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On the slow steady motion of a circular cylinder in a viscous fluid

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In memoriam: Witold Pogorzelski

Summary. A new method is given for determining the reactive force on a circular cylinder in uniform translatory motion, and for calculating the flow field near the surface. The underlying idea consists in the exploitation of special properties of a general representation formula for the solution. The result for the force is identical to one originally obtained by Lamb, although the present analysis is based on different principles. The analysis is formally justified by a uniqueness theorem.

1. An infinite circular cylinder Z immersed in a viscous incompressible fluid which fills all space outside the cylinder, is translated with (small) constant velocity $-U$ in a direction orthogonal to its axis. One seeks to determine as much as possible about the motion. It will be assumed that the motion proceeds in the planes of orthogonal sections of Z and that the velocity field is the same in each such section. It will further be supposed that at points whose position relative to Z remain fixed in the motion, the vector velocity is time independent. Viewing the motion from a Galilean frame attached to the cylinder, one is then to find a solution $[\boldsymbol{w}(x); p(x)] = [\{w_1(x_1, x_2), w_2(x_1, x_2)\}; p(x_1, x_2)]$ of the "Navier-Stokes equations"

$$\mu\Delta\boldsymbol{w} - \varrho\boldsymbol{w}\cdot\nabla\boldsymbol{w} - \nabla p = 0, \quad \nabla\cdot\boldsymbol{w} = 0 \tag{1}$$

such that $\boldsymbol{w} = 0$ on the circumference Z: $x_1^2+x_2^2 = a^2$, and such that $\boldsymbol{w}\to(U, 0)$ at $x = \infty$. Here $\boldsymbol{w}$ = velocity, p = pressure, μ = viscosity coefficient, ϱ = density. Since the motion is supposed slow, it is natural to study first the same problem for the approximating "Stokes equations",

$$\mu\Delta\boldsymbol{w} - \nabla p = 0, \quad \nabla\cdot\boldsymbol{w} = 0 \tag{2}$$

which are obtained from (1) by neglecting terms which are quadratic in $\boldsymbol{w}$ and its derivatives.

As is known (cf. [1], [2]), there is no solution to this problem; in fact there is no field $\boldsymbol{w}(x)$ for which (2) holds outside Z, such that

$\boldsymbol{w} = 0$ on Z and $|\boldsymbol{w}| = o(\log r)$ at infinity. This is the "Stokes Paradox" of hydrodynamics.

2. The "Paradox" has been clarified by Oseen ([3], p. 165]), who showed that there is a non-uniformity in the perturbation process, so that the physical assumptions underlying the perturbation are violated at infinity. As a way to avoid the difficulty, Oseen proposed the linearization of (1) about the flow at infinity, which leads to the system of "Oseen equations"

$$\mu \Delta \boldsymbol{w} - \varrho U \frac{\partial \boldsymbol{w}}{\partial x_1} - \nabla p = 0\,, \qquad \nabla \cdot \boldsymbol{w} = 0\,. \tag{3}$$

A solution of (3) for which the boundary condition on Z is approximately satisfied has been given by Lamb [4], who then obtained for the force of reaction of the fluid against the body,

$$\mathfrak{T} = \frac{8\pi\mu U}{1 - 2\log \frac{\gamma Re}{4}} \tag{4}$$

where $\log\gamma =$ Euler's constant, $\sim .577$, and Re is the Reynolds number, $Re = \varrho U a/\mu$. For the (non-dimensional) "drag coefficient" $C_D = \mathfrak{T}(\frac{1}{2}\varrho U^2 a)^{-1}$, Lamb's solution thus yields

$$C_D = \frac{16\pi}{Re\left(1 - 2\log\frac{\gamma Re}{4}\right)}\,. \tag{5}$$

3. The motivation for the present paper rests on the fact that a solution of (2) in three dimensions which represents a flow past a sphere has been obtained explicitly (Stokes [1]), and that although this solution is in some ways physically unrealistic, the calculated reactive force is asymptotically correct for small Re. (This has been shown rigorously in [5], [6].) It seems therefore natural to seek a solution of (2) exterior to a cylinder Z, which, although singular at infinity, would satisfy the boundary condition exactly and would be sufficiently meaningful near Z to yield sensible values for integrals of the motion—notably the reactive force—at low Reynolds numbers. I shall show that such a solution can be selected from among a family of singular solutions

$$\begin{aligned} w_1 &= A\left[-2\log\frac{r}{a} + \cos 2\theta + a^2 r^{-2}\cos 2\theta\right], \\ w_2 &= A[\sin 2\theta + a^2 r^{-2}\sin 2\theta]\,, \\ p &= 4A\mu\frac{\cos\theta}{r} \end{aligned} \tag{6}$$

all of which satisfy $w_1 = w_2 = 0$ for $r = a$. These solutions, although not appearing in Stokes' paper, were certainly known to him. The problem centers on a judicious choice for the parameter A.

4. One computes easily that the reactive force on Z, per unit length, is directed along the x-axis and has the magnitude

$$\mathfrak{T} = -8\pi A \,. \tag{7}$$

With this in mind, we may now state a basic uniqueness property in a form appropriate for this paper.

UNIQUENESS THEOREM. *Let $\boldsymbol{w}^{(\alpha)}, \boldsymbol{w}^{(\beta)}$ be solutions of* (2) *exterior to Z such that $\boldsymbol{w}^{(\alpha)} = \boldsymbol{w}^{(\beta)}$ on Z, such that $|\boldsymbol{w}^{(\alpha)}|, |\boldsymbol{w}^{(\beta)}| = o(r)$ as $r \to \infty$ and such that both solutions yield identical values of reactive force on Z. Then $\boldsymbol{w}^{(\alpha)} \equiv \boldsymbol{w}^{(\beta)}$ throughout the flow field.*

Thus, since (7) shows that A may be chosen to yield any desired value of $\mathfrak{T}$, we find that (modulo a rotation of coordinates) *the family* (6) *contains every solution* $[\boldsymbol{w}(x), p(x)]$ *of* (2), *such that* $|\boldsymbol{w}| = 0$ *on Z and such that* $|\boldsymbol{w}| = o(r)$ *at infinity.*

Proof. I appeal to Theorem 1 of [2], in which the general representation

$$\boldsymbol{w}(x) = \boldsymbol{w}_0 + \chi(x) \cdot \mathfrak{T} + O(r^{-1}) \tag{8}$$

is established for any solution $\boldsymbol{w}(x)$ of (2) in a neighborhood of infinity, for which $\boldsymbol{w}(x) = o(r)$. Here $\mathfrak{T}$ is the vector net reactive force on an inner bounding surface, $\chi(x)$ a fundamental solution tensor associated with the system (2), and $\boldsymbol{w}_0$ a uniquely determined constant vector.

The difference $\boldsymbol{w}(x)$ of two solutions which are equal on Z, both $o(r)$ at infinity, and for which $\mathfrak{T}$ is the same, would then admit the representation

$$\boldsymbol{w}(x) = \boldsymbol{w}_0 + O(r^{-1}) \,.$$

From Theorem 2 of [2] one then concludes $\boldsymbol{w}(x) \equiv 0$ in the flow region, which establishes the stated result.

5. In what follows, it seems conceptually best to write (2) in non-dimensional variables. In order to avoid cumbersome notation, the same symbols will be used. Thus $x, \boldsymbol{w}, p$ will now signify, in the old variables, $\frac{x}{a}, \frac{\boldsymbol{w}}{U}, \frac{a}{\mu U} p$, respectively. Equations (2) become

$$\Delta \boldsymbol{w} - \nabla p = 0 \,, \quad \nabla \cdot \boldsymbol{w} = 0 \tag{2*}$$

and the Oseen equations (3) take the form

$$\Delta \boldsymbol{w} - Re \frac{\partial \boldsymbol{w}}{\partial x_1} - \nabla p = 0 \,, \quad \nabla \cdot \boldsymbol{w} = 0 \,. \tag{3*}$$

The general representation for solutions of (2*) which are $o(r)$ appears as before,

$$\boldsymbol{w}(x) = \boldsymbol{w}_0 + \chi(x)\cdot\mathfrak{T} + O(r^{-1}) \tag{8*}$$

where all quantities are now non-dimensional.

The crucial remark is the following: in the case of three-dimensional solutions of (2*) or of (3*), the constant $\boldsymbol{w}_0$ has the physical interpretation of (non-dimensional) velocity at infinity, that is, $\boldsymbol{w}_0 = \lim_{x\to\infty} \boldsymbol{w}(x)$. In the two-dimensional case, even though the solutions of (2*)

$$\begin{aligned} w_1 &= A[-2\log r + \cos 2\theta + r^{-2}\cos 2\theta], \\ w_2 &= A[\sin 2\theta + r^{-2}\sin 2\theta], \\ p &= 4A\,\frac{\mu}{a}\cdot\frac{\cos\theta}{r} \end{aligned} \tag{6*}$$

do not approach limits at infinity, the same formal procedure leads again to a constant $\boldsymbol{w}_0$, and it is natural to interpret this constant as a velocity at infinity in a generalized sense. Recalling the non-dimensional character of the equation, we seek a solution $\boldsymbol{w}(x)$ of (2*) from the family (6*), such that $\boldsymbol{w}(x)\to(1,0)$ at infinity in that sense. The normalizing vector $(U,0)$ is then to be interpreted as the physical velocity at infinity.

6. A difficulty appears in the fact that in the case considered, the value of $\boldsymbol{w}_0$ in (8*) depends on the particular choice of the fundamental tensor $\chi(x)$. It is natural to choose this tensor to be defined everywhere except at the singular point, and such that all components are $o(r)$ at infinity. In this case, the material of [2] shows that the components χ_{ij} necessarily have the form

$$\chi_{ij} = \frac{1}{4\pi}\left[\delta_{ij}\log r - \frac{x_i x_j}{r^2} + C_{ij}\right]$$

where C_{ij} are constants which may depend on Reynolds number, $C_{ij} = C_{ij}(Re)$. For any choice of C_{ij}, one computes from

$$\mathfrak{T} = -8\pi A \tag{7*}$$

and from (6*) and (8*) that corresponding to the desired limiting velocity must hold $C_{21} = 0$, and $A = -\dfrac{1}{1-2C_{11}}$. Thus for the non-dimensional reaction on the cylinder,

$$\mathfrak{T} = \frac{8\pi}{1-2C_{11}}.$$

It is evident that nothing has been accomplished except to transfer the difficulty from the choice of the arbitrary constant A to be choice of the arbitrary constant C_{11}. However, there is a natural way to choose this latter constant, which simultaneously takes account of the inertial reaction of the fluid elements. To do so, consider the fundamental solution tensor $\bar{\chi}$ of (3*). In this case $\bar{\chi}$ is uniquely determined by the condition that its components vanish at infinity. Near the point $x = 0$, these components admit the representations (cf. [3], p. 38)

$$\bar{\chi}_{ij} = \frac{1}{4\pi}\left[\delta_{ij}\log r - \frac{x_i x_j}{r^2} + \delta_{ij}\log\gamma\frac{Re}{4} + O(r^{-1})\right].$$

Comparison with the above formula for χ_{ij} suggests the identification

$$C_{ij} = \delta_{ij}\log\frac{\gamma Re}{4}$$

from which one obtains the drag coefficient,

$$C_D = \frac{16\pi}{Re\left(1 - 2\log\dfrac{\gamma Re}{4}\right)}.$$

This result is identical to the one obtained by Lamb (equation (5)), but its derivation has been based on different principles.

7. There is apparently a contact between the procedure given above and the method of Kaplun [7], in which it is sought to approximate a solution of (1) (supposed to exist) by two separate expansions, valid respectively near the body and near infinity. The coefficients in these expansions are determined by a matching procedure in an intermediate region. The similarity appears in the determination of the C_{ij} by a process which can be considered as a matching of the fundamental tensors associated with (2*) and with (3*). However, the particular interpretation I have given to the constant $\mathfrak{w}_0$ does not seem to have any counterpart in other theories. It should be noted that the reactive force is calculated from a solution which satisfies exactly the condition on Z, whereas in the work of Lamb this boundary condition is fulfilled only asymptotically as $Re \to 0$. A comparison of this and of other results with experimental measurements has been given by D. J. Tritton [8].

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References

[1] G. G. Stokes, *On the effect of the internal friction of fluids on the motion of pendulums*, Math. and Phys. Papers, vol. III, 1851, p. 1.

[2] I. Dee Chang and R. Finn, *On the solutions of a class of equations occurring in continuum mechanics, with application to the Stokes Paradox*, Arch. Rat. Mech. and Anal. 7 (1961), pp. 388-401.

[3] C. W. Oseen, *Hydrodynamik*, Leipzig 1927.

[4] H. Lamb, *On the uniform motion of a sphere through a viscous fluid*, Phil. Mag. 6 (1911), p. 112. See also *Hydrodynamics*, New York 1945, p. 614.

[5] R. Finn, *On the steady-state solutions of the Navier-Stokes equations III*, Acta Math. 105 (1961), pp. 197-244.

[6] — *On the Stokes Paradox and related questions*: *Nonlinear Problems*, Madison, Univ. of Wisconsin Press (1963), pp. 99-115.

[7] S. Kaplun, *Low Reynolds number flow past a circular cylinder*, J. Math. Mech. 6 (1957), pp. 595-603.

[8] D. J. Tritton, *Experiments on the flow past a circular cylinder at low Reynolds numbers*, J. Math. Phys. 6 (1959), pp. 547-567.

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