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Nonlinear eigenvalue problems with the parameter near resonance

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Abstract. This paper is concerned with some nonlinear Sturm–Liouville problems subject to either periodic or Dirichlet boundary conditions. The nonlinear perturbation term is bounded and satisfies a sign condition and we establish the existence of at least one or at least three solutions depending upon the sign of $\lambda-\lambda_1$, where λ_1 is the principle eigenvalue of the linear part of the problem and λ is the eigenvalue parameter of the problem.

1. Introduction

Consider the nonlinear periodic boundary value problem

(1) $\quad u''+g(x, u) = h(x), \quad 0 \leqslant x \leqslant 2\pi, \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),$

where h is a 2π-periodic function and the nonlinear term g is a 2π-periodic function of the independent variable x. One is interested in finding conditions on h and on g which guarantee that problem (1) have a solution. This problem area has a long history and many results in this direction have been found (see e.g. [5]). (Opial has contributed extensively to this area and many of his beautiful papers have been devoted to this subject.)

In this paper, we imbed problem (1) into the one parameter family of problems

(2) $u''+\lambda u+g(x, u) = h(x), \quad 0 \leqslant x \leqslant 2\pi \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),$

where λ is a parameter, and study the existence and multiplicity of solutions of problem (2) for λ in a neighborhood of 0. As a byproduct, we, of course, shall obtain the existence of a solution of (1). Our method of treating problem (2) will consist of employing techniques from bifurcation and continuation theory. We shall see that similar techniques will apply in the study of the two point boundary value problem

(3) $\quad u''+\lambda u+g(x, u) = h(x), \quad 0 \leqslant x \leqslant \pi, \quad u(0) = 0 = u(\pi),$

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or problems with more general boundary conditions, or, in fact boundary value problems for forced semilinear elliptic equations on bounded domains with smooth boundary. In this paper, however, we shall, for the ease of exposition, confine ourselves to problems for ordinary differential equations.

The methods employed here are similar to those employed in [2], [3], and [6] and the results obtained here complement those in [3].

In the next section, we shall state the abstract tools which will be employed. In sections 3 and 4 we then shall study problems (2) and (3), respectively.

2. Bifurcation and continuation

Let E be a real Banach space and let

$$h\colon E\times \boldsymbol{R}\to E$$

be a completely continuous mapping. We consider the equation

$$u-h(u,\lambda)=0. \tag{4}$$

The following result is an easy consequence of the generalized homotopy principle of Leray–Schauder degree and may be found in [1], [4], or [7].

THEOREM 1. *Let there exist a bounded open set O in E such that*

$$\deg(id-h(\cdot,a),O,0)\neq 0. \tag{5}$$

Then there exist continua C^-, C^+, with

$$C^-\subset E\times(-\infty,a]\cap(id-h)^{-1}(0),\qquad C^+\subset E\times[a,\infty)\cap(id-h)^{-1}(0),$$

and for both $C=C^-$ and $C=C^+$ the following are valid:

(i) $C\cap O\times\{a\}\neq\emptyset$.

(ii) *Either C is unbounded or else $C\cap E\backslash\bar{O}\times\{a\}\neq\emptyset$.*

The above result has as an easy consequence the following corollary which will be useful for our purposes.

COROLLARY 2. *Assume the conditions of Theorem 1, where O is given by*

$$O=B_R(0)=\{u\in E\colon \|u\|<R\}.$$

Furthermore assume that there exists $b>a$ such that for any λ, $a\leqslant\lambda\leqslant b$ we have that $\|u\|<R$, where (u,λ) is a solution of (4). Then there exists $\delta>0$, such that for $b\leqslant\lambda\leqslant b+\delta$ there exists $(u,\lambda)\in C^+$ with $\|u\|\leqslant 2R$.

A similar statement holds for values of λ to the left of a.

As a further tool we need a result which guarantees bifurcation from *infinity*. For this we shall assume a particular form for the completely continuous mapping h, namely

$$h(u,\lambda)=L(\lambda)u+H(u,\lambda), \tag{6}$$

where the family $L(\lambda)$ is a family of compact linear operators, and the perturbation term H has the property that

$$\frac{H(u, \lambda)}{\|u\|} \to 0 \quad \text{as } \|u\| \to \infty. \tag{7}$$

THEOREM 3. *Assume (6) and (7) hold and assume that for $\lambda = \lambda_1$ the generalized nullspace of $id - L(\lambda_1)$ has odd dimension. Then there exists a continuum $C \subset (id - h)^{-1}(0)$ which bifurcates from infinity at $\lambda = \lambda_1$. That is: For any $\varepsilon > 0$, there exists $(u, \lambda) \in C$ with*

$$|\lambda - \lambda_1| < \varepsilon \quad \text{and} \quad \|u\| > 1/\varepsilon.$$

In many instances, particularly in applications of the above result to nonlinear ordinary differential equations, it is the case that the above eigenspace in one dimensional and further that one is confronted with the situation that the theory of positive operators (Krein–Rutman) may be applied to deduce certain positivity properties of the continuum C, in which case one often obtains two different kinds of solutions of large norm, a positive one and a negative one. We shall have occasion to use such a result. Since such arguments have been employed earlier (see [6]) we shall not go into details here but rather refer to [6].

3. Periodic boundary value problems

In this section we shall consider the periodic boundary value problem (2) for values of λ in a neighborhood of 0, which is the principle eigenvalue of the linear problem and which is of multiplicity one.

We shall make the following assumptions:

(a) $h \in L^1(0, 2\pi)$ and

$$\int_0^{2\pi} h(x)dx = 0. \tag{8}$$

(b) g satisfies Carathéodory conditions and there exists $\gamma \in L_1(0, 2\pi)$ with

$$|g(x, u)| \leqslant \gamma(x).$$

(c) There exists a constant $R > 0$ such that

$$g(x, u)u > 0, \quad |u| \geqslant R. \tag{9}$$

Even though it is very standard, we shall briefly indicate how problem (2) may be written in an equivalent operator form (4). To this end, we note that the linear problem

$$u'' - u = h(x),$$

subject to 2π-periodic boundary conditions, has, for any $h \in L_1(0, 2\pi)$ a unique solution which defines a continuous linear operator

$$L\colon\ L_1(0, 2\pi) \to C^1_{2\pi}[0, 2\pi]$$

(the subscript 2π signifies the periodic boundary conditions) and hence, by the compact imbedding $C^1[0, 2\pi] \to C[0, 2\pi]$ and the continuous imbedding $C[0, 2\pi] \to L_1(0, 2\pi)$ we may regard L as a compact operator of $L_1(0, 2\pi)$ to itself. Further problem (2) is then equivalent to

$$u = (\lambda+1)Lu + L(g(\cdot, u) - h). \tag{10}$$

It is easy to see that $\lambda_1 = 0$ is the principle eigenvalue of the linear part which has a one dimensional eigenspace associated with it. We hence may apply the results of the previous section.

THEOREM 4. *Assume conditions* (a), (b), (c). *Then for all* $\lambda \in (-\infty, 1)$ *problem* (2) *has at least one solution. Whereas for* $\lambda < 0$ *close to* 0, *problem* (2) *has at least three solutions.*

Proof. If $\lambda \in (-\infty, 0) \cup (0, 1)$ we may, since g is bounded, apply the Schauder fixed point theorem. To obtain the other parts of the result, we shall employ Corollary 2 and Theorem 3. We first claim, that there exists a constant $R_0 > 0$, such that, if

$$0 \leqslant \lambda \leqslant \delta < 1, \quad 0 \leqslant \mu \leqslant 1,$$

then any solution u satisfying the periodic boundary conditions and

$$u'' + \lambda u + \mu g(x, u) = \mu h(x), \tag{11}$$

satisfies

$$\|u\|_\infty = \max_{0 \leqslant x \leqslant 2\pi} |u(x)| < R_0.$$

To see this, we let

$$\bar{u} = \frac{1}{2\pi} \int_0^{2\pi} u(x)dx$$

and

$$u = \bar{u} + \tilde{u}.$$

We multiply equation (11) by $\tilde{u}$ and integrate, to obtain

$$\int_0^{2\pi} \{-(u')^2 + \lambda \tilde{u}^2 + \mu g(x, u)\tilde{u}\}dx = \mu \int_0^{2\pi} h\tilde{u}\, dx. \tag{12}$$

Since Wirtinger's inequality

$$\int_0^{2\pi} \tilde{u}^2 dx \leqslant \int_0^{2\pi} u'^2 dx$$

holds, it follows from (12) that

$$\int_0^{2\pi} u'^2 dx \leqslant \delta \int_0^{2\pi} u'^2 dx + \mu(\|\gamma\|_{L_1} + \|h\|_{L_1})\|\tilde{u}\|_\infty. \tag{13}$$

On the other hand, it follows from Sobolev's imbedding theorem that there exists a constant c such that

$$\|\tilde{u}\|_\infty^2 \leqslant c(\|u'\|_{L_2}^2 + \|\tilde{u}\|_{L_2}^2)$$

and hence, (13) and Wirtinger's inequality imply that

$$\|\tilde{u}\|_\infty^2 \leqslant \frac{2c}{1-\delta}(\|\gamma\|_{L_1} + \|h\|_{L_1})\|\tilde{u}\|_\infty. \tag{14}$$

We hence have obtained a bound on $\|\tilde{u}\|_\infty$. To obtain a bound on $\bar{u}$ we integrate equation (11) and obtain

$$\lambda\bar{u} + \frac{1}{2\pi}\int_0^{2\pi} g(x, \bar{u}+\tilde{u})dx = 0,$$

and therefore

$$2\pi\lambda\bar{u}^2 + \int_0^{2\pi} \bar{u}g(x, \bar{u}+\tilde{u})dx = 0. \tag{15}$$

A bound on $\bar{u}$ hence follows from (9), the already established bound on $\tilde{u}$ and an easy indirect argument. The necessary L_1 bound, of course follows immediately. Furthermore, since the established bound is independent of the parameter μ, we may, for $0 < \lambda \leqslant \delta$ compute the Leray–Schauder degree (5) as that of a linear homeomorphism, which is nonzero (in our case -1). We therefore may apply Corollary 2, to deduce the existence of solutions for $0 \leqslant \lambda < 1$. On the other hand, since $\lambda = 0$ is a simple eigenvalue, Theorem 3 also applies and we may conclude that there is bifurcation from infinity at $\lambda = 0$. Since, however, we have established a priori bounds for solutions when $0 \leqslant \lambda \leqslant \delta < 1$, the continua bifurcating from infinity, must do so for $\lambda < 0$. We now may employ arguments as [6] to deduce that for $\lambda < 0$ but close to 0, there must in fact exist a large positive and a large negative solution. This coupled with Corollary 2 and what has been proved above, yields the existence of three solutions for $\lambda < 0$ close to 0.

We note that it follows immediately from our proof above that condition (9) may be replaced by the less stringent requirement:

$$\begin{aligned} &\int_0^{2\pi} g(x, u(x))dx > 0, \quad \forall u, \quad \inf u(x) \geqslant R, \\ &\int_0^{2\pi} g(x, u(x))dx < 0, \quad \forall u, \quad \sup u(x) \leqslant -R. \end{aligned} \tag{16}$$

Furthermore, it is easy to see, how to obtain a result dual to Theorem 4, once one reverses the inequalities in (9) or (16). By carefully analyzing (12), one also sees how to obtain results similar to Theorem 4 in case g is not necessarily bounded, but is allowed to grow sublinearly or linearly with $\|\gamma\|_{L_1} < 1$. We shall not dwell on such points here.

4. Dirichlet boundary value problems

We now turn to the Dirichlet problem (3). As in the previous section this problem is equivalent to an operator equation (4) and as above, in order to prove similar existence results it will suffice to establish the existence of certain a priori bounds. Before stating our main result let us note that the principle eigenvalue of the linear part of problem (3) is given by $\lambda_1 = 1$ with associated eigenspace spanned by $\sin x$.

We shall make the following assumptions:

(d) $h \in L_1(0, \pi)$ and

$$\int_0^\pi h(x)\sin x \; dx = 0. \tag{17}$$

(e) g satisfies Carathéodory conditions as in (b) with $\gamma \in L_1(0, \pi)$.

(f) $ug(x, u) > 0, \; u \neq 0$.

We then have the following theorem.

THEOREM 5. *Assume that conditions* (d)–(f) *hold. Then for every* λ, $-\infty < \lambda < 4$, *problem* (3) *has at least one solution. Furthermore if* $\lambda < 1$ *but close to* 1 *problem* (3) *has at least three solutions.*

Proof. We claim that given δ, $1 < \delta < 4$, there exists a constant R_0 such that for every λ, $1 \leqslant \lambda \leqslant \delta$ and every μ, $0 \leqslant \mu \leqslant 1$, every solution u of

$$u'' + \lambda u + \mu g(x, u) = \mu h(x), \tag{18}$$

which satisfies the zero Dirichlet boundary conditions, is such that

$$\|u\|_\infty < R_0.$$

Once this claim is established we may proceed as in the proof of Theorem 4. Thus, we shall confine ourselves to this point only. We write

$$u(x) = a\sin x + v(x),$$

with

$$\int_0^\pi v(x)\sin x dx = 0.$$

Then v satisfies the equation

$$v'' + \lambda v + (\lambda - 1)a\sin x + \mu g(x, u) = \mu h(x). \tag{19}$$

Multiplying (19) by v and integrating, we obtain the inequality

$$\|v'\|_{L_2}^2 \leqslant \lambda \|v\|_{L_2}^2 + (\|\gamma\|_{L_1} + \|h\|_{L_1})\|v\|_\infty . \tag{20}$$

Using Poincaré's inequality

$$\|v\|_{L_2}^2 \leqslant \tfrac{1}{4}\|v'\|_{L_2}^2,$$

in (20), we obtain

$$\|v'\|_{L_2}^2 \leqslant \frac{4c}{4-\delta}\|v\|_\infty, \tag{21}$$

where c is a constant that only depends upon γ and h. It follows again from Sobolev's imbedding theorem that $\|v\|_\infty^2 \leqslant \bar{c}\|v'\|_{L_2}^2$, and hence from (21) that

$$\|v\|_\infty \leqslant \frac{4c\bar{c}}{4-\delta}. \tag{22}$$

It follows from (22) and the fact that v satisfies the Dirichlet boundary conditions, that there exists a constant d so that

$$|v(x)| \leqslant d\sin x, \qquad 0 \leqslant x \leqslant \pi. \tag{23}$$

We next multiply (18) by $\sin x$ and integrate to obtain

$$\lambda a + \mu \int_0^\pi g(x, a\sin x + v)\sin x \; dx = 0, \tag{24}$$

or

$$\lambda a^2 + \int_0^\pi a\sin x g\left(x, \sin x\left(a + \frac{v(x)}{\sin x}\right)\right)dx = 0. \tag{25}$$

Using (23), we may now deduce from (25) the desired bound on a (recall (f)).

A typical nonlinearity which satisfies the above conditions is

$$g(u) = \frac{u}{1+u^2}.$$

We note that this nonlinearity is such that the usual *Landesman–Lazer* conditions are not satisfied (see [3]). Also it is easy to see that the strict inequality in (f) cannot be replaced by a weak inequality.

As before, Theorem 5 has an obvious dual result in case the sign in (f) is reversed.

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