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On Ruscheweyh derivatives

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Abstract. Let K_n be the classes of regular functions $f(z) = z + a_2z^2 + \ldots$, defined in the unit disc E and satisfying

$$\operatorname{Re}\frac{D^{n+1}f(z)}{D^nf(z)} > \frac{1}{2}, \quad z \in E,\ n = 0, 1, 2, \ldots,$$

where

$$D^nf(z) = f(z) * \frac{z}{(1-z)^{n+1}},$$

and $(*)$ is the Hadamard convolution.

(i) The author determines certain real values α and β such that whenever

$$\operatorname{Re}\left(\frac{D^{n+1}f(z)}{D^nf(z)} - \frac{1}{2}\right)^{\alpha}\left(\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - \frac{1}{2}\right)^{\beta} > 0,$$

$z \in E$, $n = 0, 1, 2, \ldots$, then $f \in K_n$.

(ii) Let $h_n(z) = \dfrac{D^nf(z)}{D^ng(z)}$. The author determines the set of real values α and β such that whenever f and g belong to K_n

$$\operatorname{Re} h_n^{\alpha}(z) h_{n+1}^{\beta}(z) > 0$$

holds for $z \in E$ and $n = 0, 1, 2, \ldots$

1. Introduction. Let A denote the class of functions $f(z)$ regular in the unit disc $E = \{z\colon |z| < 1\}$ and normalized by $f(0) = 0$, $f'(0) = 1$.

By $\{K_n\}$ we mean the subclasses of A satisfying for every $f \in K_n$ the inequality

$$\operatorname{Re}\frac{\left(z^nf(z)\right)^{(n+1)}}{\left(z^{n-1}f(z)\right)^{(n)}} > \frac{(n+2)}{2}, \tag{1}$$

where $n \in N_0$, $N_0 = 0, 1, 2, \ldots$, and $z \in E$.

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S. Ruscheweyh [6] introduced the classes K_n and showed the basic property

$$K_{n+1} \subset K_n, \quad n \in N_0.$$

Thus elements of K_n are univalent and starlike of order $\frac{1}{2}$ ($K_0 \equiv S^*_{1/2}$).

Let

$$D^n f(z) = z\big(z^{n-1}f(z)\big)^{(n)}/n!, \quad n \in N_0. \tag{2}$$

We shall refer to $D^n f$ as the nth order Ruscheweyh derivative of f. Note that $D^0 f = f$, $Df(z) = zf'(z)$.

Ruscheweyh cleverly observed that

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z), \tag{3}$$

where the operation $(*)$ is the usual Hadamard product of series (i.e., if $g(z) = \sum\limits_0^\infty a_n z^n$, $f(z) = \sum\limits_0^\infty b_n z^n$, then $f*g = \sum\limits_0^\infty a_n b_n z^n$). This lead him to an equivalent but more practical definition for K_n, namely $f \in K_n$ if and only if $f \in A$ and

$$\operatorname{Re}\frac{D^{n+1}f(z)}{D^n f(z)} > \frac{1}{2}, \quad n \in N_0, \tag{4}$$

is satisfied for $z \in E$.

Now we introduce the following classes:

DEFINITION. We say that $f \in S_n(\alpha, \beta)$, $n \in N_0$, if $f \in A$ and

$$P_n\big(f(z); \alpha, \beta\big) = \left(\frac{D^{n+1}f(z)}{D^n f(z)} - \frac{1}{2}\right)^{\alpha}\left(\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - \frac{1}{2}\right)^{\beta}, \tag{5}$$

where α, β are real numbers, then

$$\operatorname{Re}P_n\big(f(z); \alpha, \beta\big) > 0, \quad n \in N_0,\ z \in E.$$

The powers appearing in (5) are meant as principal values. For every $n \in N_0$, $S_n(\alpha, \beta)$ contains many interesting classes of univalent functions; $S_n(1, 0) = K_n$, $S_n(0, 1) = K_{n+1}$, and $S_0(\alpha, 0)$ is contained in class of strongly starlike [1] when $|\alpha| \geqslant 1$.

In section 3 we will determine a set of values of real numbers α and β for which $S_n(\alpha, \beta) \subset K_n$, $n \in N_0$. Similar problem was treated in [5].

Next, let

$$h_n(z) = \frac{D^n f(z)}{D^n g(z)}, \quad f, g \in A,\ n \in N_0.$$

In section 4 we will determine the set of the non-negative real numbers α and β such that

$$\operatorname{Re} h_n^{\alpha}(z) h_{n+1}^{\beta}(z) > 0, \quad n \in N_0,\ z \in E,$$

whenever $f, g \in K_n$. Special cases of this section reduces to results in [2], Theorems 1 and 2.

2. Preliminaries. We need the following results.

LEMMA 1. *Let $w \in A$ with $w(z) \neq 0$, $z \neq 0$. If $z_0 = r_0 e^{i\theta_0}$, $0 < r_0 < 1$ and $\max_{|z| \leqslant r_0} |w(z)| = |w(z_0)|$, then*

$$z_0 w'(z_0) = m w(z_0), \quad m \geqslant 1. \tag{7}$$

Lemma 1 may be found in [3].

LEMMA 2. *Let $p(z)$ be regular in E with $p(0) = 1$ and $\operatorname{Re} p(z) > 1/2$ in E. Then*

$$\left| p(z) - \frac{1}{1-r^2} \right| < \frac{r}{1-r^2}.$$

Lemma 2 can be deduced from the geometrical properties of $q(z)$, where $q(z) = 2p(z) - 1$.

LEMMA 3. *If $f \in K_n$, $n \in N_0$, then*

$$\left| \arg \frac{D^k f(z)}{z} \right| \leqslant (k+1) \sin^{-1} r, \quad 0 \leqslant k \leqslant n+1. \tag{8}$$

Proof. Since $f \in K_n$ implies $f \in K_j$, $0 \leqslant j \leqslant n$, then

$$\frac{D^{j+1} f(z)}{D^j f(z)} = p_j(z), \quad \text{with } \operatorname{Re} p_j(z) > 1/2,\ z \in E,\ 0 \leqslant j \leqslant n.$$

Lemma 2 yields

$$\left| \arg \frac{D^{j+1} f(z)}{D^j f(z)} \right| \leqslant \sin^{-1} r, \quad 0 \leqslant j \leqslant n,\ |z| = r < 1. \tag{9}$$

Moreover, $f \in K_n \Rightarrow f \in S^*_{1/2} \Leftrightarrow \int \left(\frac{f(z)}{z} \right)^2 dz \in K$, and for $F \in K$ we have [4] that $|\arg F'(z)| \leqslant 2 \sin^{-1} r$, $|z| = r < 1$. Consequently, $f \in K_n$ implies

$$\left| \arg \frac{f(z)}{z} \right| \leqslant \sin^{-1} r. \tag{10}$$

Applying (9) and (10) to the identity

$$\frac{D^k f(z)}{z} = \frac{f(z)}{z} \prod_{j=0}^{k-1} \frac{D^{j+1} f(z)}{D^j f(z)},$$

$1 \leqslant k \leqslant n+1$, we arrive at (8).

3. The classes $S_n(\alpha, \beta)$. Let

$$\begin{aligned} G_1 &= \{(\alpha, \beta) \mid \ (\alpha + 2\beta \leqslant 4k+3) \cap (\alpha + \beta \geqslant 4k+1),\ \beta \geqslant 0,\ k \in I\}, \\ G_2 &= \{(\alpha, \beta) \mid \ (\alpha + \beta \leqslant 4k+3) \cap (\alpha + 2\beta \geqslant 4k+1),\ \beta \leqslant 0,\ k \in I\}, \\ G_3 &= \{(\alpha, 0) \mid \ |\alpha| \geqslant 1\} \cup \{(0, \beta) \mid \ |\beta| \geqslant 1\}, \\ G &= G_1 \cup G_2 \cup G_3, \end{aligned} \tag{11}$$

where I is the set of integers and α, β being real numbers. In this section we show that for $(\alpha, \beta) \in G$ and $f \in S_n(\alpha, \beta)$, then $f \in K_n$. The region G is independent of n.

We shall use the technique of Miller [5] to prove the following

THEOREM 1. $S_n(\alpha, \beta) \subset K_n$ *if* $(\alpha, \beta) \in G$, $n \in N_0$.

Proof. The case where $(\alpha, \beta) \in G_3$ is trivial. Suppose $f \in S_n(\alpha, \beta)$ and

$$\frac{D^{n+1}f(z)}{D^n f(z)} = \frac{1}{1-w(z)}, \tag{12}$$

where $z \in E$. Then $w(z)$ is regular in E with $w(0) = 0$, $w(z) \neq \pm 1$. To complete the proof we need to show that $\operatorname{Re} 1/(1-w(z)) > 1/2$, $z \in E$ and $(\alpha, \beta) \in G$. To this end, it is sufficient to show $|w(z)| < 1$, $z \in E$ and $(\alpha, \beta) \in G$.

Differentiating (12) and using an easy to verify identity

$$z(D^k f(z))' = (k+1)D^{k+1}f(z) - kD^k f(z), \qquad k \in N_0,$$

one gets

$$z(D^{n+1}f(z))' = \frac{z(D^n f(z))'}{1-w(z)} + \frac{zw'(z)}{(1-w(z))^2} D^n f(z),$$

$$(n+2)D^{n+2}f(z) - (n+1)D^{n+1}f(z)$$
$$= \frac{(n+1)D^{n+1}f(z) - nD^n f(z)}{1-w(z)} + \frac{zw'(z)D^n f(z)}{(1-w(z))^2}.$$

Thus

$$\frac{D^{n+2}f(z)}{D^{n+1}f(z)} = \frac{1}{n+2}\left[1 + \frac{n+1}{1-w(z)} + \frac{zw'(z)}{1-w(z)}\right]. \tag{13}$$

Substituting (12) and (13) in (5) we have

$$P_n(f(z); \alpha, \beta) = C\left(\frac{1+w(z)}{1-w(z)}\right)^\alpha \left(1 + (n+1)\frac{1+w(z)}{1-w(z)} + \frac{2zw'(z)}{1-w(z)}\right)^\beta, \tag{14}$$

where $C = 2^{-\alpha-\beta}(n+2)^{-\beta} > 0$.

Now suppose to the contrary that there is $z_0 \in E$ such that $\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1$, $w(z_0) \neq \pm 1$. Then Lemma 1 shows

$$z_0 w'(z_0)/(1-w(z_0)) = mw(z_0)/(1-w(z_0)).$$

Let $w(z_0) = e^{i\theta_0}$. Then

$$\frac{1+w(z_0)}{1-w(z_0)} = \frac{\sin\theta_0}{2(1-\cos\theta_0)} i,$$

$$\frac{w(z_0)}{1-w(z_0)} = \frac{\cos\theta_0 - 1}{2(1-\cos\theta_0)} + \frac{\sin\theta_0}{2(1-\cos\theta_0)} i = \tfrac{1}{2}[-1+\lambda i].$$

Consequently (14) becomes

$$(15)\quad P_n\big(f(z_0);\,\alpha,\,\beta\big) = C(\lambda i)^\alpha\big(1-m+(1+m+n)\lambda i\big)^\beta = C\,|\lambda|^\alpha\big((1-m)^2+(1+m+n)^2\big)^{\beta/2}\cos(\alpha\theta_1+\beta\theta_2),$$

where $\theta_1 = \arg\lambda i$, $\theta_2 = \arg\big(1-m+(1+m+n)\lambda i\big)$.

Case 1. $\lambda > 0$, $\theta_1 = \pi/2$, and since $1-m \leqslant 0$, $\pi/2 \leqslant \theta_2 \leqslant \pi$.

(i) If $(\alpha,\beta)\in G_1$, then

$$(4k+1)\pi/2 \leqslant (\alpha+\beta)\pi/2 \leqslant \alpha\theta_1+\beta\theta_2 \leqslant (\alpha+2\beta)\pi/2 \leqslant (4k+3)\pi/2.$$

Hence $\cos(\alpha\theta_1+\beta\theta_2)\leqslant 0$. This shows that $\operatorname{Re}\big(f(z_0);\,\alpha,\,\beta\big)\leqslant 0$ which contradicts $f\in S_n(\alpha,\beta)$.

(ii) Similarly if $(\alpha,\beta)\in G_2$, then

$$(4k+1)\pi/2 \leqslant (\alpha+2\beta)\pi/2 \leqslant \alpha\theta_1+\beta\theta_2 \leqslant (\alpha+\beta)\pi/2 \leqslant (4k+3)\pi/2$$

which leads to same contradiction.

Case 2. $\lambda < 0$. Let $\theta_3 = \arg\lambda$, $\theta_4 = \arg\big(1-m+(1+m+n)\lambda i\big)$. Then $\theta_3 = -\theta_1 = -\pi/2$, $\theta_4 = -\theta_3$, $\cos(\alpha\theta_3+\beta\theta_4) = \cos(\alpha\theta_1+\beta\theta_2)\leqslant 0$, contradiction.

This completes the proof of Theorem 1.

Remark 1. Since $S_n(0,1) = K_{n+1}$, Theorem 1 shows the basic inclusion relationship of Ruscheweyh $K_{n+1}\subset K_n$. Also $S_n(\alpha,\beta)\subset S_n(1,0)$, $(\alpha,\beta)\in G$. We will generalize this latter relation in the next theorem. The set G is given by (11).

THEOREM 2. *$S_n(\alpha,\beta)\subset S_n\big((\alpha-1)t+1,\,\beta t\big)$, $0\leqslant t\leqslant 1$ and $(\alpha,\beta)\in G$.*

Proof. Let $f\in S_n(\alpha,\beta)$, and

$$(16)\qquad \left(\frac{D^{n+1}f(z)}{D^nf(z)}-\frac{1}{2}\right)^\alpha\left(\frac{D^{n+2}f(z)}{D^{n+1}f(z)}-\frac{1}{2}\right)^\beta = q_n(z).$$

Then $\operatorname{Re}p_n(z) > 0$, $z\in E$, $(\alpha,\beta)\in G$. Also by Theorem 1,

$$(17)\qquad \frac{D^{n+1}f(z)}{D^nf(z)}-\frac{1}{2} = p_n(z),$$

where $\operatorname{Re}q_n(z) > 0$ for $z\in E$, $n\in N_0$. It follows from (16) and (17) that

$$\left(\frac{D^{n+1}f(z)}{D^nf(z)}-\frac{1}{2}\right)^{(\alpha-1)t+1}\left(\frac{D^{n+2}f(z)}{D^{n+1}f(z)}-\frac{1}{2}\right)^{\beta t} = \big(p_n(z)\big)^{1-t}\big(q_n(z)\big)^t = p(z).$$

Here $p(0) = 1$ and

$$|\arg p(z)| \leqslant (1-t)\,|\arg p_n(z)| + t\,|\arg q_n(z)| \leqslant \pi/2,$$

which shows $\operatorname{Re}p(z) > 0$. This completes the proof of Theorem 2.

4. Ratios of Ruscheweyh derivatives. In [2] Burdick and Merkes obtained sharp bounds on $\alpha > 0$ and $\beta > 0$ such that

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right)^{\alpha} > 0 \quad \text{and} \quad \operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right)^{\beta} > 0$$

for $z \in E$ and f, g varies in the classes K and S^* (starlike). In this section certain generalizations and extensions of these results which involve the Ruscheweyh derivatives will be obtained.

THEOREM 3. *Let*

$$h_n(z) = \frac{D^n f(z)}{D^n g(z)}, \quad n \in N_0. \tag{18}$$

If f and g belong to K_n, then

$$\operatorname{Re}\{h_n^{\alpha}(z) h_{n+1}^{\beta}(z)\} > 0, \tag{19}$$

$z \in E$ and $\alpha \geqslant 0$, $\beta \geqslant 0$ satisfying

$$2(n+1)\alpha + 2(n+2)\beta = 1. \tag{20}$$

Here $h_n^{\alpha}(0) = h_{n+1}^{\beta}(0) = 1$. The result is sharp.

Proof. Using Lemma 3 and (15) we have, when α and β satisfying (20),

$$\begin{aligned}|\arg h_n^{\alpha}(z) h_{n+1}^{\beta}(z)| &\leqslant \alpha|\arg D^n f(z) - \arg D^n g(z)| + \\ &\quad + \beta|\arg D^{n+1} f(z) - \arg D^{n+1} g(z)| \\ &= \alpha\left|\arg\frac{D^n f(z)}{z} - \arg\frac{D^n g(z)}{z}\right| + \\ &\quad + \beta\left|\arg\frac{D^{n+1} f(z)}{z} - \arg\frac{D^{n+1} g(z)}{z}\right| \\ &\leqslant 2\alpha(n+1)\sin^{-1} r + 2\beta(n+2)\sin^{-1} r \\ &= \sin^{-1} r < \pi/2.\end{aligned}$$

From this inequality follows (19).

To show sharpness of results, let

$$f(z) = \frac{z}{1-z}, \quad g(z) = \frac{z}{1+e^{it}z},$$

$-\pi < t \leqslant \pi$. Using (3) we easily compute

$$D^k f(z) = \frac{z}{(1-z)^{k+1}} * \frac{z}{1-z} = \frac{z}{(1-z)^{k+1}}, \quad D^k g(z) = \frac{z}{(1+e^{it}z)^{k+1}},$$

and hence

$$h_n^{\alpha}(z) h_{n+1}^{\beta}(z) = \left(\frac{1+ze^{it}}{1-z}\right)^{(n+1)\alpha+(n+2)\beta}.$$

Now since $\frac{1+ze^{it}}{1-z}$ maps the unit disc onto the half plane bounded by the line through the origin with angle of inclination $=\frac{t+\pi}{2}$, any choice of $\alpha \geqslant 0$, $\beta \geqslant 0$, satisfying $\alpha(n+1)+\beta(n+2) > \frac{1}{2}$, there exists a choice of t, $-\pi < t < \pi$ for which

$$\operatorname{Re}\{h_n^\alpha(z) h_{n+1}^\beta(z)\} < 0,$$

for some $z \in E$. Thus (20) cannot be improved.

COROLLARY. *If f and g are convex in E, then*

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right)^\alpha \left(\frac{f'(z)}{g'(z)}\right)^\beta > 0$$

for $\alpha \geqslant 0$, $\beta \geqslant 0$ satisfying the relation

$$2\alpha + 4\beta = 1.$$

Proof. Since $K \subset S^*_{1/2}$, the Corollary follows from Theorem 3 when $n = 0$.

Remark 2. For $\alpha = 0$, $\beta > 0$, and $\beta = 0$, $\alpha > 0$, our Corollary reduces to [2], Theorem 1.

Since

$$\arg \frac{f'(z)}{g'(z)} = \arg \frac{zf'(z)}{f(z)} - \arg \frac{zg'(z)}{g(z)} + \arg \frac{f(z)}{g(z)},$$

then we can easily show the following theorem.

THEOREM 4. *If f and g are starlike in E, then*

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right)^\alpha \left(\frac{f'(z)}{g'(z)}\right)^\beta > 0$$

for $z \in E$, $\alpha \geqslant 0$, $\beta \geqslant 0$ and when

$$4\alpha + 6\beta = 1.$$

This result is sharp.

The sharpness can be established by

$$f(z) = \frac{z}{(1-z)^2}, \quad g(z) = \frac{z}{(1-e^{it}z)^2} \quad -\pi < t \leqslant \pi.$$

Remark 3. For $\alpha = 0$, $\beta > 0$ and $\beta = 0$, $\alpha > 0$, Theorem 4 reduces to [2], Theorem 2.

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