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Approximation and interpolation of entire functions and generalized orders

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Abstract. The problem of generalized orders of an entire function f, defined by a series of polynomials and coinciding with the partial sum of series at a finite number of points, is studied. The same problem is considered when (i) the given function f is continuous on a compact set such that its complement with respect to the complex plane is connected and the approximation error tends to zero rapidly, (ii) f is holomorphic in the unit disc Δ and $f \in L^2(\Delta)$, (iii) f is entire and defined by a Newton series.

1. Introduction. In the first part of this paper, we consider the expansion of an entire function $f(z)$ in a series of polynomials such that the n-th partial sum of the series coincides with $f(z)$ at a given set of points. Let

$$p(z) = \prod_{i=1}^{N} (z-z_i), \tag{1.1}$$

and let $f(z)$ be entire. Then $f(z)$ can be expanded in a series

$$f(z) = \sum_{k=1}^{\infty} q_k(z)\big(p(z)\big)^{k-1}, \tag{1.2}$$

where $q_k(z)$ is a uniquely determined polynomial of degree $N-1$ or less ([14], p. 56). Further, $S_n(z) = \sum_{k=1}^{n} q_k(z)\big(p(z)\big)^{k-1}$ coincides with $f(z)$ at the points z_i, $i = 1, 2, \ldots, N$.

Rice [8] and Juneja and Kapoor [4] (see also Winiarski [15]) have studied entire functions $f(z)$ defined by (1.2) and have obtained expressions for the order [8], Lemma 3, type [8], Lemma 5, and the q-order and the lower q-order [4] of $f(z)$, analogous to those for entire functions defined by a power series (see [1], p. 9–12, [10], [11]). In Theorem 1 of this paper we obtain expressions for the generalized order $\varrho(a, \beta, f)$ (defined below) for the function $f(z)$ defined by (1.2). These results extend some of the theorems proved in [4] and [8].

Let μ be a non-negative number and let the functions a and β satisfy the following:

(H,i) $\alpha(x)$ and $\beta(x)$ are positive, strictly increasing and differentiable on $[a, \infty]$ $(a > 0)$ and tend to infinity as $x \to \infty$.

(H,ii) $\alpha(x)$ is slowly oscillating, that is, $\lim_{x\to\infty} \alpha(tx)/\alpha(x) = 1$ for every positive constant t.

(H,iii) $\beta(x)/x^\mu$ is slowly oscillating for some $\mu \geqslant 0$. If $\mu = 0$, we further assume that $\beta(x\psi(x))/\beta(e^x)$ tends to zero as $x \to \infty$ for some function ψ tending to ∞ (however, slowly) as $x \to \infty$.

(This implies that the growth of β is not "too slow".)

(H,iv) $F(x, t) = \beta^{-1}(t\alpha(x))$ satisfies, for every positive constant t, $\dfrac{dF(x,t)}{d(\log x)} = O(1)$, as $x \to \infty$.

Let $f(z)$ be any entire function and write

$$\begin{matrix} \varrho(\alpha, \beta, f) \\ \lambda(\alpha, \beta, f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log M(r,f))}{\beta(\log r)}.$$

Then $\varrho(\alpha, \beta, f)$ is called the *generalized order*, and $\lambda(\alpha, \beta, f)$ the *generalized lower order*, of $f(z)$ (cf. [9], [11]). Let Γ_R be the lemniscate

$$\Gamma_R = \{z \mid |p(z)| = R\} \quad \text{and} \quad \|\Gamma_R\| = \text{length of } \Gamma_R.$$

Then $\|\Gamma_R\| = 2\pi R^{1/N}(1+o(1))$, as $R \to \infty$. Further there exists [8] a polynomial $Q(z)$ of degree $N-1$, independent of n and R, such that for $R > c > 0$,

$$\|q_n(z)\|_{\Gamma_c} \leqslant \{\|\Gamma_R\| M(\Gamma_R, f) \|Q(z)\|_{\Gamma_R}\}/(2\pi R^n), \tag{1.3}$$

where $f(z)$ is defined by (1.2), $M(\Gamma_R, f) = \max_{z\in\Gamma_R} |f(z)|$, $\|Q(z)\|_{\Gamma_R} = \max_{z\in\Gamma_R} |Q(z)|$ and $\|q_n(z)\|_{\Gamma_c} = \max_{z\in\Gamma_c} |q_n(z)|$. In what follows we suppose that $c > 1$ is a fixed constant and write $\|q_n\|$ for $\|q_n(z)\|_{\Gamma_c}$. We prove

THEOREM 1. *Let $f(z)$ be an entire function defined by* (1.2). *Then we have:*

$$\varrho(\alpha, \beta, f) = N^\mu \limsup_{n\to\infty} \alpha(n)/\beta\left(\frac{-1}{n}\log\|q_n\|\right). \tag{1.4}$$

$$\lambda(\alpha, \beta, f) \geqslant N^\mu \liminf_{n\to\infty} \alpha(n)/\beta\left(\frac{-1}{n}\log\|q_n\|\right). \tag{1.5}$$

(1.6) *If $\{m_k\}$ is any strictly increasing sequence of natural numbers, then*

$$\lambda(\alpha, \beta, f) = N^\mu \sup_{\{m_k\}} \liminf_{k\to\infty} \alpha(m_{k-1})/\beta\big((-1/m_k)\log\|q_{m_k}\|\big).$$

(1.7)
$$\lambda(\alpha, \beta, f) = N^\mu \sup_{\{m_k\}} \{\liminf_{k\to\infty} \alpha(m_{k-1})/\beta\big((1/(m_k - m_{k-1}))\log(\|q_{m_{k-1}}\|)/(\|q_{m_k}\|)\big),$$

where supremum, in (1.6) *and* (1.7), *is taken over all sequences* $\{m_k\}$ *and we define* $\|q_{m_{k-1}}\|/\|q_{m_k}\|$ *equal to* ∞, *when* $\|q_{m_k}\| = 0$ *and* $\|q_{m_{k-1}}\|$ *zero or otherwise.*

Remarks. (a) Let $\alpha(x) = \log x$, $\mu = 1$, $\beta(x) = x$. Then conditions (H,i)–(H,iv) are satisfied and we get an extension of Lemma 4 of [8]. Note that if the right-hand side of (1.4) is infinite, then $\varrho(\alpha, \beta, f)$ is infinite and conversely.

(b) Let $\beta(x) = x$ and $\alpha(x) = l_p x$ (p-th iterate of the logarithm, $l_1 x = \log x$). Then we get Theorems 1, 3 and 4 of [4].

2. Approximation and interpolation. Next we consider, in Theorems 2 and 3, two approximation problems and, in Theorem 4, a function defined by Newton interpolation series.

(a) Let E be a compact set of points in the complex plane C and let $K = C\setminus E$. We assume that K is connected and the transfinite diameter $d(E)$, of the set E, is positive. Let $f(z)$ be continuous on E and write

$$\|f\|_E = \sup_{z\in E} |f(z)|. \tag{2.1}$$

Let P_n denote the set of all polynomials in z of degree not exceeding n. Then for every $f(z)$ continuous on E, there is exactly one polynomial $\pi_n \in P_n$ such that the approximation error

$$E_n(f, E) = \inf_{p\in P_n} \|f-p\|_E = \|f-\pi_n\|_E. \tag{2.2}$$

We prove

THEOREM 2. *Let $f(z)$ be continuous on E and suppose that $\{E_n(f, E)\}^{1/n} \to 0$ as $n \to \infty$. Then $f(z)$ has an analytic extension $\tilde{f}(z)$ which is an entire function. Furthermore, the function $g(z)$ defined by*

$$g(z) = \sum_{n=0}^{\infty} E_n(f, E) z^n \tag{2.3}$$

is entire and we have

$$\begin{aligned} \varrho(\alpha, \beta, g) &= \varrho(\alpha, \beta, \tilde{f}) = \varrho(\alpha, \beta, f), \\ \lambda(\alpha, \beta, g) &= \lambda(\alpha, \beta, \tilde{f}) = \lambda(\alpha, \beta, f); \end{aligned} \tag{2.4}$$

and formulae similar to (1.4) *through* (1.7), *with N replaced by* 1 *and* $\|q_p\|$ *by* $E_p(f, E)$, *hold.*

Remark. In (2.4) and the next theorem, we identify $f(z)$ with its analytic extension $\tilde{f}(z)$.

(b) Let $L^2(\Delta)$ denote the class of functions $f(z)$ which are holomorphic in the unit disc Δ and for which $\iint_\Delta |f(z)|^2 dx\,dy < \infty$, $z = x+iy$. Let

$$D_n(f) = \Big\{\min_{a_j} \iint_\Delta \Big|f(z) - \sum_0^n a_j z^j\Big|^2 dx\,dy\Big\}^{1/2}. \tag{2.5}$$

It is known that if $f(z) \in L^2(\Delta)$ and $\{D_n(f)\}^{1/n} \to 0$, then $f(z)$ has an analytic extension $\tilde{f}(z)$ which is an entire function. (See [7] and the references given there.)

THEOREM 3. *Let* $f(z) \in L^2(\Delta)$ *and* $\{D_n(f)\}^{1/n} \to 0$ *as* $n \to \infty$. *Then* $f(z)$ *has an analytic extension* $\tilde{f}(z)$ *which is an entire function. Furthermore,*

$$G(z) = \sum_{n=0}^{\infty} D_n(f) z^n \tag{2.6}$$

is entire and we have

$$\varrho(a, \beta, G) = \varrho(a, \beta, \tilde{f}), \quad \lambda(a, \beta, G) = \lambda(a, \beta, \tilde{f}); \tag{2.7}$$

and formulae similar to (1.4) *through* (1.7), *with* N *replaced by* 1 *and* $\|q_p\|$ *by* $D_p(f)$, *hold.*

(c) In Theorem 1, we considered the expansion (1.2), where $f(z)$ is assumed to be entire and the number of points z_i is finite. We now let $\{z_n\}_0^{\infty}$ be a bounded sequence of points and consider the series

$$f(z) = \sum_{n=0}^{\infty} a_n w_{n-1}(z), \tag{2.8}$$

where $|a_n|^{1/n} \to 0$ as $n \to \infty$, and

$$w_n(z) = (z-z_0)(z-z_1)\dots(z-z_n) \quad (n = 0, 1, \dots), \; w_{-1}(z) = 1. \tag{2.9}$$

Then $f(z)$ defined by (2.8) is entire. The order and type of $f(z)$, when $\log M(r, f) = O(r^{\varrho})$, $0 < \varrho < \infty$, have been studied by Winiarski [15] and Neidleman [5]. We consider here the generalized orders of $f(z)$.

THEOREM 4. *Let* $f(z)$ *defined by* (2.8) *be an entire function and let* $h(z) = \sum_{n=0}^{\infty} a_n z^n$. *Then* $h(z)$ *is entire and*

$$\varrho(a, \beta, f) = \varrho(a, \beta, h), \quad \lambda(a, \beta, f) = \lambda(a, \beta, h).$$

Further, the formulae similar to (1.4) *through* (1.7), *with* N *replaced by* 1 *and* $\|q_p\|$ *by* $|a_p|$, *hold.*

Remarks. (i) For the Newton series, with integer points $z_n = n$, see [1], Chapter 9.

(ii) For the interpolation problem associated with an entire function and its derivatives, see [12].

3. Proof of Theorem 1. If $f(z)$ is a polynomial, then (H,ii) through (H,iv) show that $\varrho(a, \beta, f) = \lambda(a, \beta, f) = 0$. Further (1.3) shows that the right-hand side expressions of (1.4) through (1.7) are all zero. We suppose therefore that $f(z)$ is not a polynomial. Let c be a positive constant not less than one and

$$F(z) = \sum_{n=1}^{\infty} \|q_n\| z^n, \tag{3.1}$$

where $\|q_n\| = \|q_n(z)\|_{\Gamma_c}$. From (1.3) we get for all large $R, R > R_0$ say,

$$\|q_n(z)\| \leqslant c_1 M(2R^{1/N}, f)/R^{n-1},$$

where c_1 is a constant independent of n and R. Consequently for $|z| \leqslant R/2$, $R > R_0$,

$$\sum_{n=1}^{\infty} \|q_n\| \, |z|^n < c_1 R M(2R^{1/N}, f) \sum_{n=1}^{\infty} |z/R|^n \leqslant c_1 R M(2R^{1/N}, f).$$

This implies

$$F(R/2) \leqslant c_1 R M(2R^{1/N}, f). \tag{3.2}$$

Further

$$M(\Gamma_R, f) \leqslant \sum_{k=1}^{\infty} \|q_k\|_{\Gamma_R} R^{k-1} < \sum_{k=1}^{\infty} \|q_k\|_{\Gamma_c} R^{k+N-1}$$

wherein we have used Walsh inequality ([14], Lemma, p. 77; [8]). Hence for all sufficiently large R

$$M(R^{1/N}/2, f) < R^{N-1} F(R); \tag{3.3}$$

and from (3.2) and (3.3) we get

$$\varrho(\alpha, \beta, f) = N^{\mu} \varrho(\alpha, \beta, F), \qquad \lambda(\alpha, \beta, f) = N^{\mu} \lambda(\alpha, \beta, F). \tag{3.4}$$

By (1.9) and (1.11) of [11] applied to F, we get first and second parts of Theorem 1. The remaining two parts of Theorem 1 follow from Theorem 4 of [11]. We sketch here an alternate proof of (1.6) and (1.7).

LEMMA 1. *Let $a_n \in C$ and let $\{m_k\}_1^{\infty}$ be any sequence of natural numbers. Write $a(m_k)$ for a_{m_k} and suppose that $|a_n|^{1/n} \to 0$ as $n \to \infty$. Write*

$$\lambda_0(\{m_k\}) = \liminf_{k\to\infty} \alpha(m_{k-1})/\beta\left(\frac{-1}{m_k} \log|a(m_k)|\right), \tag{3.5}$$

$$\lambda_1(\{m_k\}) = \liminf_{k\to\infty} \alpha(m_{k-1})/\beta\left(\frac{1}{m_k - m_{k-1}} \log\left|\frac{a(m_{k-1})}{a(m_k)}\right|\right). \tag{3.6}$$

Then

$$\lambda_0(\{m_k\}) \geqslant \lambda_1(\{m_k\}). \tag{3.7}$$

The proof is straightforward and omitted. Note the convention, we made in the statement of Theorem 1, when $a(m_k) = 0$.

LEMMA 2. *Let* $H(z) = \sum_{n=0}^{\infty}{}' a_n z^n$ *be an entire function and let* $\{n_k\}_0^{\infty}$ *be the range of the central index* $\nu(r, H)$ *of* H. *Let* $G(z) = \sum_{k=0}^{\infty} a(n_k) z^{n_k}$. *Then* G *is entire and*

$$\lambda(\alpha, \beta, H) = \lambda(\alpha, \beta, G) = \lambda_1(\{n_{k+1}\}). \tag{3.8}$$

This follows from Lemma 4 of [3] and Theorem 1 of [11].

LEMMA 3. *Let* $H(z) = \sum_{n=0}^{\infty} a_n z^n$ *be entire. Then for any sequence* $\{m_k\}_1^{\infty}$

$$\lambda(\alpha, \beta, H) \geqslant \lambda_0(\{m_k\}). \tag{3.9}$$

Proof. Write $\lambda_0(\{m_k\}) = \lambda_0$. We may suppose that $\lambda_0 > 0$. Given $\varepsilon > 0$, let $\mu^* = \lambda_0 - \varepsilon$ if $\lambda_0 \neq \infty$, $\mu^* = T$ if $\lambda_0 = \infty$, where T is an arbitrary large number. By the Cauchy inequality

$$\log M(r, H) \geqslant \log|a(m_k)| + m_k \log r$$
$$\geqslant m_k \log r - m_k \beta^{-1}\big(\alpha(m_{k-1})/\mu^*\big), \qquad k > n_0.$$

Choose $R_k = e\exp\big(\beta^{-1}\big(\alpha(m_{k-1})/\mu^*\big)\big)$ and $R_k \leqslant R < R_{k+1}$. Then $\alpha\big(\log M\ (R, H)\big)/\beta(\log R) \geqslant \alpha(m_k)/\beta(\log R_{k+1})$ and (3.9) follows.

LEMMA 4. *Let* $H(z) = \sum_{n=0}^{\infty} a_n z^n$ *be entire. Then*

$$\lambda(\alpha, \beta, H) = \sup_{\{m_k\}} \lambda_0(\{m_k\}) = \sup_{\{m_k\}} \lambda_1(\{m_k\}) \tag{3.10}$$

where the supremum is taken over all sequences $\{m_k\}$.

The proof follows from Lemmas 1 through 3. The parts (1.6) and (1.7) of Theorem 1 follow if we apply (3.10) to F.

4. Proof of Theorem 2. By our hypothesis, E has an infinite number of points. Following Winiarski [15] we write

$$\xi^{(n)} = \{\xi_{n0}, \xi_{n1}, \ldots, \xi_{nn}\}$$

for a sytem of $(n+1)$ points of E, and

$$V(\xi^{(n)}) = \prod_{0 \leqslant j < k \leqslant n} |\xi_{nj} - \xi_{nk}|,$$

$$\Delta^{(j)}(\xi^{(n)}) = \prod_{\substack{k=0\\k\neq j}}^{n} |\xi_{nj} - \xi_{nk}|, \qquad j = 0, 1, \ldots, n.$$

Let the system of points $\eta^{(n)} = \{\eta_{n0}, \ldots, \eta_{nn}\}$ of E satisfy the relations: (i) $V(\eta^{(n)}) = \sup_{\xi^{(n)} \in E} V(\xi^{(n)})$, (ii) $\Delta^{(0)}(\eta^{(n)}) \leqslant \Delta^{(j)}(\eta^{(n)})$, $j = 1, 2, \ldots, n$. Write

$$L^{(j)}(z, \eta^{(n)}) = \prod_{\substack{k=0\\k\neq j}}^{n} (z - \eta_{nk})/(\eta_{nj} - \eta_{nk}), \qquad j = 0, 1, \ldots, n.$$

Then there exists a finite limit

$$\lim_{n\to\infty} |L^{(0)}(z, \eta^{(n)})|^{1/n} = L(z) \geqslant 1 \tag{4.1}$$

for every z in $C\setminus E$ (see [15] and the references mentioned there). The convergence in (4.1) is uniform on every compact subset of $C\setminus E$, and L is a modulus of an analytic function φ in $C\setminus E$ which has a univalent branch

$$\varphi(z) = \gamma z + \gamma_0 + \gamma_1/z + \ldots, \qquad |\gamma| = 1/d(E) \tag{4.2}$$

in a neighborhood of infinity. Moreover, $\log L(z)$ is the Green's function for K with singularity at ∞.

By Bernstein–Walsh inequality (see [15], p. 264, [13]) and our hypothesis on $E_n(f, E)$, the function $\tilde{f}(z)$ defined by

$$\tilde{f}(z) = \pi_1(z) + \sum_{n=2}^{\infty}\big(\pi_n(z) - \pi_{n-1}(z)\big) \tag{4.3}$$

is entire (see the details following (4.5) below; see also [15], p. 269). Further on E, $\tilde{f}(z) = f(z)$ and so $\tilde{f}(z)$ provides the analytic extension of $f(z)$ to the entire plane. By our hypothesis on $E_n(f, E)$, $g(z)$ is entire and we may suppose that it is not a polynomial. Hence $\tilde{f}(z)$ is not a polynomial. Let $r > 1$ and

$$E_r = \{z \in C,\ d(E)L(z) = r\}, \quad \tilde{M}(r, \tilde{f}) = \sup_{z\in E_r} |\tilde{f}(z)|. \tag{4.4}$$

Then for all sufficiently large r [15]

$$M(r/2, \tilde{f}) \leqslant \tilde{M}(r, \tilde{f}) \leqslant M(2r, \tilde{f}). \tag{4.5}$$

Now write $\pi_0 = 0$. Then

$$|\tilde{f}(z)| \leqslant \sum_{n=1}^{\infty} |\pi_n(z) - \pi_{n-1}(z)|.$$

Hence by Bernstein–Walsh inequality we have for $z \in E_r$, $r > r_0$,

$$|\tilde{f}(z)| \leqslant \sum_{n=1}^{\infty} \|\pi_n - \pi_{n-1}\|_E L^n(z) \leqslant 2\sum_{n=1}^{\infty} E_{n-1}(f, E)(r/d)^n$$

wherein we have utilized (4.4), and $\|p\|_E = \sup_{z\in E} |p(z)|$. Since

$$g(r/d) = \sum_{n=0}^{\infty} E_n(f, E)(r/d)^n,$$

we have for all sufficiently large r,

$$g(r/d) \geqslant (d/2r)\tilde{M}(r, \tilde{f}) \geqslant (d/2r)M(r/2, \tilde{f}). \tag{4.6}$$

Now consider

$$L_n(z) = \sum_{j=0}^{n} L^{(j)}(z, \eta^{(n)}) f(\eta_{nj}).$$

Given $\varepsilon > 0$ we have for all $r > R_0(\varepsilon)$, $n > n_1$ [15]

$$\|\tilde{f} - L_n\|_E \leqslant c_2 \frac{\tilde{M}(r, \tilde{f})}{r^n} (de^{\varepsilon})^n,$$

where c_2 is a constant. Since

$$E_n(f, E) \leqslant \|\tilde{f} - L_n\|_E,$$

we have

$$|g(z)| \leqslant |p(z)| + c_2 \sum_{n=n_1+1}^{\infty} \tilde{M}(r, \tilde{f}) \left(\frac{|z| de^{\varepsilon}}{r}\right)^n,$$

where $p(z)$ is a polynomial. Choosing r such that $|z| de^{\varepsilon} \leqslant r/2$, we get

$$M\left(r/(2de^{\varepsilon}), g\right) \leqslant \left(1 + o(1)\right) 2c_2 \tilde{M}(r, \tilde{f}). \tag{4.7}$$

This combined with (4.6) and (4.5) gives (2.4). The formulae similar to (1.4) through (1.7) follow, as in Theorem 1. The auxiliary function is now $g(z)$.

5. Proof of Theorem 3. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad |z| < 1, \; f \in L^2(\Delta).$$

We may suppose, as in Theorem 1, that $f(z)$ is not a polynomial. Since

$$D_n^2(f) = \sum_{k=n+1}^{\infty} \frac{\pi}{k+1} |a_k|^2$$

and $\left(D_n(f)\right)^{1/n} \to 0$, it follows that $|a_n|^{1/n} \to 0$ and $f(z)$ is entire. From (5.2) and Cauchy's inequality we get for $n \geqslant 0$, $R \geqslant r_0$,

$$D_n(f) \leqslant M(R, f)/R^n.$$

Hence, taking $2r \leqslant R$,

$$M(r, G) \leqslant \sum_{n=0}^{\infty} \frac{M(R, f)}{R^n} r^n,$$

and so we have for all sufficiently large R

$$M(R/2, G) \leqslant M(R, f).$$

Further

$$M(r, G') = \sum_{n=1}^{\infty} nD_n r^{n-1} \geqslant \frac{1}{r^2}\sum_{n=1}^{\infty} n\sqrt{\pi/(n+2)}|a_{n+1}|r^{n+1}$$

$$\geqslant \frac{1}{r^2}\sum_{1}^{\infty}|a_{n+1}|r^{n+1} \geqslant 1+o(1)\big)M(r, f)/r^2;$$

and so

$$M(r+1, G) > \big(1+o(1)\big)M(r, f)/r^2.$$

Relations in (2.7) follow from (5.4) and (5.5). The remaining statement follows, as in Theorems 1 and 2.

6. Proof of Theorem 4. We may assume that $f(z)$ is not a polynomial. Let

$$h(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{6.1}$$

By (2.9), $h(z)$ is entire. Further, if $L = \sup_{n\geqslant 0} |z_n|$,

$$M(|z|, f) \leqslant \sum_{n=0}^{\infty} |a_n|(|z|+L)^n \leqslant M(r, h)\sum_{n=0}^{\infty}\left(\frac{|z|+L}{r}\right)^n.$$

Hence

$$M(r, f) \leqslant 2M(2r+2L, h). \tag{6.2}$$

Further [15] for $r > L$,

$$|a_n| \leqslant \big(r/(r-L)\big)^{n+1} M(r, f)/r^n$$

and so

$$M(|z|, h) \leqslant \sum_{n=0}^{\infty}|a_n|\,|z|^n \leqslant \big(rM(r, f)\big)/(r-L-|z|).$$

Consequently

$$M\left(\frac{r-L}{2}, h\right) \leqslant \frac{2rM(r, f)}{r-L}. \tag{6.3}$$

From (6.2) and (6.3) we get the required results.

COROLLARY. *If $|a_n/a_{n+1}|$ is a non-decreasing function of n for $n > n_0$, then*

$$\lambda(a, \beta, h) = \lambda(a, \beta, f) = \liminf_{n\to\infty} a(n)/\beta\left(\frac{-1}{n}\log|a_n|\right). \tag{6.4}$$

This follows from Theorem 2 of [11].

Finally we thank the referee for pointing out that the regularity of the set K, in Theorem 2, assumed in our earlier *ms*, is unnecessary.

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