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H^p spaces on bounded symmetric domains

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Abstract. Let D be a bounded symmetric domain in C^N ($N > 1$) with Bergman–Šilov boundary b and H^p ($p > 0$) the Hardy space of functions on D. If $f \epsilon H^p$ ($p \geqslant 1$), a Fourier series expansion is obtained for f which gives Cauchy and Poisson integral formulas for f. H^p ($p \geqslant 1$) can be identified with the class $\tilde{f} \epsilon L^p(b)$ whose Cauchy and Poisson integrals are the equal. H^2 can be identified with the class $\tilde{f} \epsilon L^2(b)$ whose Fourier coefficients $a_{k\nu} = 0$ for $k < 0$. Several properties of weak convergence in H^p are proved. In particular, if a bounded sequence converges pointwise on D, then it converges weakly. D. J. Newman's result on pseudo-uniform convexity of H^1 for the disc is extended to D.

1. Introduction. Let D be a bounded symmetric domain in the complex vector space C^N ($N > 1$), $0 \epsilon D$, with Bergman–Šilov boundary b, Γ the group of holomorphic automorphisms of D and Γ_0 its isotropy group. It is known that D is circular and star-shaped with respect to 0 and that b is circular. The group Γ_0 is transitive on b and b has a unique normalized Γ_0-invariant measure $V^{-1}ds_t$, V the euclidean volume of b and ds_t euclidean volume element at $t \epsilon b$. See [11], [17].

The Hardy space $H^p = H^p(D)$, $0 < p < \infty$, is the set of holomorphic functions on D with

$$\|f\|_p = \sup_{0 \leqslant r < 1} \left\{ \frac{1}{V} \int_b |f(rt)|^p ds_t \right\}^{1/p} < \infty.$$

For $p \geqslant 1$ H^p is a Banach space and for $0 < p < 1$ a complete linear Hausdorff space [6].

In Section 2 we derive a Fourier series representation for any holomorphic function in D. If $f \epsilon H^p$ ($p \geqslant 1$) a better representation is obtained which gives Cauchy and Poisson integral formulas for f. The space H^p can be identified with the class of functions $\tilde{f} \epsilon L^p(b)$ whose Cauchy and Poisson integrals are the equal (Theorem 2). Theorem 3 gives another characterization of H^2. Theorem 4 proves that if a bounded sequence in H^p ($p > 1$) converges pointwise on D, then it converges weakly; thus

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generalizing a result of Rudin for the unit disc [13]. Theorem 5 extends D. J. Newman's result on the pseudo-uniform convexity of H^1 [12] to all bounded symmetric domains, using a method of proof due to L. D. Hoffman [8] for the polydisc in C^N.

Remarks.

1. The Bergman–Šilov boundary b is the smallest closed set $\subset \bar{D}$ on which functions holomorphic on $\bar{D}$ take their maximum [5], p. 215. The real dimension of b is $\geqslant N$. If $N = 2$ any bounded symmetric domain is biholomorphically equivalent to the bidisc $\{|z_1| < 1, |z_2| < 1\}$ or to the ball $\{|z_1|^2+|z_2|^2 < 1\}$. Their Bergman–Šilov boundaries are $\{|z_1| = 1, |z_2| = 1\}$ of real dimension 2 and $\{|z_1|^2+|z_2|^2 = 1\}$ of real dimension 3, respectively. If $N = 3$ an example of a bounded symmetric domain besides the polydisc and ball is $\{|z_1|^2+|z_2|^2 < 1, |z_3| < 1\}$ with Bergman–Šilov boundary $\{|z_1|^2+|z_2|^2 = 1, |z_3| = 1\}$ of real dimension 4 [5], p. 313.

2. Bergman and Weil have generalized Cauchy's integral formula for functions holomorphic on closed analytic polyhedra [2], [15].

2. Cauchy and Poisson formulas for functions of class H^p ($p \geqslant 1$). Let $Z_{k\nu}$ denote the monomial $z_1^{\nu_1}\dots z_N^{\nu_N}$ ($k = \nu_1+\dots+\nu_N$, $k = 0, 1, 2, \dots$, $\nu = 1, \dots, m_k = \binom{N+k-1}{k}$). From the set $\{Z_{k\nu}\}$ Hua constructed by group representation theory a system $\Phi_0 = \{\varphi_{k\nu}\}$ of homogeneous polynomials, complete and orthogonal on D and orthonormal on b [9]. For each k the sets $\{\varphi_{k\nu}\}$ and $\{Z_{k\nu}\}$ are in 1-1 correspondence so that to a set of constants $\{a_{k\nu}\}$ corresponds a set $\{A_{k\nu}\}$ with $\sum_{\nu=1}^{m_k} a_{k\nu}\varphi_{k\nu} = \sum_{\nu=1}^{m_k} A_{k\nu}Z_{k\nu}$ and conversely. Let f be holomorphic on D. Then f_ϱ, defined by $f_\varrho(z) = f(\varrho z)$, $0 < \varrho < 1$, is holomorphic on $\bar{D}$ and has the series expansions

$$f_\varrho(z) = \sum_{k,\nu} a_{k\nu}(f_\varrho)\varphi_{k\nu}(z) = \sum_{k,\nu} A_{k\nu}(f_\varrho)Z_{k\nu} \tag{1}$$

($\sum_{k,\nu} = \sum_{k=0}^{\infty}\sum_{\nu=1}^{m_k}$), both series converging uniformly to f_ϱ on compact subsets of D. Term by term differentiation in (1) gives

$$A_{k\nu}(f_\varrho) = \frac{\varrho^k}{\nu_1!\dots\nu_N!}\frac{\partial^k f}{\partial z_1^{\nu_1}\dots\partial z_N^{\nu_N}}(0) = \varrho^k A_{k\nu}(f)$$

so that

$$f_\varrho(z) = \sum_{k,\nu} \varrho^k a_{k\nu}(f)\varphi_{k\nu}(z). \tag{2}$$

To find $a_{k\nu}(f)$ set $z = \varrho' t$ ($t \in b$, $0 < \varrho' < 1$) in (2), multiply by $\bar{\varphi}_{j\mu}(t)$ and integrate over b. This gives

$$a_{j\mu}(f)r^j = \int_b f(rt)\bar{\varphi}_{j\mu}(t)\,ds_t \equiv (f_r, \varphi_{j\mu}) \tag{3}$$

$(r = \varrho\varrho')$. Replacing ϱz by z in (2) gives

LEMMA. *Any holomorphic function f on D has a Fourier series expansion*

$$f(z) = \sum_{k,\nu} a_{k\nu}(f)\varphi_{k\nu}(z), \qquad a_{k\nu}(f) = \lim_{r\to 1}(f_r, \varphi_{k\nu}), \tag{4}$$

which converges uniformly on compact subsets of D.

THEOREM 1. *Let $f \in H^p$ $(p \geqslant 1)$ with boundary values f^* on b. Then f has a Cauchy integral representation*

$$f(z) = \int_b S(z, \bar{t}) f^*(t)\, ds_t \equiv (f^*, S_z) \qquad (z \in D) \tag{5}$$

and a Poisson integral representation

$$f(z) = \int_b P(z, t) f^*(t)\, ds_t \equiv (f^*, P_z) \qquad (z \in D). \tag{6}$$

Proof. By a theorem of Bochner on circular sets [3] if $f \in H^p$ $(p > 0)$ there exists $f^* \in L^p(b)$ such that $\lim_{r\to 1}\|f_r - f^*\|_p = 0$. Since $\varphi_{k\nu}$ is bounded independently of r on b, Hölder's inequality for $p > 1$ and (4) give

$$a_{k\nu}(f) = \lim_{r\to 1}(f_r, \varphi_{k\nu}) = (f^*, \varphi_{k\nu}). \tag{7}$$

This also holds for $p = 1$. (5) follows from (4) and (7) and the fact that the series $\sum_{k,\nu}\varphi_{k\nu}(z)\bar{\varphi}_{k\nu}(t)$ for $S(z, \bar{t})$ converges uniformly for z on compact subsets of D and $t \in b$ [9]. (6) follows by writing (5) for the function $g \in H^p$ defined by $g(\zeta) = f(\zeta)S(\zeta, \bar{z})/S(z, \bar{z})$, $\zeta \in D$.

By (6) if f^* is real, then f is real on D, and a real holomorphic function is a constant. Thus

COROLLARY. *If f^* is real on b and $f \in H^p$ $(p \geqslant 1)$, then f is constant on D.*

3. Characterization of $H^p(D)$ $(p \geqslant 1)$. Set

$$S^p(b) = \{\tilde{f} \in L^p(b) \colon (\tilde{f}, S_z) = (\tilde{f}, P_z)\}.$$

Then

THEOREM 2. *For $p \geqslant 1$ $S^p(b)$ is a closed subspace of $L^p(b)$ which is isometrically isomorphic to $H^p(D)$ under the correspondence $f \to \tilde{f}$ given by $f(z) = (\tilde{f}, P_z)$, $\tilde{f} \in S^p(b)$. Also if f^* is the boundary value of f, then $f^* = \tilde{f}$ a.e. on b.*

Proof. From Theorem 1, $f \in H^p(D)$ implies $f^* \in S^p(b)$. Conversely let $\tilde{f} \in S^p(b)$ and set $f(z) = (\tilde{f}, S_z)$. Then f is holomorphic on D. Since $P(z, t) \geqslant 0$ and $\int_b P(z, t)\, ds_t = 1$, from $f(z) = (\tilde{f}, P_z)$ follows by Hölder's inequality for $p > 1$

$$|f(z)| = |(\tilde{f}, P_z)| \leqslant (|\tilde{f}|^p, P_z)^{1/p}, \tag{1}$$

where $|\tilde{f}|^p \in L^1(b)$. (1) also holds for $p = 1$. Now $h(z) = (|\tilde{f}|^p, P_z) \in \mathfrak{H}(D)$ and $f \in \tilde{H}^p(D)$ ([6], p. 521) and by Theorem 3 of [6] $f \in H^p(D)$. Thus $S^p(b)$ and $H^p(D)$ are in 1–1 correspondence. For $\tilde{f} \in S^p(b)$ set $f(z) = (P_z, \tilde{f})$. By [10], Proposition 2.5; $\|f_r - \|\tilde{f}\|_p \to 0$ as $r \to 1$ if $p \geqslant 1$. By [3] there exists $f^* \in L^p(b)$ such that $\|f_r - f^*\|_p \to 0$ as $r \to 1$. Hence $\tilde{f} = f^*$ a.e. on b. Clearly $\|\tilde{f}\|_p = \|f^*\|_p = \|f\|_p$ so that $S^p(b)$ is isometrically isomorphic to $H^p(D)$. Since $H^p(D)$ is complete, $S^p(b)$ is a closed subspace of $L^p(b)$.

A second characterization of $H^2(D)$ follows easily from Hilbert space theory. By Weyl [16] the orthonormal system Φ_0 can be extended to a complete orthonormal system of continuous functions on b: $\Phi = \{\varphi_{k\nu}, k = 0, \pm 1, \pm 2, \ldots; 1 \leqslant \nu \leqslant m_k \text{ if } k \geqslant 0,\ \nu = 0 \text{ if } k < 0\}$, where the additional terms have been indexed by negative indices.

Set

$$T^2(b) = \{\tilde{f} \in L^2(b):\ a_{k\nu}(\tilde{f}) = (\tilde{f}, \varphi_{k\nu}) = 0 \text{ for } k < 0\}.$$

Then

THEOREM 3. *$T^2(b)$ is a closed subspace of $L^2(b)$ which is isometrically isomorphic to $H^2(D)$. If f^* is the boundary value of f, then $f^* = \tilde{f}$ a.e. on b.*

Proof. If $f \in H^2(D)$, then $f \to \tilde{f} \in T^2(b)$ by (2.4) and (2.7). Conversely let $\tilde{f} \in T^2(b)$ and set

$$f(z) = \sum_{k,\nu} a_{k\nu}(\tilde{f})\varphi_{k\nu}(z) \qquad (k \geqslant 0). \tag{2}$$

From the Schwarz inequality and Bessel's inequality follow that the series in (2) converges absolutely and uniformly on compact subsets of D. Hence f is holomorphic on D. By a calculation

$$\|f_r\|_2^2 = \sum_{k,\nu} |a_{k\nu}(\tilde{f})|^2 r^{2k} \leqslant \sum_{k,\nu} |a_{k\nu}(\tilde{f})|^2$$

so that $f \in H^2(D)$. Also $\|f\|_2 = \|\tilde{f}\|_2$. The rest of Theorem 3 follows as in the proof of Theorem 2.

Schmid obtained an analogous characterization of $H^2(D)$ when D is a non-compact hermitian symmetric space by using Lie group theory [14].

4. Convergence in H^p. The following properties of weak convergence are known or are easy to prove:

If $f_n \to^w f$ in H^p, then $f_n \to f$ uniformly on compact subsets of D for every $p > 0$ ([6], Theorem 9). *If $f_n \to f$ strongly in H^p $(p > 0)$, then $f_n \to^w f$ in H^p.*

This follows from the inequality $|\gamma(f_n) - \gamma(f)| \leqslant \|\gamma\|\,\|f_n - f\|_p\ (\gamma \in (H^p)^*)$.

Since H^p $(p \geqslant 1)$ is a Banach space, the norms of the elements of a weakly convergent sequence are bounded. Let $\{f_n\}$ be a bounded sequence in H^p $(p > 0)$. Then $f_n \to f$ pointwise on D if and only if $f_n \to f$ uniformly on compact subsets of D; also $f \in H^p$.

Proof. By Lemma 3 of [6], boundedness of $\{f_n\}$ in H^p implies that $\{f_n\}$ is uniformly bounded in compact subsets of D. Then by Lemma 4 of [6] $f_n \to f$ pointwise on D implies uniform convergence of $\{f_n\}$ to f on compact subsets of D. The converse is trivial. Since $\{f_n\}$ is bounded in H^p and $f_n \to f$ uniformly on $b_r = \{rt: t \in b\}$, $0 < r < 1$, $\|f_r\|_p$ is bounded independently of r so that $f \in H^p$.

The next theorem generalizes to bounded symmetric domains a result of Rudin [13] for the disc.

THEOREM 4. *Let $\{f_n\}$ be a bounded sequence in H^p $(p \geqslant 1)$. If $f_n \to f$ pointwise in D, then $f_n \to^{w} f$ in H^p for $p > 1$ but not for $p = 1$.*

Proof. See [13] for a counter-example when $p = 1$. Assume that $\|f_n\|_p < 1$ for all n. By Lemma 3 of [6] the boundedness of $\{\|f_n\|_p\}$ implies that $\{f_n(z)\}$ is bounded independently of n and z on $\bar{D}_r$ $(0 < r < 1)$. Hence by Vitali's theorem [6] $f_n \to f$ uniformly on compact subsets of D. Thus $f \in H^p$ and we may assume that $f = 0$. Show that $f_n \to^{w} 0$ in H^p.

$f_n \in H^p$ has the series representation (2.4) with Fourier coefficients $a_{k\nu}(f_n)$ given by (2.3). Since $\{f_{r,n}\}$ converges uniformly to 0 on the compact set b, (2.3) gives $\lim\limits_n a_{k\nu}(f_n) = 0$ for all $k \geqslant 0$ and ν. Hence by (2.7) $\lim\limits_n (f_n^*, \varphi_{k\nu}) = 0$ for $k \geqslant 0$. In (2.4) with $f = f_n$ set $z = rt$, multiply by $\bar{\varphi}_{k\nu}(t)$ $(k < 0)$ and integrate over b. By orthogonality of Φ $(f_{r,n}, \varphi_{k\nu}) = 0$ for all $k < 0$ and n. Since $\bar{\varphi}_{k\nu} \in C(b)$ as in (2.7) $(f_n^*, \varphi_{k\nu}) = \lim\limits_{r\to 1} (f_{r,n}, \bar{\varphi}_{k\nu}) = 0$ for $k < 0$. Hence

$$\lim_{n\to\infty} \big(f_n^*, P(\Phi)\big) = 0,$$

where $P(\Phi)$ is any linear combination of the $\varphi_{k\nu}$.

Let $\gamma \in (H^p)^*$. Since H^p is a closed subspace of $L^p(b)$ by the Hahn–Banach theorem every bounded linear functional on H^p can be extended to $L^p(b)$. Then by a well-known representation theorem for $p > 1$ [7] there exists a function $g \in L^p(b)$, $1/p + 1/q = 1$, such that $\gamma(F) = (F, g)$ for all $F \in L^p(b)$. In particular $\gamma(f_n) = (f_n^*, g)$. Now approximate g in $L^q(b)$ by a continuous function h. By [16] h can be approximated on b in the sup norm by a linear combination $P(\Phi)$ of $\varphi_{k\nu}$'s. These approximations along with Hölder's inequality and the equality $\|f_n\|_p = \|f_n^*\|_p$ give $\lim\limits_n \gamma(f_n) = 0$, which proves the theorem.

5. Pseudo-uniform convexity of $H^1(D)$.

THEOREM 5. *Let $f_n \to f$ uniformly on compact subsets of D and $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$, where $f_n, f \in H^p(D)$ $(p \geqslant 1)$. Then $\|f_n - f\|_p \to 0$ as $n \to \infty$.*

Proof. For $p > 1$ the result follows from Theorem 4 and the local uniform convexity of H^p ([7], p. 233). $H^1(D)$ is not locally uniformly convex but a proof due to L. D. Hoffman [8] in case D is the unit polydisc or ball in C^N can be extended to all bounded symmetric domains in C^N.

If $f \in H^1(D)$, then the function f_t, defined on $\Delta^1 = \{z: |z| < 1\}$ by $f_t(z) = f(tz)$ for any $t \in b$, belongs to $H^1(\Delta^1)$ for almost all $t \in b$ and

$$\|f\|_1 = \frac{1}{V}\int_b \|f_t\|_{1,1} ds_t,$$

where $\| \ \|_{1,1}$ is the H^1 norm on Δ^1.

Proof. Since $f \in H^1(D)$ and the rotation $t = t' e^{i\theta}$ preserves b and the measure $V^{-1} ds_t$

$$\|f\|_1 \geqslant \|f_r\|_1 = (2\pi)^{-1}\int_0^{2\pi} d\theta \|f_r\|_1 = V^{-1}\int_b ds_{t'} I_{r,t'} \tag{1}$$

by Fubini, where $I_{r,t} = \|f_{rt}\|_{1,1}$ so that

$$\sup_{0 \leqslant r < 1} \frac{1}{V}\int_b I_{r,t} ds_t \leqslant \|f\|_1. \tag{2}$$

(2) implies that $I_{r,t}$ is bounded independently of r for $0 \leqslant r < 1$ and almost all $t \in b$. Since also f_t is holomorphic on Δ^1, $f_t \in H^1(\Delta^1)$ for almost all $t \in b$. Thus $I_{r,t}$ is monotone in r. Interchanging sup and $\int_b$ on the left-hand side of (2) gives $V^{-1}\int_b \|f_t\|_{1,1} ds_t$. By the transformation in (1) the left-hand side of (2) equals $\|f\|_1$. Similarly $f_{t,n}$ has these properties.

It follows as in Hoffman's paper by means of his lemma in integration theory and D. J. Newman's theorem [12] for the case $N = 1$ that $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

Professor Charles Chui independently obtained the same proof of Theorem 5.

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