f) \geqslant Q$. But for the original function

$$F(x+iy) = \begin{cases} U(x)+iy & \text{if } x \geqslant 0, \\ -U(-x)+iy & \text{if } x < 0, \end{cases}$$

it is clear from the form of the quasi-conformal dilatation given on p. 125 of [2] that the point dilatation of F at the point $z = x+iy$ is given by

$$D(x+iy) + 1/D(x+iy) = U'(|x|) + 1/U'(|x|)$$

wherever U' exists (i.e., almost everywhere). Thus, since $0 < L \leqslant U' \leqslant M < \infty$, we must have $D(x+iy) \leqslant \max\{M, 1/L\} = Q$ and hence also

$$K(F) = \operatorname*{ess\,sup}_{z\in G} D(z) \leqslant Q \leqslant K(f).$$

That is, F is extremal quasi-conformal.

Remark. If, in Theorem 3 we take $h(\eta) = \eta^L$, $L > 1$, and $U(x) = Qx$, $Q \geqslant 1$, then we get the class of extremal quasi-conformal mappings obtained in [3] and [4] as a special case.

5. A condition for non-uniqueness. The question of uniqueness and non-uniqueness for the class of extremal quasi-conformal mappings introduced in the last section is partially answered by the following result:

THEOREM 4. *Let U and h satisfy the hypotheses of Theorem 3, so that the map F is extremal for the boundary values it assumes. If there is some $\varepsilon > 0$ and some interval $(a, b) \subset (0, \infty)$ on which*

$$1/(Q-\varepsilon) \leqslant U'(x) \leqslant Q-\varepsilon,$$

then F is not unique extremal.

Proof. The non-uniqueness will be proved by constructing a different extremal mapping with the same boundary values as F. Let y_0 be so large that $h^{-1}(y_0) > b$. Define a map $g: G \to G'$ as

$$g(x+iy) = \begin{cases} \left(\dfrac{2y_0 - y}{y_0} U(x) + \dfrac{y-y_0}{y_0} W(x)\right) + iy & \text{if } y_0 \leqslant y \leqslant 2y_0, \\ \left(\dfrac{3y_0 - y}{y_0} W(x) + \dfrac{y-2y_0}{y_0} U(x)\right) + iy & \text{if } 2y_0 \leqslant y \leqslant 3y_0, \\ F(x+iy) & \text{if } y \notin [y_0, 3y_0], \end{cases}$$

where $W(x) = U(x) + \sigma(x)$ for $x \in [0, \infty)$, $W(x) = U(x)$ for $x < 0$, and σ will be picked as we go along to have the following properties: $\sigma(x) > 0$ on (a, b), $\sigma(x) = 0$ on $[0, a]$ and $[b, \infty)$, $\sigma(x)$ differentiable on $(0, \infty)$ with $|\sigma'| < L/2$. It is clear that $W(x)$ is strictly increasing so g is a homeomorphism. Also that g agrees with F on the boundary of G. We will use the analytic definition of quasi-conformality again to show that g is also an extremal quasi-conformal mapping of G to G'. Since $\sigma(x) > 0$ on (a, b) it is clear from the definition that g is different from F on a set of positive measure, and this will prove the non-uniqueness.

(i) g is absolutely continuous on all vertical lines in G: Fix x_0 and look at the vertical line $\operatorname{Re} z = x_0$. If $x_0 \notin (a, b)$, then $g = F$ on this line and, as we saw in the proof of Theorem 3, g is absolutely continuous on this vertical line in G. Thus, we can assume that $x_0 \in (a, b)$. Let $x_0 + iy_1$,

x_0+iy_2 be any two points on this line with $y_1 \leqslant y_2$. If $y_0 \leqslant y_1 \leqslant y_2 \leqslant 2y_0$, then

$$|g(x_0+iy_2)-g(x_0+iy_1)| = \left|\left(\frac{y_2-y_1}{y_0}\right)\sigma(x_0)+i(y_2-y_1)\right|$$
$$\leqslant \left|\frac{\sigma(x_0)}{y_0}(y_2-y_1)\right| + |i(y_2-y_1)| = \left(1+\frac{\sigma(x_0)}{y_0}\right)(y_2-y_1)$$
$$\leqslant \left(1+\frac{L(b-a)}{2y_0}\right)(y_2-y_1).$$

If $2y_0 \leqslant y_1 \leqslant y_2 \leqslant 3y_0$, on the other hand, then

$$|g(x_0+iy_2)-g(x_0+iy_1)| = \left|\left(\frac{y_1-y_2}{y_0}\right)\sigma(x_0)+i(y_2-y_1)\right|$$
$$\leqslant \left(1+\frac{L(b-a)}{2y_0}\right)(y_2-y_1).$$

Finally, if either $y_1 \leqslant y_2 \leqslant y_0$ or $3y_0 \leqslant y_1 \leqslant y_2$, then we know from the proof of Theorem 3 that

$$|g(x_0+iy_2)-g(x_0+iy_1)| = |F(x_0+iy_2)-F(x_0+iy_1)| \leqslant (y_2-y_1).$$

Hence, if we let

$$A = \max\left\{1, 1+\frac{L(b-a)}{2y_0}\right\} = 1+\frac{L(b-a)}{2Y_0} < \infty,$$

then we have, by the triangle inequality,

$$\frac{|g(x_0+iy_2)-g(x_0+iy_1)|}{|(x_0+iy_2)-(x_0+iy_1)|} \leqslant A < \infty,$$

for any points x_0+iy_1, x_0+iy_2 on the line $\operatorname{Re} z = x_0$ with $y_1 \leqslant y_2$. Thus g is Lipschitz continuous, and so also absolutely continuous, on every vertical line in G.

(ii) g is absolutely continuous on all horizontal lines in G: Fix $y_1 > 0$ and look at the horizontal line $\operatorname{Im} z = y_1$. If $y_1 \notin (y_0, 3y_0)$, then $g = F$ and we know from the proof of Theorem 3 that g is absolutely continuous on $\operatorname{Im} z = y_1$. Thus we may assume that $y_1 \in (y_0, 3y_0)$. Let x_1+iy_1, x_2+iy_1 be any two points on this horizontal line with $a \leqslant x_1 \leqslant x_2 \leqslant b$. If $y_0 \leqslant y_1 \leqslant 2y_0$, then

$$|g(x_2+iy_1)-g(x_1+iy_1)| = \left|U(x_2)-U(x_1)+\left(\frac{y_1}{y_0}-1\right)(\sigma(x_2)-\sigma(x_1))\right|$$
$$\leqslant |U(x_2)-U(x_1)| + \left|\left(\frac{y_1}{y_0}-1\right)(\sigma(x_2)-\sigma(x_1))\right|.$$

But as we showed in the proof of Theorem 3

$$|U(x_2)-U(x_1)| \leqslant M(x_2-x_1),$$

and it is easy to see that

$$\left|\left(\frac{y_1}{y_0}-1\right)(\sigma(x_2)-\sigma(x_1))\right| \leqslant \left(\frac{2y_0}{y_0}-1\right)(\sigma(x_2)-\sigma(x_1))$$

$$= \sigma(x_2)-\sigma(x_1) \leqslant \frac{L(x_2-x_1)}{2} \quad \text{(by the mean value theorem).}$$

Hence

$$|g(x_2+iy_1)-g(x_1+iy_1)| \leqslant \left(M+\frac{L}{2}\right)(x_2-x_1).$$

Similarly, we also find if $2y_0 \leqslant y_1 \leqslant 3y_0$ that

$$|g(x_2+iy_1)-g(x_1+iy_1)| \leqslant (M+L/2)(x_2-x_1).$$

Finally, if either $x_1 \leqslant x_2 \leqslant a$ or $b \leqslant x_1 \leqslant x_2$, then by the proof of Theorem 3 we know that

$$|g(x_2+iy_1)-g(x_1+iy_1)| = |F(x_2+iy_1)-F(x_1+iy_1)| \leqslant M(x_2-x_1).$$

If we now let $A = \max\{M, M+L/2\} = M+L/2 < \infty$, then we have, by the triangle inequality,

$$\frac{|g(x_2+iy_1)-g(x_1+iy_1)|}{|(x_2+iy_1)-(x_1+iy_1)|} \leqslant A < \infty$$

for any two points x_1+iy_1, x_2+iy_1 on the line $\operatorname{Im} z = y_1$ with $x_1 \leqslant x_2$ and $y_0 \leqslant y_1 \leqslant 3y_0$. Hence g is Lipschitz continuous, and thus absolutely continuous, on all horizontal lines in G.

(iii) The quasi-conformal dilatation of g on G is finite: Let $Q = \max\{M, 1/L\}$, and let x_1+iy_1 be an arbitrary point of G. If either $x_1 \notin [a, b]$ or $y_1 \notin [y_0, 3y_0]$, then there is some neighborhood of x_1+iy_1 in which $g = F$. Hence $D_g(x_1+iy_1) = D_F(x_1+iy_1) \leqslant Q$ except on a set of measure 0, where D_g and D_F represent the point dilatation of g and F, respectively. Now assume that $x_1 \in [a, b]$ and $y_1 \in [y_0, 2y_0]$. We will treat only this case since the case of $y_1 \in [2y_0, 3y_0]$ is similar. Since $K(g)$ is not affected by the behavior of g on a set of measure 0 and the set of points with either $x_1 = a$, or $x_1 = b$, or $y_1 = y_0$, or $y_1 = 2y_0$ or where U' does not exist form a set of measure 0, we may assume that $x_1 \in (a, b)$, $y_1 \in (y_0, 2y_0)$ and $U'(x_1)$ exists.

It is clear that there is some neighborhood of x_1+iy_1 in which $x_1 \in (a, b)$ and $y_1 \in (y_0, 2y_0)$, and that in this neighborhood the definition of g simplifies to $g(x+iy) = U(x)+(y/y_0-1)\sigma(x)+iy$. We will use this simplified form of g and the form of the point dilatation D given on p. 125 of [2] to calculate $D(x_1+iy_1)$.

$$D+\frac{1}{D} = U'(x)+\left(\frac{y}{y_0}-1\right)\sigma'(x)+\frac{1}{U'(x)+\left(\frac{y}{y_0}-1\right)\sigma'(x)}+\frac{\left(\sigma(x)/y_0\right)^2}{U'(x)+\left(\frac{y}{y_0}-1\right)\sigma'(x)}.$$

From the hypotheses of this theorem we know that $1/(Q-\varepsilon) \leqslant U'(x) \leqslant Q-\varepsilon$. Hence, by choosing both $\sigma(x)$ and $|\sigma'(x)|$ small enough we can insure, for an arbitrarily given $\hat{\varepsilon} > 0$, that

$$U'(x)+\left(\frac{y}{y_0}-1\right)\sigma'(x)+\frac{1}{U'(x)+\left(\frac{y}{y_0}-1\right)\sigma'(x)} \leqslant Q+\frac{1}{Q}+\frac{\hat{\varepsilon}}{2}$$

and

$$\frac{\left(\sigma(x)/y_0\right)^2}{U'(x)+\left(\frac{y}{y_0}-1\right)\sigma'(x)} \leqslant \frac{\hat{\varepsilon}}{2}.$$

Letting $\hat{\varepsilon}$ tend to 0 this shows that $D+1/D \leqslant Q+1/Q$, or equivalently $D \leqslant Q$, at the point x_1+iy_1. Therefore

$$K(g) = \operatorname*{ess\,sup}_{z\in G} D(z) \leqslant Q < \infty,$$

which shows that g is quasi-conformal. In the proof of Theorem 3, however, we showed that any quasi-conformal mapping of G to G' that agrees with F on the boundary must have dilatation greater than or equal to $Q = \max\{M, 1/L\}$. Hence g must also be extremal with $K(g) = K(F) = Q$, and $g \neq F$ on a set of positive measure. This proves that while F is extremal it is not unique extremal.

References

[1] L. V. Ahlfors, *Lectures on quasi-conformal mappings*, D. Van Nostrand, Princeton, New Jersey, 1966.

[2] — and A. Beurling, *The boundary correspondence under quasi-conformal mappings*, Acta Math. 96 (1956), p. 125–142.

[3] E. Reich and K. Strebel, *On the extremality of certain Teichmüller mappings*, Comment. Math. Helv. 45 (1970), p. 353–362.

[4] K. Strebel, *Zur Frage der Eindeutigkeit extremaler quasikonformer Abbildungen des Einheitskreises*, ibidem 36 (1962), p. 306–323.

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