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Limitations of solutions of non-linear parabolic equations in unbounded domains

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Introduction. J. Szarski [11], [12] has established certain limitations of the difference between two solutions of Fourier's problems for systems of equations of the form

$$\frac{\partial z_i}{\partial t} = F_i\left(t, x, z_1, \ldots, z_n, \frac{\partial z_i}{\partial x_j}, \frac{\partial^2 z_i}{\partial x_j \partial x_k}\right) \quad \binom{i = 1, \ldots, n}{j, k = 1, \ldots, m}, \qquad x = (x_1, \ldots, x_m)\,. \tag{0.1}$$

Those limitations are given by solutions of ordinary differential equations. The author has assumed that the solutions are defined in domains whose intersections with planes $t = \text{const.}$ are bounded. This paper is concerned with similar theorems under stronger assumptions regarding the right-hand members of (0.1) but in unbounded domains. We also give some limitations of the solutions themselves of (0.1), which constitute generalizations of theorems proved by M. Krzyżański [6], [7] for a linear equation and which are similar to those given by W. Mlak [8] for bounded domains.

The methods applied by J. Szarski and by W. Mlak are based on properties of ordinary differential inequalities. We derive some limitations from differential inequalities of parabolic type, which are included here without proofs (Theorems 1 and 8), in a slightly more general form than in papers [1], [2]. Other limitations are proved by using methods similar to those applied in the proofs of Theorem II of [1] and Theorem II of [2].

We make use of a very general notion introduced by J. Szarski and concerning the definition of parabolic equations and the so called parabolic solution.

Part I

§ 1. Let D be an open unbounded domain of the space $(t, x_1, \ldots, x_m)$ contained in the zone $0 < t < T$ $(T \leqslant +\infty)$. The parts of the boundary of D lying on planes $t = 0$, $t = T$ will be denoted by S^0 and S^T, respectively, and the part contained in the zone $0 < t < T$ will be denoted by Σ.

Let $F_i(t, x, y_1, \dots, y_n, p_j, p_{jk})$ $(i = 1, \dots, n; j, k = 1, \dots, m)$ be functions defined for $(t, x) \in D$ and arbitrary y_s, p_j, p_{jk} $(s = 1, \dots, n; j, k = 1, \dots, m)$. We will say that the function $F_i(t, x, y_1, \dots, y_n, p_j, p_{jk})$ *satisfies the condition* $(\mathfrak{L})$ if the following Lipschitz condition is satisfied:

$$|F_i(t, x, y_1, \dots, y_n, p_j, p_{jk}) - F_i(t, x, y_1, \dots, y_n, \bar{p}_j, \bar{p}_{jk})| \leqslant L_0 \sum_{j,k=1}^{m} |p_{jk} - \bar{p}_{jk}| + (L_1|x| + L_2) \sum_{j=1}^{m} |p_j - \bar{p}_j| ,$$

L_0, L_1, L_2 being positive constants, $|x| = \left(\sum_{i=1}^{m} x_i^2\right)^{1/2}$.

If there exist positive constants L_3, L_4 such that for every $y_s, \bar{y}_s$ $(s = 1, \dots, n)$, p_j, p_{jk} $(j, k = 1, \dots, m)$ the inequality

$$[F_i(t, x, y_1, \dots, y_n, p_j, p_{jk}) - F_i(t, x, \bar{y}_1, \dots, \bar{y}_n, p_j, p_{jk})] \operatorname{sgn}(y_i - \bar{y}_i) \leqslant (L_3|x|^2 + L_4) \sum_{s=1}^{n} |y_s - \bar{y}_s|$$

is fulfilled, then we say that the function F_i satisfies the condition $(\mathfrak{L}_0)$.

The function F_i is called *elliptic* (in the sense of J. Szarski) with respect to a sequence $z_1(t, x), \dots, z_n(t, x)$ of functions of class C^1 if for every $p_{jk}, \bar{p}_{jk}$ $(j, k = 1, \dots, m)$, $p_{jk} = p_{kj}$, $\bar{p}_{jk} = \bar{p}_{kj}$, such that $\sum_{j,k=1}^{m} (p_{jk} - \bar{p}_{jk}) \lambda_j \lambda_k \leqslant 0$ for each real vector $(\lambda_1, \dots, \lambda_m)$, we have

$$F_i\left(t, x, z_1(t, x), \dots, z_n(t, x), \frac{\partial z_i(t, x)}{\partial x_j}, p_{jk}\right) - F_i\left(t, x, z_1(t, x), \dots, z_n(t, x), \frac{\partial z_i(t, x)}{\partial x_j}, \bar{p}_{jk}\right) \leqslant 0$$

for $(t, x) \in D$.

The solution of (0.1) is called *parabolic* if every function F_i is elliptic with respect to this solution.

Let us consider a system of functions $\chi_1(y_1, \dots, y_n, \xi), \dots, \chi_n(y_1, \dots, y_n, \xi)$; ξ being a sequence of variables different from $y_1, \dots, y_n$. The function χ_i is said to *satisfy the condition* $W[y_1, \dots, y_n]$ if for $y_s \leqslant \bar{y}_s$ $(s = 1, \dots, n)$, $y_i = \bar{y}_i$, the inequality

$$\chi_i(y_1, \dots, y_n, \xi) \leqslant \chi_i(\bar{y}_1, \dots, \bar{y}_n, \xi)$$

holds. It is said *to satisfy the condition* $\Lambda[y_1, \dots, y_n]$ if there exists a positive constant L such that for every $y_s, \bar{y}_s$ $(s = 1, \dots, n)$, we have

$$[\chi_i(y_1, \dots, y_n, \xi) - \chi_i(\bar{y}_1, \dots, \bar{y}_n, \xi)] \operatorname{sgn}(y_i - \bar{y}_i) \leqslant L \sum_{s=1}^{n} |y_s - \bar{y}_s| .$$

A function $z(t, x)$ will be called *regular* if it is continuous in the set $D+S^0+\Sigma$ and if it possesses the derivative $\partial z/\partial t$ and the continuous derivatives $\partial z/\partial x_j$, $\partial^2 z/\partial x_j \partial x_k$ $(j, k = 1, \ldots, m)$ in D.

By $\bar{E}_2(M(t), K)$ $(\underline{E}_2(M(t), K))$ or shortly $\bar{E}_2$ ($\underline{E}_2$ respectively) we denote the class of functions $\varphi(t, x)$, defined in D, for which there exist a positive function $M(t)$, bounded on every interval $0 \leqslant t \leqslant t_0$, $t_0 < T$, and a positive constant K such that

$$\varphi(t, x) \leqslant M(t)\exp(K|x|^2) \qquad (\varphi(t, x) \geqslant -M(t)\exp(K|x|^2) \text{ respectively})$$

in D. The class of functions belonging to $\bar{E}_2(M(t), K)$ and $\underline{E}_2(M(t), K)$ simultaneously will be denoted by $E_2(M(t), K)$ or E_2.

§ 2. Now we formulate Theorem 1, whose proof is an easy modification of the proof of Theorem II included in [1] ([1]).

THEOREM 1. *If*

1^0 $u_i(t, x)$, $v_i(t, x)$ *are regular*, $u_i \in \bar{E}_2$, $v_i \in \underline{E}_2$ $(i = 1, \ldots, n)$, *and atisfy the systems of inequalities*

$$\begin{aligned}
&(2.1) \qquad \frac{\partial u_i}{\partial t} \leqslant F_i^{(1)}\left(t, x, u_1, \ldots, u_n, \frac{\partial u_i}{\partial x_j}, \frac{\partial^2 u_i}{\partial x_j \partial x_k}\right), \\
&(2.2) \qquad \frac{\partial v_i}{\partial t} \geqslant F_i^{(2)}\left(t, x, v_1, \ldots, v_n, \frac{\partial v_i}{\partial x_j}, \frac{\partial^2 v_i}{\partial x_j \partial x_k}\right),
\end{aligned} \qquad (i = 1, \ldots, n)$$

respectively,

2^0 *for each* i $(i = 1, \ldots, n)$ *the function* $F_i^{(1)}(t, x, y_1, \ldots, y_n, p_j, p_{jk})$ *is elliptic with respect to the sequence* $\{u_s(t, x)\}$ *or the function* $F_i^{(2)}(t, x, y_1, \ldots, y_n, p_j, p_{jk})$ *is elliptic with respect to* $\{v_s(t, x)\}$ $(s = 1, \ldots, n)$,

3^0 *for every fixed* i *at least one function* $F_i^{(1)}, F_i^{(2)}$ *satisfies the condition* $(\mathfrak{L})$, *at least one of them satisfies the condition* $(\mathfrak{L}_0)$, *and at least one satisfies the condition* $W[y_1, \ldots, y_n]$,

4^0 $F_i^{(1)}(t, x, y_1, \ldots, y_n, p_j, p_{jk}) \leqslant F_i^{(2)}(t, x, y_1, \ldots, y_n, p_j, p_{jk})$ $(i = 1, \ldots, n)$ *in the domain of existence of these functions*,

5^0 $u_i(0, x) \leqslant v_i(0, x)$ $(i = 1, \ldots, n)$ *for* $x \in \bar{S}^0$,

6^0 $u_i(t, x) \leqslant v_i(t, x)$ $(i = 1, \ldots, n)$ *for* $(t, x) \in \Sigma$,

then the inequalities

$$(2.3) \qquad u_i(t, x) \leqslant v_i(t, x) \qquad (i = 1, \ldots, n)$$

hold true everywhere in D.

([1]) In Theorem II of [1] instead of the inequalities (2.1), (2.2) the corresponding equations have been dealt with and it has been assumed that for each i the function $F_i^{(1)}$ or the function $F_i^{(2)}$ satisfies the three conditions: $(\mathfrak{L})$, $(\mathfrak{L}_0)$ and $W[y_1, \ldots, y_n]$ simultaneously (cf. assumption 3^0 of Theorem 1 of this paper).

§ 3. Take a system of functions $\sigma_i(t, y_1, \ldots, y_n)$ $(i = 1, \ldots, n)$ defined for $0 < t < T$. $-\infty < y_s < +\infty$ $(s = 1, \ldots, n)$. Let $y_i = \omega_i(t, \eta_1, \ldots, \eta_n)$ $(i = 1, \ldots, n)$ be an arbitrary right-hand solution of the system of ordinary differential equations

$$\frac{dy_i}{dt} = \sigma_i(t, y_1, \ldots, y_n) \quad (i = 1, \ldots, n) \tag{3.1}$$

such that $\omega_i(0, \eta_1, \ldots, \eta_n) = \eta_i$, η_i being arbitrary constants, and assume that the solution exists in the interval $0 \leqslant t < T$.

§ 4. The proofs of the following Theorems 2-5 will be based on Theorem 1.

THEOREM 2. *If*

1° *there exist regular solutions* $\{u_i(t, x)\}$, $\{v_i(t, x)\}$ *of the systems of equations*

$$\frac{\partial u_i}{\partial t} = f_i^{(1)}\left(t, x, u_1, \ldots, u_n, \frac{\partial u_i}{\partial x_j}, \frac{\partial^2 u_i}{\partial x_j \partial x_k}\right), \tag{4.1}$$

$$\frac{\partial v_i}{\partial t} = f_i^{(2)}\left(t, x, v_1, \ldots, v_n, \frac{\partial v_i}{\partial x_j}, \frac{\partial^2 v_i}{\partial x_j \partial x_k}\right), \quad (i = 1, \ldots, n) \tag{4.2}$$

respectively, where the functions $f_i^{(1)}, f_i^{(2)}$ *are defined for* $(t, x) \in D$, *the remaining variables being arbitrary,*

2° $u_i \in \bar{E}_2$ $(u_i \in \underline{E}_2)$, $v_i \in \underline{E}_2$ $(v_i \in \bar{E}_2$ *respectively*$)$ $(i = 1, \ldots, n)$,

3° *all the functions* $f_i^{(1)}(t, x, y_1, \ldots, y_n, p_j, p_{jk})$ *are elliptic with respect to the solution* $\{u_s(t, x)\}$ $(s = 1, \ldots, n)$ *and satisfy the conditions* $(\mathfrak{L})$, $(\mathfrak{L}_0)$ *and* $W[y_1, \ldots, y_n]$ *or all* $f_i^{(2)}(t, x, y_1, \ldots, y_n, p_j, p_{jk})$ *are elliptic with respect to* $\{v_s(t, (t, x)\}$ *and satisfy the conditions* $(\mathfrak{L})$, $(\mathfrak{L}_0)$, $W[y_1, \ldots, y_n]$,

4° $f_i^{(1)}(t, x, \bar{y}_1, \ldots, \bar{y}_n, p_j, p_{jk}) - f_i^{(2)}(t, x, \bar{\bar{y}}_1, \ldots, \bar{\bar{y}}_n, p_j, p_{jk})$
$\leqslant \sigma_i(t, \bar{y}_1 - \bar{\bar{y}}_1, \ldots, \bar{y}_n - \bar{\bar{y}}_n)$ $(i = 1, \ldots, n)$
$(f_i^{(1)}(t, x, \bar{y}_1, \ldots, \bar{y}_n, p_j, p_{jk}) - f_i^{(2)}(t, x, y_1, \ldots, y_n, p_j, p_{jk})$
$\geqslant \sigma_1(t, \bar{y}_1 - \bar{\bar{y}}_1, \ldots, \bar{y}_n - \bar{\bar{y}}_n)$ *respectively*$)$,

5° $u_i(0, x) - v_i(0, x) \leqslant \eta_i$ $(i = 1, \ldots, n)$ *for* $x \in \bar{S}^\circ$,
$(u_i(0, x) - v_i(0, x) \geqslant \eta_i$ *respectively*$)$,

6° $u_i(t, x) - v_i(t, x) \leqslant \omega_i(t, \eta_1, \ldots, \eta_n)$ $(i = 1, \ldots, n)$ *for* $(t, x) \in \Sigma$ $(u_i(t, x) - v_i(t, x) \geqslant \omega_i(t, \eta_1, \ldots, \eta_n)$ *respectively*$)$ (*the functions* $\sigma_i(t, y_1, \ldots, y_n)$ *and* $\omega_i(t, \eta_1, \ldots, \eta_n)$ $(i = 1, \ldots, n)$ *being defined as in* § 3), *then we have*

$$\begin{gathered} u_i(t, x) - v_i(t, x) \leqslant \omega_i(t, \eta_1, \ldots, \eta_n) \quad (i = 1, \ldots, n) \\ (u_i(t, x) - v_i(t, x) \geqslant \omega_i(t, \eta_1, \ldots, \eta_n) \text{ respectively}) \end{gathered} \tag{4.3}$$

in the domain D.

Proof. We shall restrict ourselves to the case where the first member of alternative 3° holds true. In the second case the proof is similar. We shall begin by proving the first inequality of (4.3).

Adding the respective equations of (3.1) and (4.2) we have

$$\frac{\partial(v_i+\omega_i)}{\partial t} = f_i^{(2)}\left(t, x, v_1, \dots, v_n, \frac{\partial v_i}{\partial x_j}, \frac{\partial^2 v_i}{\partial x_j \partial x_k}\right) + \sigma_i(t, \omega_1, \dots, \omega_n). \tag{4.4}$$

It follows from assumption 4° that

$$f_i^{(1)}\left(t, x, v_1+\omega_1, \dots, v_n+\omega_n, \frac{\partial v_i}{\partial x_j}, \frac{\partial^2 v_i}{\partial x_j \partial x_k}\right) - f_i^{(2)}\left(t, x, v_1, \dots, v_n, \frac{\partial v_i}{\partial x_j}, \frac{\partial^2 v_i}{\partial x_j \partial x_k}\right) \leqslant \sigma_i(t, \omega_1, \dots, \omega_n). \tag{4.5}$$

By (4.4), (4.5) and the identities

$$\frac{\partial \omega_i}{\partial x_j} \equiv 0, \quad \frac{\partial^2 \omega_i}{\partial x_j \partial x_k} \equiv 0 \qquad (i = 1, \dots, n;\ j, k = 1, \dots, m) \tag{4.6}$$

we get

$$\frac{\partial(v_i+\omega_i)}{\partial t} \geqslant f_i^{(1)}\left(t, x, v_1+\omega_1, \dots, v_n+\omega_n, \frac{\partial(v_i+\omega_i)}{\partial x_j}, \frac{\partial^2(v_i+\omega_i)}{\partial x_j \partial x_k}\right) \tag{4.7}$$

$(i = 1, \dots, n)$. $\{u_i(t, x)\}$ is the solution of (4.1), therefore, the functions u_i, $v_i+\omega_i$ fulfil the systems of inequalities of the form (2.1), (2.2), respectively, with $F_i^{(2)} \equiv F_i^{(1)} \equiv f_i^{(1)}$.

Evidently, u_i and $v_i+\omega_i$ $(i = 1, \dots, n)$ belong to classes $\bar{E}_2$ and $\underline{E}_2$, respectively. It is easy to verify that the remaining assumptions of Theorem 1 with v_i replaced by $v_i+\omega_i$ are satisfied, whence $u_i \leqslant v_i+\omega_i$ $(i = 1, \dots, n)$ in D.

In order to show that $u_i - v_i \geqslant \omega_i$ we add the respective equations of (3.1) and (4.2), we make use of the second part of assumption 4° and of (4.6), and we obtain

$$\frac{\partial(v_i+\omega_i)}{\partial t} \leqslant f_i^{(1)}\left(t, x, v_1+\omega_1, \dots, v_n+\omega_n, \frac{\partial(v_i+\omega_i)}{\partial x_j}, \frac{\partial^2(v_i+\omega_i)}{\partial x_j \partial x_k}\right)$$

$(i = 1, \dots, n)$. As before, from Theorem 1 it follows that $v_i+\omega_i \leqslant u_i$ $(i = 1, \dots, n)$, and Theorem 2 is thus proved.

THEOREM 3. *Suppose that*

1° $\{u_i(t, x)\}$, $\{v_i(t, x)\}$ $(i = 1, \dots, n)$ *are two regular solutions of the system*

$$\frac{\partial z_i}{\partial t} = f_i\left(t, x, z_1, \dots, z_n, \frac{\partial z_i}{\partial x_j}, \frac{\partial^2 z_i}{\partial x_j \partial x_k}\right) \qquad (i = 1, \dots, n), \tag{4.8}$$

2° $u_i \in \bar{E}_2$ $(u_i \in E_2)$, $v_i \in \underline{E}_2$ $(v_i \in \bar{E}_2$ *respectively*$)$ $(i = 1, \dots, n)$, 3° *each function* $f_i(t, x, y_1, \ldots, y_n, p_j, p_{jk})$ *is elliptic with respect to the sequence* $\{u_s(t, x)\}$ *or* $\{v_s(t, x)\}$ $(s = 1, \ldots, n)$ *and satisfies the conditions* $(\mathfrak{L})$, $(\mathfrak{L}_0)$ *and* $W[y_1, \ldots, y_n]$,

4° *for* $(t, x) \in D$ *and for arbitrary* y_s, p_j, p_{jk} $(s = 1, \ldots, n; j, k = 1, \ldots, m)$

$$(4.9)\qquad f_i(t, x, y_1+\delta_1, \ldots, y_n+\delta_n, p_j, p_{jk}) \leqslant f_i(t, x, y_1, \ldots, y_n, p_j, p_{jk}) \quad (i = 1, \ldots, n)$$

$$(4.10)\qquad (f_i(t, x, y_1+\delta_1, \ldots, y_n+\delta_n, p_j, p_{jk}) \geqslant f_i(t, x, y_1, \ldots, y_n, p_j, p_{jk}) \quad \textit{respectively}),$$

δ_i *being constants,*

5° $u_i(0, x) - v_i(0, x) \leqslant \delta_i$ $(i = 1, \ldots, n)$, $x \in \bar{S}^0$
$(u_i(0, x) - v_i(0, x) \geqslant \delta_i$ *respectively*$)$,

6° $u_i(t, x) - v_i(t, x) \leqslant \delta_i$ $(i = 1, \ldots, n)$, $(t, x) \in \Sigma$
$(u_i(t, x) - v_i(t, x) \geqslant \delta_i$ *respectively*$)$.

Under these assumptions the inequalities

$$(4.11)\qquad \begin{gathered} u_i(t, x) - v_i(t, x) \leqslant \delta_i \quad (i = 1, \ldots, n) \\ (u_i(t, x) - v_i(t, x) \geqslant \delta_i \quad \textit{respectively}) \end{gathered}$$

are fulfilled in D.

Remark. When $\delta_s > 0$ $(s = 1, \ldots, n)$ then assumption (4.9) together with the condition $W[y_1, \ldots, y_n]$ require that the decrement of function f_i, if y_i increases to the value $y_i + \delta_i$, be equal to or greater than the summary increment of this function if the variables $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n$ increase to the values $y_1+\delta_1, \ldots, y_{i-1}+\delta_{i-1}, y_{i+1}+\delta_{i+1}, \ldots, y_n+\delta_n$. Inequalities (4.9) are fulfilled for example when for every fixed i there exist non-negative constants $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n$, $B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_n$ and non-positive constants A_i, B_i such that

$$\sum_{s=1}^{n} A_s\delta_s \leqslant 0\,; \qquad \sum_{s=1}^{n} B_s\delta_s \leqslant 0$$

and

$$f_i(t, x, \bar{y}_1, \ldots, \bar{y}_n, p_j, p_{jk}) - f_i(t, x, \bar{\bar{y}}_1, \ldots, \bar{\bar{y}}_n, p_j, p_{jk}) \leqslant \sum_{s=1}^{n} (A_s|x|^2+B_s)(\bar{y}_s - \bar{\bar{y}}_s)$$

for $\bar{y}_s \geqslant \bar{\bar{y}}_s$ $(s = 1, \ldots, n)$.

If the function $f_i(t, x, y_1, \ldots, y_n, p_j, p_{jk})$ has continuous derivatives $\partial f_i/\partial y_s$ $(s = 1, \ldots, n)$ and the inequalities

$$0 \leqslant \frac{\partial f_i}{\partial y_s} \leqslant A_s|x|^2+B_s; \quad A_s, B_s \geqslant 0 \quad (s = 1, \ldots, i-1, i+1 \ldots, n)\,,$$

$$\frac{\partial f_i}{\partial y_i} \leqslant A_i|x|^2+B_i; \quad A_i, B_i \leqslant 0\,,$$

and

$$\sum_{s=1}^{n} \frac{\partial f_i}{\partial y_s} \delta_s \leqslant 0$$

are fulfilled, then (4.9) is satisfied. This follows from the mean value theorem.

In case of one equation

$$\frac{\partial z}{\partial t} = f\left(t, x, z, \frac{\partial z}{\partial x_j}, \frac{\partial^2 z}{\partial x_j \partial x_k}\right)$$

inequality (4.9) is satisfied if the function $f(t, x, y, p_j, p_{jk})$ is non-increasing in y.

If all δ_s are negative, then (4.9) holds for instance for the function f_i non-decreasing in y_i.

The meaning of condition (4.10) is similar.

Proof of Theorem 3. Note that in Theorem 2 instead of the first part of assumption 4° the weaker condition (4.5) may be assumed. If we do that, then the first part of Theorem 3 follows from the first part of Theorem 2. For this purpose we should set $\sigma_i \equiv 0$, $\omega^i \equiv \delta_i$, $f_i^{(2)} \equiv f_i^{(1)} \equiv f_i$ $(i = 1, \ldots, n)$. The validity of the second part may be shown similarly.

THEOREM 4. *Suppose that*

1° $\{u_i(t, x)\}$ $(i = 1, \ldots, n)$ *is a regular parabolic solution of class* $\bar{E}_2$ $(\underline{E}_2)$ *of system* (4.8),

2° *for* $(t, x) \in D$ *and arbitrary* y_s, p_j, p_{jk} $(s = 1, \ldots, n;\ j, k = 1, \ldots, m)$ *we have*

$$f_i(t, x, y_1, \ldots, y_n, p_j, p_{jk}) \leqslant L_0 \sum_{j,k=1}^{m} |p_{jk}| + (L_1|x| + L_2) \sum_{j=1}^{m} |p_j| + \sigma_i(t, y_1, \ldots, y_n) \text{ (2)}$$

$$\left(f_i(t, x, y_1, \ldots, y_n, p_j, p_{jk}) \geqslant -\left[L_0 \sum_{j,k=1}^{m} |p_{jk}| + (L_1|x| + L_2) \sum_{j=1}^{m} |p_j|\right] + \sigma_i(t, y_1, \ldots, y_n)\right.$$

respectively) $(i = 1, \ldots, n)$, L_0, L_1, L_2 *being positive constants*,

3° *for every fixed* i *at least one function* f_i, σ_i *satisfies the condition* $\Lambda[y_1, \ldots, y_n]$ *and at least one of them satisfies the condition* $W[y_1, \ldots, y_n]$,

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(2) If the function f_i satisfies condition ($\mathfrak{L}$), then this inequality is equivalent to the following one:

$$f_i(t, x, y_1, \ldots, y_n, 0, 0) \leqslant \sigma_i(t, y_1, \ldots, y_n)\,.$$

4° $u_i(0, x) \leqslant \eta_i$ $(u_i(0, x) \geqslant \eta_i$ *respectively*) $(i = 1, \ldots, n)$, $x \in \bar{S}^0$,

5° $u_i(t, x) \leqslant \omega_i(t, \eta_1, \ldots, \eta_n)$ $(i = 1, \ldots, n)$, $(t, x) \in \Sigma$,

$(u_i(t, x) \geqslant \omega_i(t, \eta_1, \ldots, \eta_n)$ *respectively*).

Under these assumptions we have, for $(t, x) \in D$,

$$(4.12) \qquad \begin{gathered} u_i(t, x) \leqslant \omega_i(t, \eta_1, \ldots, \eta_n) \quad (i = 1, \ldots, n) \\ (u_i(t, x) \geqslant \omega_i(t, \eta_1, \ldots, \eta_n) \ \textit{respectively}) . \end{gathered}$$

Proof. To get the first part of the theorem observe that the $\omega_i(t, \eta_1, \ldots, \eta_n)$ satisfy the system

$$\frac{\partial \omega_i}{\partial t} = L_0 \sum_{j,k=1}^{m} \left| \frac{\partial^2 \omega_i}{\partial x_j \partial x_k} \right| + (L_1|x| + L_2) \sum_{j=1}^{m} \left| \frac{\partial \omega_i}{\partial x_j} \right| + \sigma_i(t, \omega_1, \ldots, \omega_n) .$$

Thus our theorem follows immediately from Theorem 1.

THEOREM 5. *Suppose hypothesis* 1° *of Theorem* 4 *to be true and the functions* f_i $(i = 1, \ldots, n)$ *to satisfy the conditions* $(\mathfrak{L})$, $(\mathfrak{L}_0)$ *and* $W[y_1, \ldots, y_n]$. *Let* $\varphi_i(t)$ $(i = 1, \ldots, n)$ *be functions possessing the derivative* $\varphi_i'(t)$ *in the interval* $\langle 0, T)$. *Suppose furthermore that*

$$(4.13) \quad u_i(0, x) \leqslant \varphi_i(0) \qquad (u_i(0, x) \geqslant \varphi_i(0) \ \textit{respectively}) \qquad (i = 1, \ldots, n)$$

for $x \in \bar{S}^0$,

$$(4.14) \quad u_i(t, x) \leqslant \varphi_i(t) \qquad (u_i(t, x) \geqslant \varphi_i(t) \ \textit{respectively}) \qquad (i = 1, \ldots, n)$$

for $(t, x) \in \Sigma$,

$$f_i\big(t, x, \varphi_1(t), \ldots, \varphi_n(t), 0, 0\big) \leqslant \varphi_i'(t) \qquad (i = 1, \ldots, n) , \ (t, x) \in D ,$$

$$\big(f_i\big(t, x, \varphi_1(t), \ldots, \varphi_n(t), 0, 0\big) \geqslant \varphi_i'(t) \ \textit{respectively}\big) .$$

These hypotheses imply that inequalities

$$(4.15) \quad u_i(t, x) \leqslant \varphi_i(t) \qquad (u_i(t, x) \geqslant \varphi_i(t) \ \textit{respectively}) \qquad (i = 1, \ldots, n)$$

are fulfilled for $(t, x) \in D$.

Proof. The first part of our theorem results from Theorem 1 by the substitution $v_i = \varphi_i$, $F_i^{(1)} = F_i^{(2)} = f_i$ $(i = 1, \ldots, n)$. To obtain the second part we should set in Theorem 1 $u_i = \varphi_i$, $v_i = u_i$ and $F_i^{(1)} = F_i^{(2)} = f_i$.

COROLLARY 1. *From Theorem* 5 *one can easily obtain certain theorems concerning the asymptotic behaviour of the solutions. Suppose, for the sake of simplicity, the equations of system* (4.8) *to be linear with respect to the unknown functions. Namely, let*

$$f_i(t, x, y_1, \ldots, y_n, 0, 0) = \sum_{s=1}^{n} c_{is}(t, x) y_s ,$$

where the coefficients $c_{is}(t, x)$ *satisfy the inequalites*

$$0 \leqslant c_{is}(t, x) \leqslant A|x|^2 + B \quad (s = 1, \ldots, i-1, i+1, \ldots, n;\ i = 1, \ldots, n),$$

$$\sum_{s=1}^{n} c_{is}(t, x) \leqslant -\varepsilon \quad (i = 1, \ldots, n),$$

A, B, ε *being positive constants. If, furthermore, the functions* $f_i(t, x, y_1, \ldots, y_n, p_j, p_{jk})$ $(i = 1, \ldots, n)$ *fulfil the condition* $(\mathfrak{L})$, *then every regular parabolic solution of* (4.8) *belonging to* E_2 *in* D *and satisfying the inequalities*

$$|u_i(t, x)| \leqslant M_0 \exp(-\varepsilon t) \quad on \quad S^0 + \Sigma, \tag{4.16}$$

satisfies (4.16) *in* D. *Hence* $u_i(t, x) \to 0$ *as* $t \to \infty$.

The above assertion results immediately from Theorem 5 if we put $\varphi_i(t) = M_0 \exp(-\varepsilon t)$ $(\varphi_i(t) = -M_0 \exp(-\varepsilon t)$ respectively).

Moreover observe that the inequalities of M. Krzyżański (Theorems 1 and 2 of [6]) for one linear equation result from Theorem 5 by the substitution $\varphi_i(t) \equiv M = \text{const} \geqslant 0$.

§ 5. Now let $\sigma_i(t, y_1, \ldots, y_n)$ $(i = 1, \ldots, n)$ be functions defined and non-negative for $0 < t < T$, $y_s \geqslant 0$ $(s = 1, \ldots, n)$, and let $y_i = \omega_i(t, \eta_1, \ldots, \eta_n)$ $(i = 1, \ldots, n)$ be an arbitrary right-hand solution of the system of ordinary differential equations

$$\frac{dy_i}{dt} = \sigma_i(t, y_1, \ldots, y_n) \quad (i = 1, \ldots, n) \tag{5.1}$$

such that $\omega_i(0, \eta_1, \ldots, \eta_n) = \eta_i$, where η_i are non-negative constants. We assume that the solution exists in the interval $0 \leqslant t < T$.

§ 6. THEOREM 6. *If*

1° *in class* E_2 *there exist regular solutions* $\{u_s(t, x)\}$, $\{v_s(t, x)\}$ $(s = 1, \ldots, n)$ *of the systems* (4.1), (4.2), *respectively*,

2° *the functions* $f_i^{(1)}(t, x, y_1, \ldots, y_n, p_j, p_{jk})$, $f_i^{(2)}(t, x, y_1, \ldots, y_n, p_j, p_{jk})$ $(i = 1, \ldots, n)$ *are defined for* $(t, x) \in D$, $y_1, \ldots, p_{mm}$ *arbitrary, and for every fixed* i $(1 \leqslant i \leqslant n)$ *the function* $f_i^{(1)}$ *is elliptic with regard to* $\{u_s(t, x)\}$ *or the function* $f_i^{(2)}$ *is elliptic with regard to* $\{v_s(t, x)\}$,

3° *for every* i *the function* $f_i^{(1)}$ *or the function* $f_i^{(2)}$ *satisfies condition* $(\mathfrak{L})$ (see § 1),

4° *for every* i *at least one function among* $f_i^{(1)}, f_i^{(2)}, \sigma_i$ *satisfies the condition* $\Lambda[y_1, \ldots, y_n]$ *and at least one of them satisfies the condition* $W[y_1, \ldots, y_n]$,

5° *for* $(t, x) \in D$ *and for arbitrary* $y_s, \bar{y}_s, p_j, p_{jk}$ $(s = 1, \ldots, n;$ $j, k = 1, \ldots, m)$ *we have*

$$|f_i^{(1)}(t, x, y_1, \ldots, y_n, p_j, p_{jk}) - f_i^{(2)}(t, x, \bar{y}_1, \ldots, \bar{y}_n, p_j, p_{jk})| \\ \leqslant \sigma_i(t, |y_1 - \bar{y}_1|, \ldots, |y_n - \bar{y}_n|) \quad (i = 1, \ldots, n),$$

6° $|u_i(0, x)-v_i(0, x)| \leqslant \eta_i$ $(i = 1, \ldots, n)$ *for* $x \in \bar{S}^0$,

7° $|u_i(t, x)-v_i(t, x)| \leqslant \omega_i(t, \eta_1, \ldots, \eta_n)$ $(i = 1, \ldots, n)$ *for* $(t, x) \in \Sigma$,

where the functions $\sigma_i(t, y_1, \ldots, y_n)$ *and* $\omega_i(t, \eta_1, \ldots, \eta_n)$ $(i = 1, \ldots, n)$ *are defined as in* § 5,

then the inequalities

$$|u_i(t, x)-v_i(t, x)| \leqslant \omega_i(t, \eta_1, \ldots, \eta_n) \qquad (i = 1, \ldots, n) \tag{6.1}$$

hold for $(t, x) \in D$.

Proof. It follows from our assumption that the functions

$$w_i(t, x) \stackrel{\mathrm{df}}{=} |u_i(t, x)-v_i(t, x)|-\omega_i(t, \eta_1, \ldots, \eta_n) \qquad (i = 1, \ldots, n) \tag{6.2}$$

belong to a class $E_2\big(M(t), K_0\big)$. We shall make use of the properties of the function

$$H(t, x; K) = \exp\left\{\frac{K|x|^2}{1-\mu(K)t}+\nu(K)t\right\} \tag{6.3}$$

constructed by M. Krzyżański (cf. [3]), with $K > K_0$ and

$$\mu(K) = 4KL_0m^2+4m(L_1+L_2)+K^{-1},$$
$$\nu(K) = 4\big(2KL_0m+2Km(L_1+L_2)+nL+1\big).$$

Put

$$h = \frac{1}{2\mu(K)}.$$

By D^h, Σ^h we denote the parts of the sets D, Σ, respectively, contained in the zone $0 < t < h$ and by S^h the part of boundary of D^h lying on the plane $t = h$.

It is enough to prove the theorem for the domain D^h, since then we can carry out the same argument step by step.

Put

$$\mathcal{F}H(t, x; K) = L_0\sum_{j=1}^{m}\left|\frac{\partial^2 H}{\partial x_j\partial x_k}\right|+(L_1|x|+L_2)\sum_{j=1}^{m}\left|\frac{\partial H}{\partial x_j}\right|+LnH-\frac{\partial H}{\partial t}. \tag{6.4}$$

As in [1] one can show, by a simple computation, that function (6.3) fulfils the inequality

$$\mathcal{F}H(t, x; K) \leqslant -H(t, x; K) \quad \text{in the domain } D^h. \tag{6.5}$$

Let $\{R_a\}$ be an increasing sequence, $R_a > 0$, $R_a \to \infty$ as $a \to \infty$. Further, denote by D_a^h, Σ_a^h, S_a^0, S_a^h the parts of the sets D_a^h, Σ_a^h, S_a^0, S_a^h respectively, lying inside the cylindrical surface C_a with the equation $|x| = R_a$ and by C_a^h denote the part of C_a which is contained in D^h.

Introduce the transformations

$$(6.6) \qquad u_i = \bar{u}_i H\,, \quad v_i = \bar{v}_i H\,, \quad \omega_i = \bar{\omega}_i H \quad (i = 1, \ldots, n)\,.$$

Putting $\bar{w}_i = |\bar{u}_i - \bar{v}_i| - \bar{\omega}_i$ we have

$$(6.7) \qquad w_i = \bar{w}_i H \quad (i = 1, \ldots, n)\,.$$

Let us consider the sequence

$$A_a = \max_{(i)} \max_{(t,x)\in \overline{D_a^h}} \bar{w}_i(t, x) \quad (a = 1, 2, \ldots)\,.$$

The theorem will be proved if we show that $A_a \leqslant 0$ for every a. For this purpose notice that for any a there exist an index i_a and a point $(t_a, x_a) \in \overline{D_a^h} = (D_a^h + S_a^h) + (\overline{S_a^0} + \Sigma_a^h) + C_a^h$ such that $A_a = \bar{w}_{i_a}(t_a, x_a)$. We shall show that the relation $(t_a, x_a) \in D_a^h + S_a^h$ implies $\bar{w}_{i_a}(t_a, x_a) \leqslant 0$. Indeed, suppose on the contrary that $\bar{w}_{i_a}(t_a, x_a) > 0$. This supposition implies $w_{i_a}(t_a, x_a) > 0$. Since $\omega_{i_a} \geqslant 0$ and $\bar{\omega}_{i_a} \geqslant 0$, by the definition of w_i and $\bar{w}_i$ we have $|u_{i_a}(t_a, x_a) - v_{i_a}(t_a, x_a)| > 0$ as well as $|\bar{u}_{i_a}(t_a, x_a) - \bar{v}_{i_a}(t_a, x_a)| > 0$. Hence it follows that the functions $w_{i_a}(t, x)$, $\bar{w}_{i_a}(t, x)$ possess the derivatives

$$(6.8) \qquad \frac{\partial w_{i_a}}{\partial t}, \frac{\partial w_{i_a}}{\partial x_j}, \frac{\partial^2 w_{i_a}}{\partial x_j \partial x_k}; \quad \frac{\partial \bar{w}_{i_a}}{\partial t}, \frac{\partial \bar{w}_{i_a}}{\partial x_j}, \frac{\partial^2 \bar{w}_{i_a}}{\partial x_j \partial x_k} \quad (j, k = 1, \ldots, m)$$

at the point (t_a, x_a).

Put

$$\varepsilon_i = \begin{cases} 1\,, & \text{if} \quad \bar{u}_i(t_a, x_a) > \bar{v}_i(t_a, x_a)\,, \\ -1\,, & \text{if} \quad \bar{u}_i(t_a, x_a) \leqslant \bar{v}_{i_a}(t_a, x_a)\,. \end{cases}$$

At (t_a, x_a) we have, by (4.1), (4.2) and (5.1),

$$\frac{\partial w_{i_a}}{\partial t} = \varepsilon_{i_a}\Big[f_{i_a}^{(1)}\Big(t_a, x_a, u_1, \ldots, u_n, \frac{\partial u_{i_a}}{\partial x_j}, \frac{\partial^2 u_{i_a}}{\partial x_j \partial x_k}\Big) - $$
$$- f_{i_a}^{(2)}\Big(t_a, x_a, v_1, \ldots, v_n, \frac{\partial v_{i_a}}{\partial x_j}, \frac{\partial^2 v_{i_a}}{\partial x_j \partial x_k}\Big)\Big] - \sigma_{i_a}(t_a, \omega_1, \ldots, \omega_n)\,,$$

which, by identities (4.6), may be written in the form

$$(6.9) \qquad \frac{\partial w_{i_a}}{\partial t} = \varepsilon_{i_a}\Big[f_{i_a}^{(1)}\Big(t_a, x_a, u_1, \ldots, u_n, \frac{\partial u_{i_a}}{\partial x_j}, \frac{\partial^2 u_{i_a}}{\partial x_j \partial x_k}\Big) - $$
$$- f_{i_a}^{(2)}\Big(t_a, x_a, v_1, \ldots, v_n, \frac{\partial (v_{i_a} + \varepsilon_{i_a}\omega_{i_a})}{\partial x_j}, \frac{\partial^2 (v_{i_a} + \varepsilon_{i_a}\omega_{i_a})}{\partial x_j \partial x_k}\Big)\Big] - \sigma_{i_a}(t_a, \omega_1, \ldots, \omega_n)\,.$$

Let us introduce the following notation:

$$(6.10)\qquad \begin{aligned} \tau_j(u) &= \frac{\partial u}{\partial x_j} H + u\frac{\partial H}{\partial x_j}, \\ \tau_{jk}(u, v) &= \frac{\partial^2 u}{\partial x_j \partial x_k} H + \frac{\partial v}{\partial x_j}\frac{\partial H}{\partial x_k} + \frac{\partial v}{\partial x_k}\frac{\partial H}{\partial x_j} + v\frac{\partial^2 H}{\partial x_j \partial x_k}. \end{aligned}$$

Substituting (6.6) and (6.7) into (6.9) we get

$$(6.11)\qquad \frac{\partial \overline{w}_{i_a}}{\partial t} H + \overline{w}_{i_a}\frac{\partial H}{\partial t} = \varepsilon_{i_a}\big[f^{(1)}_{i_a}\big(t_a, x_a, \bar{u}_1 H, \ldots, \bar{u}_n H, \tau_j(\bar{u}_{i_a}), \tau_{jk}(\bar{u}_{i_a}, \bar{u}_{i_a})\big) - $$
$$ - f^{(2)}_{i_a}\big(t_a, x_a, \bar{v}_1 H, \ldots, \bar{v}_n H, \tau_j(\bar{v}_{i_a} + \varepsilon_{i_a}\overline{\omega}_{i_a}), \tau_{jk}(\bar{v}_{i_a} + \varepsilon_{i_a}\overline{\omega}_{i_a}, \bar{v}_{i_a} + \varepsilon_{i_a}\overline{\omega}_{i_a})\big)\big] - $$
$$ - \sigma_{i_a}(t_a, \overline{\omega}_1 H, \ldots, \overline{\omega}_n H).$$

The function $\overline{w}_{i_a}(t, x)$ reaches a maximum at the point $(t_a, x_a) \in D^h_a + S^h_a$, whence

$$(6.12)\qquad \frac{\partial \overline{w}_{i_a}(t_a, x_a)}{\partial t} \geqslant 0,$$

$$(6.13)\qquad \frac{\partial \overline{w}_{i_a}(t_a, x_a)}{\partial x_j} = 0 \quad (j = 1, \ldots, m)$$

and

$$(6.14)\qquad \sum_{j,k=1}^{m} \frac{\partial^2 \overline{w}_{i_a}(t_a, x_a)}{\partial x_j \partial x_k}\lambda_j\lambda_k \leqslant 0.$$

Let q $(1 \leqslant q \leqslant n)$ denote the number of those functions among $\overline{w}_1, \ldots, \overline{w}_n$ which admit a positive value at point (t_a, x_a). Without loss of generality we may assume that these are the functions $\overline{w}_1, \ldots, \overline{w}_q$ (therefore $i_a \leqslant q$). We may also assume that the first part of the alternative of assumption 2° holds true, i.e. that the function $f^{(1)}_{i_a}$ is elliptic (with respect to $\{u_s(t, x)\}$) since in the contrary case we may change the role of systems (4.1), (4.2).

Our theorem will be proved in the following two cases, defined by assumptions 3° and 4°.

Case I. The function $f^{(1)}_{i_a}$ satisfies the condition $\Lambda[y_1, \ldots, y_n]$, the function $f^{(2)}_{i_a}$ satisfies the condition $(\mathfrak{L})$, and the function σ_{i_a} satisfies the condition $W[y_1, \ldots, y_n]$.

Case II. The function $f^{(2)}_{i_a}$ satisfies the condition $(\mathfrak{L})$ and the function σ_{i_a} satisfies the conditions $\Lambda[y_1, \ldots, y_n]$ and $W[y_1, \ldots, y_n]$.

In the remaining cases the proof is similar.

Let us consider case I. By the definition of an elliptic function we have

$$(6.15)\qquad \varepsilon_{i_a}\big[f_{i_a}^{(1)}\big(t_a, x_a, \bar{u}_1 H, \ldots, \bar{u}_n H, \tau_j(\bar{u}_{i_a}), \tau_{jk}(\bar{u}_{i_a}, \bar{u}_{i_a})\big) - -f_{i_a}^{(1)}\big(t_a, x_a, \bar{u}_1 H, \ldots, \bar{u}_n H, \tau_j(\bar{u}_{i_a}), \tau_{jk}(\bar{v}_{i_a} + \varepsilon_{i_a}\bar{\omega}_{i_a}, \bar{u}_{i_a})\big)\big] \leqslant 0$$

since

$$\varepsilon_{i_a}(\bar{u}_{i_a} - \bar{v}_{i_a} - \varepsilon_{i_a}\bar{\omega}_{i_a}) = |\bar{u}_{i_a} - \bar{v}_{i_a}| - \bar{\omega}_{i_a} = \bar{w}_{i_a}\,,$$

$$\varepsilon_{i_a}[\tau_{jk}(\bar{u}_{i_a}, \bar{u}_{i_a}) - \tau_{jk}(\bar{v}_{i_a} + \varepsilon_{i_a}\bar{\omega}_{i_a}, \bar{u}_{i_a})] = H\frac{\partial^2 \bar{w}_{i_a}}{\partial x_j \partial x_k}$$

and (6.14) holds. Making use of the condition $\Lambda[y_1, \ldots, y_n]$ we obtain

$$(6.16)\qquad \varepsilon_{i_a}\big[f_{i_a}^{(1)}\big(t_a, x_a, \bar{u}_1 H, \ldots, \bar{u}_n H, \tau_j(\bar{u}_{i_a}), \tau_{jk}(\bar{v}_{i_a} + \varepsilon_{i_a}\bar{\omega}_{i_a}, \bar{u}_{i_a})\big) - -f_{i_a}^{(1)}\big(t_a, x_a, (\bar{v}_1 + \varepsilon_1\bar{\omega}_1)H, \ldots, (\bar{v}_q + \varepsilon_q\bar{\omega}_q)H, \bar{u}_{q+1}H, \ldots \ldots, \bar{u}_n H, \tau_j(\bar{u}_{i_a}), \tau_{jk}(\bar{v}_{i_a} + \varepsilon_{i_a}\bar{\omega}_{i_a}, \bar{u}_{i_a})\big)\big]$$

$$\leqslant L\sum_{s=1}^{q}|\bar{u}_s - \bar{v}_s - \varepsilon_s\bar{\omega}_s|H = L\sum_{s=1}^{q}\bar{w}_s H \leqslant Lq\bar{w}_{i_a}H \leqslant Ln\bar{w}_{i_a}H\,.$$

From assumption 5° it follows that

$$(6.17)\qquad \varepsilon_{i_a}\big[f_{i_a}^{(1)}\big(t_a, x_a, (\bar{v}_1 + \varepsilon_1\bar{\omega}_1)H, \ldots, (\bar{v}_q + \varepsilon_q\bar{\omega}_q)H, \bar{u}_{q+1}H, \ldots \ldots, \bar{u}_n H, \tau_j(\bar{u}_{j_a}), \tau_{jk}(\bar{v}_{i_a} + \varepsilon_{i_a}\bar{\omega}_{i_a}, \bar{u}_{i_a})\big) - -f_{i_a}^{(2)}\big(t_a, x_a, \bar{v}_1 H, \ldots, \bar{v}_n H, \tau_j(\bar{u}_{i_a}), \tau_{jk}(\bar{v}_{i_a} + \varepsilon_{i_a}\bar{\omega}_{i_a}, \bar{u}_{i_a})\big)\big]$$

$$\leqslant \sigma_{i_a}(t_a, \bar{\omega}_1 H, \ldots, \bar{\omega}_q H, |\bar{u}_{q+1} - \bar{v}_{q+1}|H, \ldots, |\bar{u}_n - \bar{v}_n|H)$$

$$\leqslant \sigma_{i_a}(t_a, \bar{\omega}_1 H, \ldots, \bar{\omega}_n H)\,.$$

The last inequality has been obtained by virtue of the condition $W[y_1, \ldots, y_n]$ imposed on the function σ_{i_a}.

According to condition ($\mathfrak{L}$) and relation (6.13) we get

$$(6.18)\qquad \varepsilon_{i_a}\big[f_{i_a}^{(2)}\big(t_a, x_a, \bar{v}_1 H, \ldots, \bar{v}_n H, \tau_j(\bar{u}_{i_a}), \tau_{jk}(\bar{v}_{i_a} + \varepsilon_{i_a}\bar{\omega}_{i_a}, \bar{u}_{i_a})\big) - -f_{i_a}^{(2)}\big(t_a, x_a, \bar{v}_1 H, \ldots, \bar{v}_n H, \tau_j(\bar{v}_{i_a} + \varepsilon_{i_a}\bar{\omega}_{i_a}), \tau_{jk}(\bar{v}_{i_a} + \varepsilon_{i_a}\bar{\omega}_{i_a}, \bar{v}_{i_a} + \varepsilon_{i_a}\bar{\omega}_{i_a}$$

$$\leqslant \bar{w}_{i_a}\left[L_0\sum_{j,k=1}^{m}\left|\frac{\partial^2 H}{\partial x_j \partial x_k}\right| + (L_1|x| + L_2)\sum_{j=1}^{m}\left|\frac{\partial H}{\partial x_j}\right|\right].$$

Adding the relations (6.15), (6.16), (6.17), and (6.18) we derive from (6.11) the inequality

$$(6.19)\qquad \frac{\partial \bar{w}_{i_a}}{\partial t}H \leqslant \bar{w}_{i_a}\mathcal{F}H\,,$$

$\mathcal{F}H$ being defined by (6.4). Hence, in view of (6.5) and (6.12), and of $\bar{w}_{i_a}(t_a, x_a) > 0$ we obtain a contradiction.

In case II the relations (6.15) and (6.18) hold true. Furthermore, we have, by assumption 5°,

$$(6.20)\quad \varepsilon_{t_a}\big[f_{i_a}^{(1)}\big(t_a, x_a, \bar{u}_1 H, \ldots, \bar{u}_n H, \tau_j(\bar{u}_{t_a}), \tau_{jk}(\bar{v}_{t_a}+\varepsilon_{t_a}\overline{\omega}_{t_a}, \bar{u}_{t_a})\big) - -f_{i_a}^{(2)}\big(t_a, x_a, \bar{v}_1 H, \ldots, \bar{v}_n H, \tau_j(\bar{u}_{t_a}), \tau_{jk}(\bar{v}_{t_a}+\varepsilon_{t_a}\overline{\omega}_{t_a}, \bar{u}_{i_a})\big)\big] \leqslant \sigma_{t_a}(t_a, |\bar{u}_1-\bar{v}_1|H, \ldots, |\bar{u}_n-\bar{v}_n|H).$$

According to the condition $\Lambda[y_1, \ldots, y_n]$

$$(6.21)\quad \sigma_{t_a}(t_a, |\bar{u}_1-\bar{v}_1|H, \ldots, |\bar{u}_n-\bar{v}_n|H) - -\sigma_{i_a}(t_a, \overline{\omega}_1 H, \ldots, \overline{\omega}_n H, |\bar{u}_{q+1}-\bar{v}_{q+1}|H, \ldots, |\bar{u}_n-\bar{v}_n|H) \leqslant L\sum_{s=1}^{q}\overline{w}_s H \leqslant LnH\overline{w}_{i_a}.$$

Using the condition $W[y_1, \ldots, y_n]$ we obtain

$$(6.22)\quad \sigma_{i_a}(t_a, \overline{\omega}_1 H, \ldots, \overline{\omega}_q H, |\bar{u}_{q+1}-\bar{v}_{q+1}|H, \ldots, |\bar{u}_n-\bar{v}_n|H) \leqslant \sigma_{i_a}(t_a, \overline{\omega}_1 H, \ldots, \overline{\omega}_q H, \overline{\omega}_{q+1} H, \ldots, \overline{\omega}_n H).$$

Now, adding the relations (6.15), (6.20), (6.21), (6.22), (6.18), and taking advantage of (6.11), we derive (6.19), which contradicts (6.5) and (6.12). Thus we have proved that if $(t_a, x_a) \in D_a^h + S_a^h$, then $\overline{w}_{i_a}(t_a, x_a) \leqslant 0$ and therefore $A_a \leqslant 0$. Further, if $(t_a, x_a) \in \bar{S}_a^0 + \Sigma_a^h$, then (by our assumption) $\overline{w}_{i_a}(t_a, x_a) \leqslant 0$ and finally if $(t_a, x_a) \in C_a^h$, then we have

$$A_a = \overline{w}_{i_a}(t_a, x_a) \leqslant \frac{M(t_a)\exp(K_0 R_a^2)}{\exp\left\{\dfrac{KR_a^2}{1-\mu t_a}+\nu t_a\right\}} \to 0 \quad \text{as} \quad a\to\infty.$$

From the previous considerations it follows that the last inequality always holds. On the other hand, the sequence $\{A_a\}$ is non-decreasing. Hence we conclude that for every a, $A_a \leqslant 0$, which completes the proof.

COROLLARY 2. *Let $\{u_s(t, x)\}$ be a regular parabolic solution of class E_2 of system* (4.8). *If the functions f_i satisfy the conditions* ($\mathfrak{L}$) *and $\Lambda[y_1, \ldots, y_n]$, then the solution $\{u_s(t, x)\}$ depends continuously on the initial and boundary data and on the right-hand members of* (4.8).

For the proof it is sufficient to put in Theorem 6 $f_i^{(2)} \equiv f_i$ and $\sigma_i \equiv L\sum_{s=0}^{n} y_s + \eta_0$, $\eta_0 > 0$ $(i = 1, \ldots, n)$ and to take $\eta_i \to 0$ $(i = 0, 1, \ldots, n)$.

A certain theorem concerning the stability problem in unbounded domains, similar to that established by W. Mlak (Theorem 3 of [10]) for bounded domains, may also be obtained from Theorem 6.

§ 7. Let $\sigma_i(t, y_1, \ldots, y_n)$ and $\omega_i(t, \eta_1, \ldots, \eta_n)$ be functions defined as in § 5.

THEOREM 7. *If*

1° $\{u_s(t, x)\}$ $(s = 1, \ldots, n)$ *is a regular parabolic solution of class* E_2 *of system* (4,8),

$$2^\circ \quad |f_i(t, x, y_1, \ldots, y_n, p_j, p_{jk})| \leqslant L_0 \sum_{j,k=1}^{m} |p_{jk}| + (L_1|x| + L_2) \sum_{j=1}^{m} |p_j| + \sigma_i(t, |y_1|, \ldots, |y_n|) \quad (i = 1, \ldots, n),$$

3° *for every fixed* i *at least one function* $f_i(t, x, y_1, \ldots, y_n, p_j, p_{jk})$, $\sigma_i(t, y_1, \ldots, y_n)$ *satisfies the condition* $\Lambda[y_1, \ldots, y_n]$ *and at least one of them satisfies the condition* $W[y_1, \ldots, y_n]$,

4° $|u_i(0, x)| \leqslant \eta_i$ $(i = 1, \ldots, n)$ *for* $x \in \bar{S}^0$,

5° $|u_i(t, x)| \leqslant \omega_i(t, \eta_1, \ldots, \eta_n)$ $(i = 1, \ldots, n)$ *for* $(t, x) \in \Sigma$, *then*

$$(7.1) \qquad |u_i(t, x)| \leqslant \omega_i(t, \eta_1, \ldots, \eta_n) \qquad (i = 1, \ldots, n)$$

for $(t, x) \in D$.

Proof. As before, it is enough to prove the theorem for the domain D^h defined as in the proof of Theorem 6.

We retain the meaning of symbols $\{R_a\}$, D_a^h, Σ_a^h, S_a^0, S_a^h, C_a^h introduced in the proof of Theorem 6.

Put

$$(7.2) \qquad w_i = |u_i| - \omega_i .$$

There exist a function $M(t) > 0$ and $K_0 > 0$ such that

$$(7.3) \qquad |w_i| \leqslant M(t)\exp(K_0|x|^2) \quad \text{for} \quad (t, x) \in D .$$

Introduce the transformations

$$(7.4) \qquad u_i = \bar{u}_i H, \quad \omega_i = \bar{\omega}_i H \quad (i = 1, \ldots, n) .$$

where $H = H(t, x; K)$, $K > K_0$, is defined by (6.3).

Writing

$$(7.5) \qquad \bar{w}_i = |\bar{u}_i| - \bar{\omega}_i$$

we have

$$(7.6) \qquad w_i = \bar{w}_i H .$$

Put

$$\bar{A}_a = \max_{(i)} \max_{(t,x) \in \overline{D_a^h}} \bar{w}_i(t, x) .$$

From (7.4), (7.5) and (7.6) it follows that in order to prove theorem (7.1) in the domain D^h it is sufficient to show that $\bar{A}_a \leqslant 0$ for every a.

For every fixed a there exist an index i_a and a point $(t_a, x_a) \in \overline{D_a^h}$ such that $\bar{A}_a = \bar{w}_{i_a}(t_a, x_a)$.

We shall show that if $(t_a, x_a) \in D_a^h + S_a^h$, then $\overline{w}_{i_a}(t_a, x_a) \leqslant 0$. In fact, suppose that $\overline{w}_{i_a}(t_a, x_a) > 0$. Therefore $w_{i_a}(t_a, x_a) > 0$. By (7.5) we obtain $|u_{i_a}(t_a, x_a)| > 0$ and $|\bar{u}_{i_a}(t_a, x_a)| > 0$. Therefore, the functions $w_{i_a}(t, x)$, $\overline{w}_{i_a}(t, x)$ have the derivatives (6.8) at point (t_a, x_a). Putting

$$\varepsilon_i = \begin{cases} 1 & \text{if} \quad \bar{u}_i(t_a, x_a) > 0\,, \\ -1 & \text{if} \quad \bar{u}_i(t_a, x_a) \leqslant 0\,. \end{cases}$$

we get, by virtue of (4.8), (5.1) and (4.6), at (t_a, x_a),

$$(7.7) \quad \frac{\partial w_{i_a}}{\partial t} = \varepsilon_{i_a} f_{i_a}\left(t_a, x_a, u_1, \ldots, u_n, \frac{\partial(u_{i_a} - \varepsilon_{i_a}\omega_{i_a})}{\partial x_j}, \frac{\partial^2(u_{i_a} - \varepsilon_{i_a}\omega_{i_a})}{\partial x_j \partial x_k}\right) - \\ - \sigma_{i_a}(t_a, \omega_1, \ldots, \omega_n)\,.$$

Applying the transformations (7.4), (7.6) and the notation (6.10) we may write relation (7.7) in the form

$$(7.8) \quad \frac{\partial \overline{w}_{i_a}}{\partial t} H + \overline{w}_{i_a} \frac{\partial H}{\partial t}$$

$$= \varepsilon_{i_a} f_{i_a}\big(t_a, x_a, \bar{u}_1 H, \ldots, \bar{u}_n H, \tau_j(\bar{u}_{i_a} - \varepsilon_{i_a}\overline{\omega}_{i_a}), \tau_{jk}(\bar{u}_{i_a} - \varepsilon_{i_a}\overline{\omega}_{i_a}, \bar{u}_{i_a} - \varepsilon_{i_a}\overline{\omega}_{i_a})\big) - \\ - \sigma_{i_a}(t_a, \overline{\omega}_1 H, \ldots, \overline{\omega}_n H)\,.$$

It may be assumed that the first q $(1 \leqslant q \leqslant n)$ functions, among $\overline{w}_1, \ldots, \overline{w}_n$, have a positive value at point (t_a, x_a).

We shall prove our theorem in the case where the function σ_{i_a} fulfils the conditions $\Lambda[y_1, \ldots, y_n]$ and $W[y_1, \ldots, y_n]$. In the other cases defined by assumption 3° the proof is similar.

The function $\overline{w}_{i_a}(t, x)$ has a maximum at $(t_a, x_a) \in D_a^h + S_a^h$, and so the relations (6.12), (6.13) and (6.14) hold. Therefore, by the definition of the elliptic function, we find

$$(7.9) \quad \varepsilon_{i_a}\big[f_{i_a}\big(t_a, x_a, \bar{u}_1 H, \ldots, \bar{u}_n H, \tau_j(\bar{u}_{i_a} - \varepsilon_{i_a}\overline{\omega}_{i_a}), \tau_{jk}(\bar{u}_{i_a} - \varepsilon_{i_a}\overline{\omega}_{i_a}, \bar{u}_{i_a} - \varepsilon_{i_a}\overline{\omega}_{i_a})\big) - \\ - f_{i_a}\big(t_a, x_a, \bar{u}_1 H, \ldots, \bar{u}_n H, \tau_j(\bar{u}_{i_a} - \varepsilon_{i_a}\overline{\omega}_{i_a}), \tau_{jk}(0, \bar{u}_{i_a} - \varepsilon_{i_a}\overline{\omega}_{i_a})\big)\big] \leqslant 0\,.$$

According to assumption 2°, we obtain, on account of (6.13),

$$(7.10) \quad \varepsilon_{i_a} f_{i_a}\big((t_a . x_a, \bar{u}_1 H, \ldots, \bar{u}_n H, \tau_j(\bar{u}_{i_a} - \varepsilon_{i_a}\overline{\omega}_{i_a}), \tau_{jk}(0, \bar{u}_{i_a} - \varepsilon_{i_a}\overline{\omega}_{i_a})\big) \\ \leqslant \overline{w}_{i_a}\left[L_0 \sum_{j,k=1}^{m} \left|\frac{\partial^2 H}{\partial x_j \partial x_k}\right| + (K_1|x| + L_2) \sum_{j=1}^{m} \left|\frac{\partial H}{\partial x_j}\right|\right] + \sigma_{i_a}(t_a, |\bar{u}_1| H, \ldots, |\bar{u}_n| H)\,.$$

Taking advantage of the condition $\Lambda[y_1, \ldots, y_n]$ we obtain

$$(7.11) \quad \sigma_{i_a}(t_a, |\bar{u}_1| H, \ldots, |\bar{u}_n| H) - \sigma_{i_a}(t_a, \overline{\omega}_1 H, \ldots, \overline{\omega}_q H, |\bar{u}_{q+1}| H, \ldots, |\bar{u}_n| H) \\ \leqslant L \sum_{s=1}^{q} |\overline{w}_s| H \leqslant L n H \overline{w}_{i_a}\,,$$

and the condition $W[y_1, \ldots, y_n]$ yields the inequality

$$(7.12)\qquad \sigma_{i_a}(t_a, \overline{\omega}_1 H, \ldots, \overline{\omega}_q H, |\overline{u}_{q+1}| H, \ldots, |\overline{u}_n| H) \leqslant \sigma_{i_a}(t_a, \overline{\omega}_1 H, \ldots, \overline{\omega}_q H, \overline{\omega}_{q+1} H, \ldots, \overline{\omega}_n H).$$

Adding the relations (7.12), (7.11), (7.10), (7.9), and (7.8) we deduce that inequality (6.19) holds, and thus, by (6.12) and (6.5), we come to a contradiction. Consequently if $(t_a, x_a) \in D_a^h + S_a^h$, then $\overline{w}_{i_a}(t_a, x_a) \leqslant 0$. Further if $(t_a, x_a) \in S_a^0 + \Sigma_a^h$, then, by our assumption, $\overline{w}_{i_a}(t_a, x_a) \leqslant 0$. Finally if $(t_a, x_a) \in C_a^h$, then we have

$$\bar{A}_a = \overline{w}_{i_a}(t_a, x_a) \leqslant \frac{M(t_a)\exp(K_0 R_a^2)}{\exp\left\{\dfrac{K R_a^2}{1-\mu t_a} + \nu t_a\right\}} \to 0 \quad \text{as} \quad a \to \infty.$$

From the preceding remark it follows that the last inequality always holds. But the sequence $\{\bar{A}_a\}$ is non-decreasing, whence we deduce that $\bar{A}_a \leqslant 0$ for any a, which completes the proof.

Part II

§ 8. Denote by Δ a bounded and closed domain of the space $(x_1, \ldots, x_m)$ and by S its complementary domain. The boundary $F\Delta$ of Δ is supposed to be represented by the equation $\Gamma(x) = 0$, $x = (x_1, \ldots, x_m)$, where $\Gamma(x)$ is a function with continuous and bounded derivatives of the second order in S, of class C^1 in the closure $\bar{S}$ and satisfies the condition

$$(8.1)\qquad |\operatorname{grad}\Gamma(x)| \geqslant \Gamma_0 = \text{const} > 0.$$

In part II of this paper by D is meant the topological product of S with the interval $(0, T)$, i.e. $D = S \times (0, T)$. Similarly, we define $\Sigma = F\Delta \times (0, T)$. The part of the boundary of D lying on the plane $t = 0$ will be denoted by S^0.

For every $(t, x) \in \Sigma$ and every i $(i = 1, \ldots, n)$ let l_i be a straight half-line entering the interior of D (at point (t, x)) and parallel to the plane $t = 0$. Suppose there exists $\gamma_0 > 0$ such that $\cos(l_i, n_0) \geqslant \gamma_0$ $(i = 1, \ldots, n)$ for $(t, x) \in \Sigma$, n_0 being the normal to Σ directed to the interior of D.

Let $G_i^{(1)}(t, x, y_1, \ldots, y_n)$, $G_i^{(2)}(t, x, y_1, \ldots, y_n)$ $(i = 1, \ldots, n)$ be functions defined for $(t, x) \in \Sigma$ and arbitrary $y_1, \ldots, y_n$.

§ 9. The following Theorem 8 may be proved by an easy modification of the proof of Theorem II included in [2].

THEOREM 8. *If the assumptions* 1°-5° *of Theorem* 1 *hold true and if*

6° $u_i(t,x)$, $v_i(t,x)$ *possess the derivatives* $\frac{du_i}{dl_i}$, $\frac{dv_i}{dl_i}$ $(i=1,\ldots,n)$ *at points of* Σ, *and satisfy the boundary inequalities*

$$\frac{du_i}{dl_i}+G_i^{(1)}(t,x,u_1,\ldots,u_n)\geqslant\frac{dv_i}{dl_i}+G_i^{(2)}(t,x,v_1,\ldots,v_n)\quad(i=1,\ldots,n), \tag{9.1}$$

7° *for every fixed* i *at least one function* $G_i^{(1)}(t,x,y_1,\ldots,y_n)$, $G_i^{(2)}(t,x,y_1,\ldots,y_n)$ *satisfies the condition* $\Lambda[y_1,\ldots,y_n]$ *and at least one of them satisfies the condition* $W[y_1,\ldots,y_n]$,

8° $G_i^{(1)}(t,x,y_1,\ldots,y_n)\leqslant G_i^{(2)}(t,x,y_1,\ldots,y_n)$ $(i=1,\ldots,n)$ *for* $(t,x)\in\Sigma$, $-\infty<y_s<+\infty$,

then inequalities (2.3) *are fulfilled in* D.

§ 10. The following Theorems 9-12 may be derived from Theorem 8.

THEOREM 9. *Suppose that the assumptions* 1°-5° *of Theorem* 2 *are fulfilled and that, for* $(t,x)\in\Sigma$, *we have the boundary inequalities*

$$\frac{d(u_i-v_i)}{dl_i}+\sum_{s=1}^{n}\psi_s^i(t,x)(u_s-v_s)\geqslant\sum_{s=1}^{n}\psi_s^i(t,x)\,\omega_s(t,\eta_1,\ldots,\eta)$$

$$\left(\frac{d(u_i-v_i)}{dl_i}+\sum_{s=1}^{n}\psi_s^i(t,x)(u_s-v_s)\leqslant\sum_{s=1}^{n}\psi_s^i(t,x)\,\omega_s(t,\eta_1,\ldots,\eta)\ \textit{respectively}\right)$$

$(i=1,\ldots,n)$, *where the functions* $\psi_s^i(t,x)$ *are assumed to be defined on* Σ *and to satisfy the conditions*

$$\psi_s^i(t,x)\geqslant0,\quad s\neq i,\quad \psi_s^i(t,x)\leqslant C\quad(s,i=1,\ldots,n),$$

C *being an arbitrary positive constant.*

Under these assumptions the limitation (4.3) *holds true in* D.

Proof. In the same manner as in the proof of Theorem 2 it can be shown that the functions $v_i+\omega_i$ satisfy inequalities (4.7) and the functions u_i satisfy, by our assumptions, the system (4.1). Note that $d\omega_i/dl_i\equiv0$ $(i=1,\ldots,n)$. Putting

$$G_i^{(1)}(t,x,y_1,\ldots,y_n)\equiv G_i^{(2)}(t,x,y_1,\ldots,y_n)\equiv\sum_{s=1}^{n}\psi_s^i(t,x)y_s\quad(i=1,\ldots,n)$$

and applying Theorem 8 we obtain $u_i\leqslant v_i+\omega_i$. The second part of the Theorem may be obtained similarly.

The validity of the following Theorems 10-12 can be proved in a similar way.

THEOREM 10. *Let the assumptions* 1°-5° *of Theorem* 3 *be satisfied and assume additionally that both solutions* $\{u_s(t,x)\}$ *and* $\{v_s(t,x)\}$ *are parabolic. Suppose furthermore that, for* $(t,x)\in\Sigma$,

$$\frac{d(u_i-v_i)}{dl_i}+g_i(t,x,u_1,\dots,u_n)-g_i(t,x,v_1+\delta_1,\dots,v_n+\delta_n)\geqslant 0\quad (i=1,\dots,n)$$

$$\left(\frac{d(u_i-v_i)}{dl_i}+g_i(t,x,u_1,\dots,u_n)-g_i(t,x,v_1+\delta_1,\dots,v_n+\delta_n)\leqslant 0\quad \textit{respectively}\right),$$

where the functions $g_i(t,x,y_1,\dots,y_n)$ *are assumed to satisfy the conditions* $\Lambda[y_1,\dots,y_n]$ *and* $W[y_1,\dots,y_n]$. *These assumptions imply the inequalities* (4.11) *in* D.

THEOREM 11. *If the assumptions* 1°-4° *of Theorem* 4 *hold true and if for* $(t,x)\in\Sigma$ *we have*

$$\frac{du_i}{dl_i}+g_i(t,x,u_1,\dots,u_n)\geqslant g_i(t,x,\omega_1,\dots,\omega_n)\qquad (i=1,\dots,n)$$

$$\left(\frac{du_i}{dl_i}+g_i(t,x,u_1,\dots,u_n)\leqslant g_i(t,x,\omega_1,\dots,\omega_n)\qquad \textit{respectively}\right),$$

$g_i(t,x,y_1,\dots,y_n)$ *being functions satisfying the conditions* $\Lambda[y_1,\dots,y_n]$ *and* $W[y_1,\dots,y_n]$, *then the limitation* (4.12) *is true in* D.

THEOREM 12. *If the assumptions of Theorem* 5 *are satisfied except that the boundary condition* (4.14) *is replaced by the condition*

$$\frac{du_i}{dl_i}+g_i(t,x,u_1,\dots,u_n)\geqslant g_i\big(t,x,\varphi_1(t),\dots,\varphi_m(t)\big)$$

$$\left(\frac{du_i}{dl_i}+g_i(t,x,u_1,\dots,u_n)\leqslant g_i\big(t,x,\varphi_1(t),\dots,\varphi_n(t)\big)\ \textit{respectively}\right)$$

$(i=1,\dots,n)$, $(t,x)\in\Sigma$, *where the* $g_i(t,x,y_1,\dots,y_n)$ *are assumed to fulfil the conditions* $\Lambda[y_1,\dots,y_n]$ *and* $W[y_1,\dots,y_n]$, *then* (4.15) *holds for* $(t,x)\in D$.

§ 11. THEOREM 13. *Let the hypotheses* 1°-6° *of Theorem* 6 *be fulfilled. Furthermore, suppose that the functions* $u_i(t,x)$, $v_i(t,x)$ *satisfy the boundary inequalities*

$$(11.1)\qquad \left|\frac{d(u_i-v_i)}{dl_i}-\sum_{s=1}^{n}\psi_s^i(t,x)(u_s-v_s)\right|\leqslant\sum_{s=1}^{n}\psi_s^i(t,x)\,\omega_s,\qquad (t,x)\in\Sigma,$$

$(i=1,\dots,n)$, *where the functions* $\psi_s^i(t,x)$ *are assumed to be defined on* Σ *and to satisfy the following conditions*

$$(11.2)\qquad -C\leqslant\psi_s^i(t,x)\leqslant 0,\qquad s\neq i,$$

$$(11.3)\qquad \sum_{s=1}^{n}\psi_s^i(t,x)\,\omega_s\geqslant 0\qquad (i=1,\dots,n),$$

C *being a positive constant.*

Under these assumptions estimations (6.1) *hold in* D.

Proof. It is sufficient to prove our theorem—just as Theorem 6—for a part D^{h_1} of domain D contained in a strip $0 < t < h_1$. Let Σ^{h_1} be the part of Σ lying in the strip $0 < t < h_1$ and S^{h_1} the part of boundary of D^{h_1} lying on the plane $t = h_1$.

The functions

$$w_i(t, x) = |u_i(t, x) - v_i(t, x)| - \omega_i(t, \eta_1, \ldots, \eta_n) \qquad (i = 1, \ldots, n)$$

belong to a class $E_2(M(t), K_0)$. We will make use of the auxiliary function

$$H_1(t, x; K) = \exp\left\{\frac{K[\Gamma(x) - p(K)]^2}{1 - \mu_1(K)t} + \nu_1(K)t\right\}, \qquad K > K_0,$$

introduced by M. Krzyżański [7], which has certain properties similar to those of function (6.3). Namely, it has been shown in [2] that, the constants $\mu_1(K)$, $\nu_1(K)$ being suitably chosen, the relation

$$\mathcal{F} H_1(t, x; K) \leqslant -H_1(t, x; K) \tag{11.4}$$

holds for $(t, x) \in D^{h_1}$, where h_1 is sufficiently small and $\mathcal{F} H_1$ is defined by (6.4). Moreover, if the constant $p(K)$ is suitably chosen, then we have

$$\frac{dH_1}{dl_i} + (n-1)CH_1 \leqslant -H_1 \quad \text{for} \quad (t, x) \in \Sigma^{h_1}. \tag{11.5}$$

The detailed computations concerning the last relation are just the same as in proof of Theorem I of [2].

We may assume that

$$\Gamma(x) \equiv |x| \quad \text{for} \quad |x| > R_0, \tag{11.6}$$

R_0 being the radius of a sphere $|x| = R_0$ situated in the space $(x_1, \ldots, x_m)$ and containing the boundary $F\Delta$ in its interior.

Let $\{R_a\}$ be an increasing sequence, $R_a > R_0$, $R_a \to \infty$ as $a \to \infty$. Denote by $D_a^{h_1}$, S_a^0, $S_a^{h_1}$ the parts of the sets D^{h_1}, S^0, S^{h_1} respectively, lying inside the cylindrical surface C_a with the equation $|x| = R_a$. By $C_a^{h_1}$ we denote the part of C_a contained in D^{h_1}. Further we introduce the transformations

$$u_i = \tilde{u}_i H_1, \quad v_i = \bar{v}_i H_1, \quad \omega_i = \bar{\omega}_i H_1 \quad (i = 1, \ldots, n). \tag{11.7}$$

If we write $\bar{w}_i = |\tilde{u}_i - \bar{v}_i| - \bar{\omega}_i$, then

$$w_i = \bar{w}_i H_1 \quad (i = 1, \ldots, n). \tag{11.8}$$

Consider the sequence with the terms

$$A_a = \max_{(i)} \max_{(t,x) \in \overline{D_a^{h_1}}} \bar{w}_i(t, x) \quad (a = 1, 2, \ldots).$$

We shall show that for every a, $A_a \leqslant 0$. For any a there exist an index i_a and a point $(t_a, x_a) \in \overline{D_a^{h_1}} = (D_a^{h_1}+S_a^{h_1})+\overline{S_a^0}+\Sigma^{h_1}+C_a^{h_1}$ such that $A_a = \overline{w}_{i_a}(t_a, x_a)$. Since the function $H_1(t, x; K)$ has the property expressed by (11.4), we may repeat the fragment of the proof of Theorem 6 for the case of (t_a, x_a) being the point of $D_a^{h_1}+S_a^{h_1}$, whence we obtain (in this case) $A_a \leqslant 0$. If $(t_a, x_a) \in \overline{S_a^0}$, then, by our assumption, $A_a \leqslant 0$. Now we shall show that if $(t_a, x_a) \in \Sigma^{h_1}$ then $A_a \leqslant 0$. In fact, suppose that $(t_a, x_a) \in \Sigma^{h_1}$ and $A_a > 0$, i.e. $\overline{w}_{i_a}(t_a, x_a) > 0$. Inequality (11.1) (for $i = i_a$, $t = t_a$, $x = x_a$) may be written in the form

$$(11.9)\qquad \left|\frac{d|u_{i_a}-v_{i_a}|}{dl_{i_a}} - \varepsilon_{i_a}\sum_{s=1}^{n}\psi_s^{i_a}(t_a, x_a)\,\varepsilon_s|u_s-v_s|\right| \leqslant \sum_{s=1}^{n}\psi_s^{i_a}(t_a, x_a)\,\omega_s\,,$$

where $\varepsilon_i = \operatorname{sgn}\big(u_i(t_a, x_a)-v_i(t_a, x_a)\big)$.

We shall show that

$$(11.10)\qquad \frac{d|u_{i_a}-v_{i_a}|}{dl_{i_a}} - \varepsilon_{i_a}\sum_{s+1}^{n}\psi_s^{i_a}(t_a, x_a)\,\varepsilon_s|u_s-v_s| < 0\,.$$

Indeed, assumption (11.2) yields

$$(11.11)\qquad -\varepsilon_{i_a}\sum_{s=1}^{n}\psi_s^{i_a}\varepsilon_s|u_s-v_s| \leqslant -\sum_{s=1}^{n}\psi_s^{i_a}|u_s-v_s|\,,$$

and by (11.3) we have

$$(11.12)\qquad -\sum_{s=1}^{n}\psi_s^{i_a}|u_s-v_s| \leqslant -\sum_{s=1}^{n}\psi_s^{i_a}w_s\,.$$

In view of (11.11), (11.12) and of the equality $d\omega_{i_a}/dl_{i_a} = 0$ we obtain

$$(11.13)\qquad \frac{d|u_{i_a}-v_{i_a}|}{dl_{i_a}} - \varepsilon_{i_a}\sum_{s=1}^{n}\psi_s^{i_a}\varepsilon_s|u_s-v_s| \leqslant \frac{dw_{i_a}}{dl_{i_a}} - \sum_{s=1}^{n}\psi_s^{i_a}w_s\,.$$

By (11.8) we obtain

$$(11.14)\qquad \frac{dw_{i_a}}{dl_{i_a}} - \sum_{s=1}^{n}\psi_s^{i_a}w_s = \frac{d\overline{w}_{i_a}}{dl_{i_a}}H_1 + \overline{w}_{i_a}\frac{dH_1}{dl_{i_a}} - \sum_{s=1}^{n}\psi_s^{i_a}\overline{w}_s H_1\,.$$

From the definition of $\overline{w}_{i_a}(t_a, x_a)$ it follows that

$$(11.15)\qquad \frac{d\overline{w}_{i_a}}{dl_{i_a}} \leqslant 0$$

and

$$-\sum_{s=1}^{n}\psi_s^{i_a}\overline{w}_s \leqslant -\sum_{s=1}^{n}\psi_s^{i_a}\overline{w}_{i_a} \leqslant (n-1)C\overline{w}_{i_a}. \tag{11.16}$$

By (11.14), (11.15), (11.16) and (11.5) we find

$$\frac{d\overline{w}_{i_a}}{dl_{i_a}} - \sum_{s=1}^{n}\psi_s^{i_a}w_s \leqslant -\overline{w}_{i_a}H_1. \tag{11.17}$$

Therefore, by (11.13), inequality (11.10) holds. Consequently, condition (11.9) may be written as

$$\frac{dw_{i_a}}{dl_{i_a}} - \varepsilon_{i_a}\sum_{s=1}^{u}\psi_s^{i_a}\varepsilon_s|u_s - v_s| \geqslant -\sum_{s=1}^{n}\psi_s^{i_a}\omega_s,$$

and hence, using (11.11) we have

$$\frac{dw_{i_a}}{dl_{i_a}} - \sum_{s=1}^{n}\psi_s^{i_a}w_s \geqslant 0,$$

which contradicts (11.17). Therefore we have proved that if $(t_a, x_a) \in \Sigma^{h_1}$, then $A_a \leqslant 0$. Finally if $(t_a, x_a) \in C_a^{h_1}$, we have, by (11.18),

$$A_a \leqslant \frac{M(t_a)\exp(K_0R_a^2)}{\exp\left\{\dfrac{K[R_a-p(K)]^2}{1-\mu_1 t} + \nu_1 t\right\}} \to 0 \quad \text{as} \quad a\to\infty.$$

Thus we have proved that the last inequality always holds. Moreover, the sequence $\{A_a\}$ is non-decreasing, whence we find that for every a, $A_a \leqslant 0$, q.e.d.

The next theorem may be proved in a similar way.

THEOREM 14. *If the assumptions* 1°-4° *of Theorem* 7 *hold and the boundary inequalities*

$$\left|\frac{du_i}{dl_i} - \sum_{s=1}^{n}\psi_s^{i_a}(t,x)u_s\right| \leqslant \sum_{s=1}^{n}\psi_s^{i}(t,x)\omega_s \quad (i = 1, \ldots, n),\ (t,x)\in\Sigma,$$

are fulfilled, the functions $\psi_s^i(t,x)$ *being the same as in Theorem* 13, *then the limitation* (7.1) *remains true in the domain* D.

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