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On the non-convergence of successive aproximations in the theory of the equation $z''_{xy} = f(x, y, z, z'_x, z'_y)$

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Abstract. Let $D = [0, a] \times [0, b]$ and let us denote by $\mathscr{F}_A(P)$ the set of all continuous and bounded functions $f\colon D \times \boldsymbol{R}^{3n} \to \boldsymbol{R}^n$ such that $\|f(x, y, z, p, q) - f(x, y, z, \overline{p}, \overline{q})\| \leqslant A(\|p-\overline{p}\| + \|q-\overline{q}\|)$ for $(x, y, z, p, q), (x, y, z, \overline{p}, \overline{q}) \in D \times \boldsymbol{R}^{3n}$, where $A > 0$. In this paper it is shown that the set $\mathscr{A} \subset \mathscr{F}_A(P)$ consisting of those $f \in \mathscr{F}_A(P)$ for which successive approximations $\{z_n^f\}$ of the Darboux problem for the equation $z''_{xy} = f(x, y, z, z'_x, z'_y)$ are non-converging together with $\{\partial z_n^f/\partial x\}$ and $\{\partial z_n^f/\partial y\}$ is of Baire's first category in the space $\mathscr{F}_A(P)$ for every $A > 0$.

1. Introduction. Let R denote the real line and let $\boldsymbol{R}^n$ be the n-dimensional linear space with a norm $\|\cdot\|$. Let $D = [0, a] \times [0, b]$ for $0 < a < b < \infty$ and let P be defined by $P = D \times \boldsymbol{R}^{3n}$. By Q_k we shall mean the subset of P of the form $Q_k = \{(x, y, z, p, q) \in P\colon \|z\|, \|p\|, \|q\| \leqslant k\}$ for $k > 0$. We will denote by $C^*(D)$ the space of all functions $v\colon D \to \boldsymbol{R}^n$ continuous together with partial derivatives v_x, v_y and v_{xy} on D. We introduce in $C^*(D)$ the norm $\|\cdot\|_{C^*}$ by the formula $\|u\|_{C^*} = \sup_D\big(\|u(x, y)\| + \|u'_x(x, y)\| + \|u'_y(x, y)\|\big)$. It will be tacitly assumed that the functions $\sigma\colon [0, a] \to \boldsymbol{R}^n$ and $\tau\colon [0, b] \to \boldsymbol{R}^n$ have continuous derivatives of first order and that $\sigma(0) = \tau(0)$.

Let $CB(P)$ denote the space of all continuous bounded functions $f\colon P \to \boldsymbol{R}^n$. For a given $f \in CB(P)$ we consider in this paper the hyperbolic equation

$$z''_{xy} = f(x, y, z, z'_x, z'_y). \tag{1}$$

It is easy to verify that (1) with the initial conditions $z(x, 0) = \sigma(x)$ and $z(0, y) = \tau(y)$ for $(x, y) \in D$ is equivalent to the integral equation

$$z(x, y) = \varphi_0(x, y) + \int_0^x \int_0^y f\big(\xi, \eta, z(\xi, \eta), z'_x(\xi, \eta), z'_y(\xi, \eta)\big) d\xi\, d\eta, \tag{2}$$

where $\varphi_0(x, y) = \sigma(x) + \tau(y) - \sigma(0)$.

It is known ([1], [2]) that, for the existence of a solution of (2), it suffices to suppose that $f \in CB(P)$ satisfies the Lipschitz condition of the form $\|f(x, y, z, p, q)-f(x, y, z, \overline{p}, \overline{q})\| \leqslant A(\|p-\overline{p}\|+\|q-\overline{q}\|)$ for (x, y, z, p, q), $(x, y, z, \overline{p}, \overline{q}) \in P$, where $A > 0$. This is not sufficient to guarantee the uniqueness of solutions. A. Alexiewicz and W. Orlicz have proved in [1] that non-uniqueness of solutions of (2) is, in some sense, a rare case.

The process of successive approximation is, in general, not successful if the function $f \in CB(P)$ satisfies only the Lipschitz condition with respect to p and q. Although the functions

(3)

$$z_0^f(x, y) = \sigma(x)+\tau(y)-\sigma(0),$$

$$\dots\dots\dots\dots\dots$$

$$z_n^f(x, y) = \varphi_0(x, y)+\int_0^x\int_0^y f\Big(\xi, \eta, z_{n-1}^f(\xi, \eta), \frac{\partial z_{n-1}^f}{\partial x}(\xi, \eta), \frac{\partial z_{n-1}^f}{\partial y}(\xi, \eta)\Big)d\xi d\eta,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

are well-defined, the sequence $\{z_n^f\}$ is not necessarily convergent. In this paper we prove that the non-convergence of the sequence $\{z_n^f\}$ is, in some sense, a rare case.

Let us observe that for every $f \in CB(P)$ there is a number $k_f > 0$ such that $\|z_n^f\|_{C^*} \leqslant k_f$ for every $n = 0, 1, 2, \dots$ Therefore the sequence $\{z_n^f\}$ defined by (3) for $f \in CB(P)$ is just the same as the sequence $\{z_n^{f|Q_{k_f}}\}$ corresponding to $f|Q_{k_f}$, where $f|Q_{k_f}$ denotes the restriction of f to the set Q_{k_f}.

2. A fundamental metric space. Let us denote by $\mathscr{F}_A(P)$ the set of all functions $f \in CB(P)$ such that

$$\|f(x, y, z, p, q)-f(x, y, z, \overline{p}, \overline{q})\| \leqslant A(\|p-\overline{p}\|+\|q-\overline{q}\|)$$

for (x, y, z, p, q), $(x, y, z, \overline{p}, \overline{q}) \in P$, where $A > 0$. We introduce in $\mathscr{F}_A(P)$ a metric ϱ in the following way: $\varrho(f_1, f_2) = \|f_1-f_2\|_{\mathscr{F}}$, where $\|f\|_{\mathscr{F}} = \sup_P \|f(x, y, z, p, q)\|$ for $f, f_1, f_2 \in \mathscr{F}_A(P)$.

LEMMA 1. *$(\mathscr{F}_A(P), \varrho)$ is a complete metric space for every $A > 0$.*

Proof. Suppose $\{f_n\}$ is a sequence of $\mathscr{F}_A(P)$ such that $\lim_{n\to\infty} \varrho(f_n, f) = 0$, where $f \in CB(P)$. For (x, y, z, p, q), $(x, y, z, \overline{p}, \overline{q}) \in P$ we have

$$\|f(x, y, z, p, q)-f(x, y, z, \overline{p}, \overline{q})\|$$
$$\leqslant \|f(x, y, z, p, q)-f_n(x, y, z, p, q)\|+\|f_n(x, y, z, p, q)-f_n(x, y, z, \overline{p}, \overline{q})\|+$$
$$+\|f^n(x, y, z, \overline{p}, \overline{q})-f(x, y, z, \overline{p}, \overline{q})\|.$$

Hence it follows

$$\|f(x, y, z, p, q)-f(x, y, z, \overline{p}, \overline{q})\| \leqslant A(\|p-\overline{p}\|+\|q-\overline{q}\|).$$

Therefore $f \in \mathscr{F}_A(P)$. This completes the proof.

For a given $f \in \mathscr{F}_A(P)$ and a positive integer n let us define, just as in [1], the function $f_n\colon Q_{k_f} \to \boldsymbol{R}^n$ having the following properties

$$\max_{Q_{k_f}} \|f_n(x, y, z, p, q) - f(x, y, z, q)\| \leqslant \frac{1}{n},$$

$$\|f_n(x, y, z, p, q) - f_n(x, y, \bar{z}, \bar{p}, \bar{q})\| \leqslant L_n(\|z - \bar{z}\| + \|p - \bar{p}\| + \|q - \bar{q}\|),$$

and

$$\|f_n(x, y, z, p, q) - f_n(x, y, z, \bar{p}, \bar{q})\| \leqslant A(\|p - \bar{p}\| + \|q - \bar{q}\|)$$

for $(x, y, z, p, q), (x, y, \bar{z}, \bar{p}, \bar{q}) \in Q_{k_f}$.

Let $\{\hat{z}_n\}$ be the sequence of successive approximations defined by (3) corresponding to f_n defined above. In virtue of theorem given in [3] there is $z \in C^*(D)$ such that $\|\hat{z}_n - z\|_{C^*} \to 0$ as $n \to \infty$. For $w \in \boldsymbol{R}^n$ and $k \in \boldsymbol{R}^+$ let $\overline{w}$ be defined by

$$\overline{w} = \begin{cases} k \cdot e & \text{for } \|w\| > k, \\ w & \text{for } \|w\| \leqslant k, \end{cases}$$

where $e = (\underbrace{1, 1, \ldots, 1}_{n})$. Then for $f \in \mathscr{F}_A(P)$ let

$$\tilde{f}(x, y, z, p, q) = (f|Q_{k_f})(x, y, \bar{z}, \bar{p}, \bar{q}).$$

It is easy to see that if f and f_n are such as above, then $\varrho(\tilde{f}, \tilde{f}_n) \leqslant 1/n$. Obviously $z_n^f = z_n^{\tilde{f}}$ for every $f \in \mathscr{F}_A(P)$ and $n = 1, 2, \ldots$

3. Non-convergence of successive approximations. Now we shall show that the non-convergence of the sequence $\{z_n^f\}$ defined by (3) is, in some sense, a rare case. More precisely, we shall prove that the set $\mathscr{A}$ of those $f \in \mathscr{F}_A(P)$ for which the sequences $\{z_n^f\}$, $\{\partial z_n^f/\partial x\}$ and $\{\partial z_n^f/\partial y\}$ are not converging is of Baire's first category in $(\mathscr{F}_A(P), \varrho)$.

For a given $z \in C^*(D)$ let $\hat{z} = (z, z'_x, z'_y)$. Then for a fixed $f \in \mathscr{F}_A(P)$ and $(x, y) \in D$ let $\Delta(f, x, y) = \limsup_{n \to \infty}\{\operatorname{diam} E[\hat{z}_n^f(x, y)]\}$, where $E[\hat{z}_n^f(x, y)] = \{\hat{z}_n^f(x, y), \hat{z}_{n+1}^f(x, y), \ldots\}$ and $\operatorname{diam} A$ denotes the diameter of a set $A \subset \boldsymbol{R}^{3n}$. Taking $|\hat{z}(x, y)| = \|z(x, y)\| + \|z'_x(x, y)\| + \|z'_y(x, y)\|$ for $z \in C^*(D)$, we have

$$\operatorname{diam} E[\hat{z}_n^f(x, y)] = \sup_{(u,v)} |\hat{z}_{n+u}^f(x, y) - \hat{z}_{n+v}^f(x, y)|.$$

Hence it follows that the sequences $\{z_n^f\}$, $\{\partial z_n^f/\partial x\}$ and $\{\partial z_n^f/\partial y\}$ are convergent on D iff $\Delta(f, x, y) = 0$ for every $(x, y) \in D$. Then $\{z_n^f\}$, $\{\partial z_n^f/\partial x\}$ and $\{\partial z_n^f/\partial y\}$ are not converging iff there is a point $(\tilde{x}, \tilde{y}) \in D$ such that $\Delta(f, \tilde{x}, \tilde{y}) > 0$.

Let $\{(x_r, y_r)\}$ be a sequence of points of D dense in D. Then let $\Omega_{Mpr} = \{f \in \mathscr{F}_A(P)\colon \|f\|_{\mathscr{F}} \leqslant M,\ \Delta(f, x_r, y_r) \geqslant 1/p\}$. In the proof of our main result we shall use following lemmas:

LEMMA 2. *Ω_{Mpr} are closed subsets of $\mathscr{F}_A(P)$ for every $M, r, p = 1, 2, \ldots$*

Proof. Suppose $\{f_k\}$ is a sequence in Ω_{Mpr} such that $\|f_k - f\|_{\mathscr{F}} \to 0$ as $k \to \infty$, where $f \in \mathscr{F}_A(P)$. It is easy to see that $\|f\|_{\mathscr{F}} \leqslant M$. For every

$k = 1, 2, \ldots$ we have $\Delta(f_k, x_r, y_r) \geqslant 1/p$. Then $\sup_m\{\operatorname{diam} E[\hat{z}^{f_k}_{n+m}(x_r, y_r)]\} \geqslant 1/p$ for $n, k = 1, 2, \ldots$ Hence it follows that for every $l = 1, 2, \ldots$ there is m_l such that $\operatorname{diam} E[\hat{z}^{f_k}_{n+m_l}(x_r, y_r)] > 1/p - 1/l$, i.e.,

$$\sup_{(u,v)} \left|\hat{z}^{f_k}_{n+m_l+u}(x_r, y_r) - \hat{z}^{f_k}_{n+m_l+v}(x_r, y_r)\right| > 1/p - 1/l.$$

Then for every $q = 1, 2, \ldots$ there is (u_q, v_q) such that

$$\left|\hat{z}^{f_k}_{n+m_l+u_q}(x_r, y_r) - \hat{z}^{f_k}_{n+m_l+v_q}(x_r, y_r)\right| > 1/p - 1/l - 1/q$$

for $n, k = 1, 2, \ldots$ Let $Z(k, n+m_l+u_q)(x, y) = z^{f_k}_{n+m_l+u_q}(x, y)$. It is easy to see that the family $X \subset C^*(D)$ defined by $X = \{Z(k, n+m_l+u_q)\}_{k,n,l,q=1,2,\ldots}$ satisfies the hypotheses of Arzela's Theorem. Then there is a subsequence $\{Z(n_k, n+m_l+u_q)\}$ of $\{Z(k, n+m_l+u_q)\}$ which is uniformly convergent on D. Suppose $\lim_{k\to\infty} Z(n_k, n+m_l+u_q) = Z(n+m_l+u_q)$.

We shall show that

$$Z(n+m_l+u_l)(x, y) = \varphi_0(x, y) + \int_0^x \int_0^y f\Big(\xi, \eta, Z(n+m_l+u_q-1)(\xi, \eta), \frac{\partial Z(n+m_l+u_q-1)}{\partial x}(\xi, \eta), \frac{\partial Z(n+m_l+u_q-1)}{\partial y}(\xi, \eta)\Big) d\xi\, d\eta$$

for every $n, l, q = 1, 2, \ldots$ and $(x, y) \in D$. Let

$$g_{\nu\mu}(n+m_l+u_q)(x, y) = \left\|\frac{\partial Z(n_\nu, n+m_l+u_q)}{\partial x}(x, y) - \frac{\partial Z(n_\mu, n+m_l-u_q)}{\partial x}(x, y)\right\| + \left\|\frac{\partial Z(n_\nu, n+m_l+u_q)}{\partial y}(x, y) - \frac{\partial Z(n_\mu, n+m_l-u_q)}{\partial y}(x, y)\right\|$$

for $(x, y) \in D$ and $\nu, \mu, n, l, q = 1, 2, \ldots$ For $(x, y) \in D$ and $\nu, \mu, n, l, q = 1, 2, \ldots$ we have

$$\frac{\partial Z(n_\nu, n+m_l+u_q}{\partial x}(x, y) - \frac{\partial Z(n_\mu, n+m_l+u_q)}{\partial x}(x, y) \tag{4}$$

$$= \int_0^y \Big\{f_{n_\nu}\Big[x, \eta, Z(n_\nu, n+m_l+u_q-1)(x, \eta), \frac{\partial Z(n_\nu, n+m_l+u_q-1)}{\partial x}(x, \eta), \frac{\partial Z(n_\nu, n+m_l+u_q-1)}{\partial y}(x, \eta)\Big] - f_{n_\mu}\Big[x, \eta, Z(n_\mu, n+m_l+u_q-1)(x, \eta), \frac{\partial Z(n_\mu, n+m_l+u_q-1)}{\partial x}(x, \eta), \frac{\partial Z(n_\mu, n+m_l+u_q-1)}{\partial y}(x, \eta)\Big]\Big\} d\eta$$

and

$$(5)\quad \frac{\partial Z(n_\nu, n+m_l+u_q)}{\partial y}(x, y) - \frac{\partial Z(n_\mu, n+m_l+u_q)}{\partial y}(x, y)$$

$$= \int_0^x \Big\{ f_{n_\nu}\Big[\xi, y, Z(n_\nu, n+m_l+u_q-1)(\xi, y), \frac{\partial Z(n_\nu, n+m_l+u_q-1)}{\partial x}(\xi, y),$$

$$\frac{\partial Z(n_\nu, n+m_l+u_q-1)}{\partial y}(\xi, y)\Big] - f_{n_\mu}\Big[\xi, y, Z(n_\mu, n+m_l+u_q-1)(\xi, y),$$

$$\frac{\partial Z(n_\mu, n+m_l+u_q-1)}{\partial x}(\xi, y), \frac{\partial Z(n_\mu, n+m_l+u_q-1)}{\partial y}(\xi, y)\Big]\Big\} d\xi .$$

Since

$$\Big\| f_{n_\nu}\Big[x, y, Z(n_\nu, n+m_l+u_q-1), \frac{\partial Z(n_\nu, n+m_l+u_q-1)}{\partial x},$$

$$\frac{\partial Z(n_\nu, n+m_l+u_q-1)}{\partial y}\Big] - f_{n_\mu}\Big[x, y, Z(n_\mu, n+m_l+u_q-1),$$

$$\frac{\partial Z(n_\mu, n+m_l+u_q-1)}{\partial x}, \frac{\partial Z(n_\mu, n+m_l+u_q-1)}{\partial y}\Big]\Big\|$$

$$\leqslant \Big\| f_{n_\nu}\Big[x, y, Z(n_\nu, n+m_l+u_q-1), \frac{\partial Z(n_\nu, n+m_l+u_q-1)}{\partial x},$$

$$\frac{\partial Z(n_\nu, n+m_l+u_q-1)}{\partial y}\Big] - f_{n_\nu}\Big[x, y, Z(n+m_l+u_q-1),$$

$$\frac{\partial Z(n_\nu, n+m_l+u_q-1)}{\partial x}, \frac{\partial Z(n_\nu, n+m_l+u_q-1)}{\partial y}\Big]\Big\| +$$

$$+ \Big\| f_{n_\nu}\Big[x, y, Z(n+m_l+u_q-1), \frac{\partial Z(n_\nu, n+m_l+u_q-1)}{\partial x},$$

$$\frac{\partial Z(n_\nu, n+m_l+u_q-1)}{\partial y}\Big] - f\Big[x, y, Z(n+m_l+u_q-1),$$

$$\frac{\partial Z(n_\nu, n+m_l+u_q-1)}{\partial x}, \frac{\partial Z(n_\nu, n+m_l+u_q-1)}{\partial y}\Big]\Big\| +$$

$$+ \Big\| f\Big[x, y, Z(n+m_l+u_q-1), \frac{\partial Z(n_\nu, n+m_l-u_q-1)}{\partial x},$$

$$\frac{\partial Z(n_\nu, n+m_l+u_q-1)}{\partial y}\Big] - f\Big[x, y, Z(n+m_l+u_q-1),$$

$$\frac{\partial Z(n_\mu, n+m_l+u_q-1)}{\partial x}, \frac{\partial Z(u_\mu, n+m_l+u_q-1)}{\partial y}\Big]\Big\| +$$

$$+\Big\| f\Big[x, y, Z(n+m_l+u_q-1), \frac{\partial Z(n_\mu, n+m_l+u_q-1)}{\partial x},$$

$$\frac{\partial Z(u_\mu, n+m_l+u_q-1)}{\partial y}\Big] - f_{n_\mu}\Big[x, y, Z(n+m_l+u_q-1),$$

$$\frac{\partial Z(n_\mu, n+m_l+u_q-1)}{\partial x}, \frac{\partial Z(n_\mu, n+m_l+u_q-1)}{\partial y}\Big]\Big\| +$$

$$+\Big\| f_{n_\mu}\Big[x, y, Z(n+m_l+u_q-1), \frac{\partial Z(n_\mu, n+m_l+u_q-1)}{\partial x},$$

$$\frac{\partial Z(n_\mu, n+m_l+u_q-1)}{\partial y}\Big] - f_{n_\mu}\Big[x, y, Z(n_\mu, n+m_l+u_q-1),$$

$$\frac{\partial Z(n_\mu, n+m_l+u_q-1)}{\partial x}, \frac{\partial Z(n_\mu, n+m_l+u_q-1)}{\partial y}\Big]\Big\|,$$

then for every $\varepsilon > 0$ there is a number $N(\varepsilon)$ such that for $\nu, \mu > N(\varepsilon)$ we have

$$(6) \quad \Big\| \frac{\partial Z(n_\nu, n+m_l+u_q)}{\partial x}(x, y) - \frac{\partial Z(n_\mu, n+m_l+u_q)}{\partial x}(x, y)\Big\|$$
$$< \frac{\varepsilon}{2} + A\int_0^y g_{\nu\mu}(n+m_l+u_q-1)(x, \eta)\,d\eta$$

for $n, l, q = 1, 2, \ldots, (x, y) \in D$.

In a similar way we obtain

$$(7) \quad \Big\| \frac{\partial Z(n_\nu, n+m_l+u_q)}{\partial y}(x, y) - \frac{\partial Z(n_\mu, n+m_l+u_q)}{\partial y}(x, y)\Big\|$$
$$< \frac{\varepsilon}{2} + A\int_0^x g_{\nu\mu}(n+m_l+u_q-1)(\xi, y)\,d\xi$$

for $n, l, q = 1, 2, \ldots, (x, y) \in D$ and $\nu, \mu \geqslant N(\varepsilon)$.

Hence it follows

$$(8) \quad g_{\nu\mu}(n+m_l+u_q)(x, y)$$
$$< \varepsilon + A\Big[\int_0^x g_{\nu\mu}(n+m_l+u_q-1)(\xi, y)\,d\xi + \int_0^y g_{\nu\mu}(n+m_l+u_q-1)(x, \eta)\,d\eta\Big]$$

for $n, l, q = 1, 2, \ldots, (x, y) \in D$ and $\nu, \mu \geqslant N(\varepsilon)$. Since $g_{\nu\mu}(1)(x, y) < \varepsilon$

for $\nu, \mu \geqslant N(\varepsilon)$ and $(x, y) \in D$, then

$$g_{\nu\mu}(n+m_l+u_q)(x, y) < \varepsilon \cdot \sum_{j=0}^{n+m_l+u_q-1} [A(a+b)^j]$$

for $n, l, q = 1, 2, \ldots, (x, y) \in D$ and $\nu, \mu \geqslant N(\varepsilon)$. Hence it is easy to see that $g_{\nu\mu}(n+m_l+u_q) \to 0$ as $\nu, \mu \to \infty$ uniformly on D and for every fixed $n+m_l+u_q$. Therefore

$$\left\{\frac{\partial Z(n_k, n+m_l+u_q)}{\partial x}\right\} \quad \text{and} \quad \left\{\frac{\partial Z(n_k, n+m_l+u_q)}{\partial y}\right\}$$

are uniformly convergent on D as $k \to \infty$ for every fixed $n+m_l+u_q$. Obviously

$$\lim_{k\to\infty} \frac{\partial Z(n_k, n+m_l+u_q)}{\partial x}(x, 0) = \sigma'(x)$$

$$\text{and} \quad \lim_{k\to\infty} \frac{\partial Z(n_k, n+m_l+u_q)}{\partial y}(0, y) = \tau'(y)$$

for $x \in [0, a]$ and $y \in [0, b]$, respectively. Since

$$(7') \quad Z(n_k, n+m_l+u_q)(x, y) = \varphi_0(x, y) + \int_0^x \int_0^y f_{n_k}\Big(\xi, \eta, Z(n_k, n+m_l+u_q-1)(\xi, \eta), \frac{\partial Z(n_k, n+m_l+u_q-1)}{\partial x}(\xi, \eta), \frac{\partial Z(n_k, n+m_l+u_q-1)}{\partial y}(\xi, \eta)\Big) d\xi\, d\eta$$

for every $k, l, n, q = 1, 2, \ldots$ and $(x, y) \in D$, then

$$(8') \quad Z(n+m_l+u_q)(x, y) = \varphi_0(x, y) + \int_0^x \int_0^y f\Big(\xi, \eta, Z(n+m_l+u_q-1)(\xi, \eta), \lim_{k\to\infty} \frac{\partial Z(n_k, n+m_l+u_q-1)}{\partial x}(\xi, \eta), \lim_{k\to\infty} \frac{\partial Z(n_k, n+m_l+u_q-1)}{\partial y}(\xi, \eta)\Big) d\xi\, d\eta.$$

Hence we obtain

$$(9) \quad \frac{\partial Z(n+m_l+u_q)}{\partial x}(x, y) = \sigma'(x) + \int_0^y f\Big(x, \eta, Z(n+m_l+u_q-1)(x, \eta), \lim_{k\to\infty} \frac{\partial Z(n_k, n+m_l+u_q-1)}{\partial x}(x, \eta), \lim_{k\to\infty} \frac{\partial Z(n_k, n+m_l+u_q-1)}{\partial y}(x, \eta)\Big) d\eta$$

and

$$(10)\qquad \frac{\partial Z(n+m_l+u_q)}{\partial y}(x,y) = \tau'(y)+\int_0^x f\Big(\xi, y, Z(n+m_l+u_q-1)(\xi,y),$$

$$\lim_{k\to\infty}\frac{\partial Z(n_k, n+m_l+u_q-1)}{\partial x}(\xi,y), \lim_{k\to\infty}\frac{\partial Z(n_k, n+m_l+u_q-1)}{\partial y}(\xi,y)\Big)d\xi$$

for every $n, l, q = 1, 2, \ldots$ and $(x, y) \in D$.

On the other hand, from (7) we get

(11)

$$\lim_{k\to\infty}\frac{\partial Z(n_k, n+m_l+u_q)}{\partial x}(x,y) = \sigma'(x)+\int_0^y f\Big(x, \eta, Z(n+m_l+u_q-1)(x,\eta),$$

$$\lim_{k\to\infty}\frac{\partial Z(n_k, n+m_l+u_k-1)}{\partial x}(x,\eta), \lim_{k\to\infty}\frac{\partial Z(n_k, n+m_l+u_q-1)}{\partial y}(x,\eta)\Big)d\eta$$

and

(12)

$$\lim_{k\to\infty}\frac{Z(n_k, n+m_l+u_q)}{\partial y}(x,y) = \tau'(y)+\int_0^x f\Big(\xi, y, Z(n+m_l+u_q-1)(\xi,y),$$

$$\lim_{k\to\infty}\frac{\partial Z(n_k, n+m_l+u_q-1)}{\partial x}(\xi,y), \lim_{k\to\infty}\frac{\partial Z(n_k, n+m_l+u_q-1)}{\partial y}(\xi,y)\Big)d\xi$$

for $n, l, q = 1, 2 \ldots$ and $(x, y), \in D$. From (9)–(12) we obtain

$$\frac{\partial Z(n+m_l+u_q)}{\partial x}(x,y) = \lim_{k\to\infty}\frac{\partial Z(n_k, n+m_l+u_q)}{\partial x}(x,y)$$

and

$$\frac{\partial z(n+m_l+u_q)}{\partial y}(x,y) = \lim_{k\to\infty}\frac{\partial Z(n_k, n+m_l+u_q)}{\partial y}(x,y)$$

for every $n, l, q = 1, 2, \ldots$ and $(x, y) \in D$. Hence it follows $\hat{Z}(n+m_l+u_q) = \hat{z}^f_{n+m_l+u_q}$ for every $n, l, q = 1, 2, \ldots$

Since

$$|\hat{Z}(n_k, n+m_l+u_q)(x_r, y_r)-\hat{Z}(n_k, n+m_l+v_q)(x_r, y_r)| > 1/p-1/l-1/q$$

for every $k, n, l, q = 1, 2, \ldots,$ then

$$|\hat{z}^f_{n+m_l+u_q}(x_r, y_r)-\hat{z}^f_{n+m_l+v_q}(x_r, y_r)| \geqslant 1/p-1/l-1/q.$$

Hence $\Delta(f, x_r, y_r) \geqslant 1/p$. Therefore $f \in \Omega_{Mpr}$.

LEMMA 3. *Ω_{Mpr} are non-dense in $\mathscr{F}_\Delta(P)$ for every $M, p, r = 1, 2, \ldots$*

Proof. Suppose there are a triple $(\bar{M}, \bar{p}, \bar{r})$ and a sphere $S_h(f_0) \subset \mathscr{F}_\Delta(P)$ with a center $f_0 \in \mathscr{F}_\Delta(P)$ and a radius $h > 0$ such that $S_h(f_0) \subset$

$\subset \overline{\Omega}_{\overline{M}\overline{p}\overline{r}}$. By Lemma 2 this means that $S_h(f_0) \subset \Omega_{\overline{M}\overline{p}\overline{r}}$. According to remarks given in Section 2, for every positive integer n we can find a function f_n: $Q_{k_{f_0}} \to \boldsymbol{R}^n$ having the properties described in Section 2. Therefore $\varrho(\tilde{f}_0, \tilde{f}_n) \leqslant 1/n$. Let N be such that $1/N < h$. We have $\tilde{f}_N \in S_h(f_0)$. In virtue of theorem given in [3] we have $\Delta(\tilde{f}_N, x, y) = 0$ for every $(x, y) \in D$. Therefore $\tilde{f}_N \notin \Omega_{\overline{M}\overline{p}\overline{r}}$. This completes the proof.

Now we can prove the main result of this paper.

THEOREM 4. *The set* $\mathscr{A}$ *of those* $f \in \mathscr{F}_A(P)$ *for which the successive approximations* $\{z_n^f\}$ *defined by* (3) *are not converging together with* $\{\partial z_n^f/\partial x\}$ *and* $\{\partial z_n^f/\partial y\}$ *is of Baire's first category in the space* $(\mathscr{F}_A(P), \varrho)$ *for every* $A > 0$.

Proof. In virtue of Lemma 3 it suffices to show that $\mathscr{A} = \bigcup_{M=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcup_{r=1}^{\infty} \Omega_{Mpr}$. It is easy to see that $\Omega_{Mpr} \subset \mathscr{A}$ for each $M, p, r = 1, 2, \ldots$ Let us observe that

$$\mathscr{A} = \{f \in \mathscr{F}_A(P) \colon \text{there is } (\tilde{x}, \tilde{y}) \in D;\ \Delta(f, \tilde{x}, \tilde{y}) > 0\}.$$

Suppose $f \in \mathscr{A}$. There exists a point $(\tilde{x}, \tilde{y}) \in D$ such that $\Delta(f, \tilde{x}, \tilde{y}) > 0$. Therefore there is a positive integer $\overline{p}$ such that $\Delta(f, \tilde{x}, \tilde{y}) \geqslant 2/\overline{p}$. It is not difficult to see that for a given $\overline{p}$ there exists an element $(\overline{x}, \overline{y})$ of a sequence $\{(x_r, y_r)\}$ such that $\Delta(f, \overline{x}, \overline{y}) \geqslant \Delta(f, \tilde{x}, \tilde{y}) - 1/\overline{p}$. Therefore there is a positive integer $\overline{r}$ such that $\Delta(f, x_{\overline{r}}, y_{\overline{r}}) \geqslant 1/\overline{p}$. Obviously we can find a positive integer $\overline{M}$ such that $\|f\|_{\mathscr{F}} \leqslant \overline{M}$. Therefore $f \in \Omega_{\overline{m}\overline{p}\overline{r}}$. Hence it follows that $\mathscr{A} \subset \bigcup_{M=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcup_{r=1}^{\infty} \Omega_{Mpr}$.

Remark. Theorem 3 and Baire's Theorem imply that the set $\mathscr{B}$ of those $f \in \mathscr{F}_A(P)$ for which the successive approximations $\{z_n^f\}$ together with $\left\{\dfrac{\partial z_n^f}{\partial x}\right\}$ and $\left\{\dfrac{\partial z_n^f}{\partial y}\right\}$ are convergent is dense and is of second category in the space $(\mathscr{F}_A(P), \varrho)$.

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