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On the stability of the functional equation of the first order

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In this paper we shall deal with the problem of the stability of the functional equation

$$\varphi[f(x)] = g[x, \varphi(x)]. \tag{1}$$

The problem of the stability of a functional equations has been first discussed by Hyers [2] with respect to the Cauchy functional equation:

$$f(x+y) = f(x)+f(y). \tag{2}$$

Hyers [2] calls equation (2) *stable* in a set E if there exists a positive constant K such that for any positive number ε and any solution $\psi(x)$ of the inequality

$$|\psi(x+y)-\psi(x)-\psi(y)| \leqslant \varepsilon \tag{3}$$

there exists a solution $f(x)$ of equation (2) such that

$$|f(x)-\psi(x)| \leqslant K\cdot\varepsilon \quad \text{for } x \in E. \tag{4}$$

Equation (1) is not stable in the above sense (see [1]). Therefore, we are going to modify the definition of the stability as follows (see [1]): Put

$$g_0(x, y) = y, \quad g_{n+1}(x, y) = g[f^n(x), g_n(x, y)], \quad n = 0, 1, 2, \ldots \tag{5}$$

Definition. Equation (1) is called *stable* in a set E in some class Φ of functions defined in E if there exists a positive number K such that for any $\varepsilon > 0$ and any solution $\psi(x) \in \Phi$ of the system of inequalities

$$|\psi[f^n(x)]-g_n[x, \psi(x)]| \leqslant \varepsilon, \quad n = 0, 1, 2, \ldots, \ x \in E, \tag{6}$$

there exists a solution $\varphi(x) \in \Phi$ of equation (1) satysfying inequality (4).

We shall assume the following hypotheses:

(H_1) The function $f(x)$ is defined in an interval E, it is continuous and strictly increasing in E and there exists such a point $\xi \in E$ that

$$\big(f(x)-x\big)(\xi-x) > 0 \quad \text{for } x\in E, x \neq \xi,$$

$$\big(f(x)-\xi\big)(\xi-x) < 0 \quad \text{for } x\in E, x \neq \xi.$$

(H_2) The function $g(x, y)$ is defined in a set $\Omega \subset E\times \Xi$ and has values in Ξ. For any fixed $x\in E$ the function $g(x, y)$, considered as a function of y, is invertible in the set $\Omega_x = \{y\colon (x, y)\in\Omega\}$.

Without any loss of generality we may assume that ξ is the left end of the interval E. Then we have, by hypothesis (H_1), $f(x) < x$ for $x\in E\setminus\{\xi\}$ and ξ is a fixpoint of $f(x)$.

Let b be the other end of the interval E.

THEOREM 1. *Let hypotheses* (H_1) *and* (H_2) *be fulfilled. If there exists a positive constant M such that*

$$(7) \qquad |g_n(x, y_1)-g_n(x, y_2)| \leqslant M|y_1-y_2| \quad \text{for } n = 0, 1, 2, \ldots,$$

then for any $\eta\in(0, b-\xi)$ equation (1) *is stable in $[\xi, b-\eta]$ in the class C of functions continuous in E.*

Proof. Let ε be a positive number and let $\psi(x)$ be a function defined and continuous in E which satisfies inequalities (6). For an arbitrary $\eta\in(0, b-\xi)$ put

$$(8) \qquad \varphi_0(b-\eta) = \psi(b-\eta),$$

$$(9) \qquad \varphi_0[f(b-\eta)] = g[b-\eta, \varphi_0(b-\eta)].$$

We have, in virtue of (6) and (8),

$$(10) \quad |\varphi_0[f(b-\eta)]-\psi[f(b-\eta)]| = |g[b-\eta, \varphi_0(b-\eta)]-\psi[f(b-\eta)]| = |g[b-\eta, \psi(b-\eta)]-\psi[f(b-\eta)]| \leqslant \varepsilon.$$

Thus there exists a function $\varphi_0(x)$ defined and continuous in $[f(b-\eta), b-\eta]$ fulfilling conditions (8) and (9) and the condition

$$(11) \qquad |\varphi_0(x)-\psi(x)| \leqslant \varepsilon \quad \text{for } x\in[f(b-\eta), b-\eta].$$

Consequently, there also exists an unique continuous solution of equation (1) in E, fulfilling the condition

$$(12) \qquad \varphi(x) = \varphi_0(x) \quad \text{for } x\in[f(b-\eta), b-\eta]$$

(see [1]). Consider an arbitrary point $x\in\big(\xi, f(b-\eta)\big)$. There exist a point $t\in[f(b-\eta), b-\eta]$ and a positive integer n, such that

$$(13) \qquad x = f^n(t)$$

(see [3]). We obtain from (10), (6) and (13):

$$\begin{aligned}|\psi(x)-\varphi(x)| &= |\psi[f^n(t)]-\varphi[f^n(t)]| = |\psi[f^n(t)]-g_n[t,\varphi(t)]| \\ &= |\psi[f^n(t)]-g_n[t,\psi(t)]+g_n[t,\psi(t)]-g_n[t,\varphi(t)]| \\ &\leqslant |\psi[f^n(t)]-g_n[t,\psi(t)]|+|g_n[t,\psi(t)]-g_n[t,\varphi(t)]| \\ &\leqslant \varepsilon+M\cdot|\psi(t)-\varphi(t)| \leqslant \varepsilon+M\cdot\varepsilon = \varepsilon\cdot(M+1).\end{aligned}$$

So inequality (4) holds for $x\in(\xi, f(b-\eta))$. It also holds for $x=\xi$ by the continuity both of $\varphi(x)$ and $\psi(x)$ at ξ. Since (4) is also fulfilled in $[f(b-\eta), b-\eta]$, on account of (10), so we have proved inequality (4) with $K=1+M$ for all $x\in[\xi, b-\eta]$.

COROLLARY. *Let hypotheses* (H_1) *and* (H_2) *be fulfilled. If there exists a constant* $N\in(0,1]$ *such that*

$$(14)\qquad |g(x,y_1)-g(x,y_2)|\leqslant N|y_1-y_2|,$$

then equation (1) *is stable in* $[\xi, b-\eta]$ *for any* $\eta\in(0, b-\xi)$.

Proof. We have from (14) by induction

$$|g_n(x,y_1)-g_n(x,y_2)|\leqslant N^n|y_1-y_2|\leqslant|y_1-y_2| \quad \text{for } n=1,2,\ldots,$$

whence

$$|\psi(x)-\varphi(x)|\leqslant 2\cdot\varepsilon.$$

Thus inequality (4) holds in E for $K=2$.

Equation (1) need not to be stable in (ξ, b), as is shown in [4].

Concerning the stability of the equation (1) in the class C^r of functions which have continuous derivatives of order up to r, we can prove the following

THEOREM 2. *Let hypotheses* (H_1) *and* (H_2) *and inequality* (7) *be fulfilled. Suppose, further, that*

(a) *the function* $f(x)$ *is of class* C^r *in* E *and* $f'(x)\neq 0$ *for* $x\in E$,

(b) *for each* $x\in E$ *the set* $\Omega_x=\{y\colon g(x,y)\in E\}$ *is an open interval,*

(c) *the function* $g(x,y)$ *is of class* C^r *and* $g_y(x,y)\neq 0$ *in the set* $\{(x,y)\colon x\in E, y\in\Omega_x\}$,

(d) $g_x(\Omega_x)=\Omega_{f(x)}$ *for* $x\in E$.

Then equation (1) *is stable in* $(\xi, b-\eta]$ *in the class* C^r *for any* $\eta\in(0, b-\xi)$.

Proof. Let $\psi(x)$ be a function of class C^r in E, satysfying inequality (6) in E. Choose a function $\varphi_0(x)$ of class C^r satysfying conditions (8) and (9) in $[f(b-\eta), b-\eta]$ and the condition

$$\varphi_0^{(k)}[f(x_0)]=G_k\big(x_0,\varphi_0(x_0),\varphi_0'(x_0),\ldots,\varphi_0^{(k)}(x_0)\big),\qquad k=1,\ldots,r,$$

where

$$G_1(x, y, y_1) = [f'(x)]^{-1}(g'_x(x, y) + g'_y(x, y)\cdot y_1),$$

$$G_{k+1}(x, y, y_1, \ldots, y_{k+1}) = [f'(x)]^{-1}(\partial G_k/\partial x + \partial G_k/\partial y\cdot y_1 + \ldots + \\ + \partial G_k/\partial y_k \cdot y_{k+1}).$$

Then there exists an unique solution $\varphi(x)$ of equation (1) in E, which is of class C^r in E and fulfills the condition $\varphi(x) = \varphi_0(x)$ in $[f(b-\eta), b-\eta]$ (see [3]) and

$$|\psi(x)-\varphi(x)| \leqslant \varepsilon\cdot(M+1).$$

References

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[2] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Mat. Acad. Sci. U. S. 27 (1941), p. 222-224.

[3] M. Kuczma, *Functional equations in a single variable*, Warszawa 1968.

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