ANNALES
POLONICI MATHEMATICI
XVI (1965)

Asymptotic properties of solutions of some integral equations and second order differential equations

by Z. Polniakowski (Poznań)

The purpose of this paper is to investigate asymptotic properties (as $x\to\infty$) of solutions of some Volterra integral equations (section 1) and some second order differential equations (section 2). The second order differential equation which appears in 2.1 is obtained by a suitable transformation of the integral equation of the following type:

$$(*)\qquad y = f(x)+\int\limits_a^x \left\{\frac{g'(t)}{g(x)}\varphi(t)-\psi(t)\right\}y(t)\,dt \qquad (|a|<\infty \text{ or } a=\infty).$$

In section 2 (Theorem 3) we give sufficient conditions for the existence of a bounded (resp. convergent) solution of the differential equation

$$(**)\qquad a_2(x)y''+a_1(x)y'+a_0(x)y = b(x) \qquad (x\geqslant x_0).$$

We also prove an analogous Theorem 5 about the asymptotic behaviour of the function $y(x)\exp\int\limits_{x_0}^{x_1} a_0(t)\,dt$, where $y(x)$ is a solution of the differential equation (**) for $a_1(x)=1$.

The theorems of section 2 follow from a theorem concerning the asymptotic properties of the solutions of the integral equation (*) (Theorem 2). Theorem 1 is a generalization of Theorem 2.

In the proofs we make use of l'Hopital's rule for complex-valued functions of the real variable. We shall use that rule in the form given by Theorem C of [1], p. 20. We also give the criterion (1.2) which is a sufficient condition for the function in the denominator (in the formulation of l'Hopital's rule) to have the property H (see below), which makes possible the application of this rule.

We consider in this paper complex-valued functions of one and two real variables. Integration is understood in the sense of Riemann.

1. We shall say that *the function* $g(x)$ *has the property* H *with the constant* K ($\geqslant 1$) *at the point* $x = \xi$ ($|\xi| \leqslant \infty$) if it is defined and differentiable in some neighbourhood I of ξ, except the point ξ at most, and if there exists a constant K such that for each pair of points $x, x_0 \in I$ we have

a) $\lim\limits_{x\to\xi} |g(x)| = \infty$ and $\Big|\int\limits_{x_0}^{x} |g'(t)|dt\Big| \leqslant K|g(x)|$ ($x_0 < x < \xi$ or $\xi < x < x_0$),

b) $\lim\limits_{x\to\xi} g(x) = 0$ and $\Big|\int\limits_{x}^{\xi} |g'(t)|dt\Big| \leqslant K|g(x)|$ ($x \neq \xi$).

If $x, x_0 < \xi$, we shall say that the function $g(x)$ *has the left-side property* H; if $x, x_0 > \xi$, we shall say that it has the *right-side property* H.

1.1. *It is easy to prove that if* $|g(x)|$ *has the property* H *with the constant* K *at* ξ, *and if* $\left|\dfrac{g'(x)}{|g(x)|'}\right| \leqslant M$ *in the neighbourhood of* ξ, *then the function* $g(x)$ *has the property* H *at* ξ *with the constant* $K_1 = KM$.

1.2. *Suppose that for some* $X < \xi$ *the following conditions are satisfied in the interval* $\langle X, \xi)$:

1) $h(x)$ *is continuous and* $h(x) \neq 0$,

2) *there exists a constant* N *such that we have* $|h(x)| \leqslant N|\mathrm{re}\, h(x)|$,

3) $\int\limits_{x_0}^{\xi} |h(x)|\,dx = \infty$ ($x_0 \in \langle X, \xi)$),

4) *the function* $g(x)$ *has a continuous derivative and* $g(x) \neq 0$,

5) $\dfrac{g'(x)}{g(x)} \sim h(x)$ ($x \uparrow \xi$).

Then for every real $p \neq 0$ *and every* $K > N$ *the function* $g^p(x)$ (¹) *has at the point* ξ *the left-side property* H (*with the constant* K). *Furthermore the function* $|g(x)|$ *is monotone in* $\langle x_1, \xi)$ *for some* $x_1 \in \langle X, \xi)$.

It suffices to prove 1.2 for $p = 1$. Let us observe that from the equality

$$\mathrm{re}\frac{g'}{g} = \mathrm{re}\, h\,\mathrm{re}\frac{g'}{gh} - \mathrm{im}\, h\,\mathrm{im}\frac{g'}{gh} \qquad (X \leqslant x < \xi),$$

it follows, in virtue of 2) and of the equality $\lim\limits_{x\to\xi} \mathrm{im}\dfrac{g'}{gh} = 0$, that

$$\frac{|g(x)|'}{|g(x)|} = \mathrm{re}\frac{g'(x)}{g(x)} \sim \mathrm{re}\, h(x) \qquad (x \uparrow \xi).$$

Next, from 1) and 2) it follows that we always have $\mathrm{re}\, h(x) > 0$ or $\mathrm{re}\, h(x) < 0$ in $\langle X, \xi)$.

(¹) In the definition of the property H only the functions $|g(x)|^p$, $|g(x)|^{p-1}$ and $|g'(x)|$ occur in this case.

a) In the case of $\mathrm{re}\,h(x) > 0$ we deduce from the last relation that for some $x_1 \in \langle X, \xi)$ in the interval $\langle x_1, \xi)$ the following inequalities hold:

$$|g(x)|' \geqslant \tfrac{1}{2}|g(x)|\,\mathrm{re}\,h(x) > 0\,,$$

$$\ln|g(x)| - \ln|g(x_1)| = \int_{x_1}^{x} \frac{|g(t)|'}{|g(t)|}\,dt \geqslant \frac{1}{2}\mathrm{re}\int_{x_1}^{x} h(t)\,dt\,.$$

In virtue of 2) and 3) we get $\lim_{x\uparrow\xi}|g(x)| = \infty$. Now the proof, for this case, follows from the relation

$$\frac{|g'(x)|}{|g(x)|'} \sim \frac{|h(x)|}{\mathrm{re}\,h(x)} \qquad (x\uparrow\xi),$$

because $|g(x)|$ is monotonic and because 2) and 1.1 are satisfied.

b) In the case of $\mathrm{re}\,h(x) < 0$ we infer as in case a) that for some $x_2 \in \langle X, \xi)$ in the interval $\langle x_2, \xi)$ the inequalities $|g(x)|' \leqslant \frac{1}{2}|g(x)|\,\mathrm{re}\,h(x) < 0$ and the equality $\lim_{x\uparrow\xi} g(x) = 0$ are satisfied. In addition we have $|g'(x)| \leqslant 2|g(x)h(x)|$ for $x \in \langle x_2, \xi)$. We obtain from this for $x \in \langle x_2, \xi)$:

$$|g(x)| \leqslant |g(x_2)|\,e^{-u(x)}, \qquad \text{where} \qquad u(x) = -\tfrac{1}{2}\mathrm{re}\int_{x_2}^{x} h(t)\,dt \uparrow \infty \qquad (x\uparrow\xi)\,.$$

For every pair x_3, x_4 $(x_2 \leqslant x_3 \leqslant x_4 < \xi)$ we have

$$\int_{x_3}^{x_4} |g'(x)|\,dx \leqslant 2\int_{x_3}^{x_4} |g(x)h(x)|\,dx \leqslant -2N|g(x_2)|\int_{x_3}^{x_4} e^{-u(x)}\mathrm{re}\,h(x)\,dx$$
$$= 4N|g(x_2)|\int_{u(x_3)}^{u(x_4)} e^{-t}\,dt\,.$$

This implies the convergence of the integral $\int_{x_2}^{\xi} |g'(x)|\,dx$. Now, we complete the proof as in case a).

Remark. If the conditions of 1.2 are satisfied in the interval $(\xi, X\rangle$ with $X > \xi$, we apply 1.2 replacing x by $-x$.

1.3. *Suppose that*

1) *the functions* $f(x)$ *and* $K(x, t)$ *are defined for* $x \geqslant a$, $t \geqslant a$,

2) $\overline{\lim}_{x\to\infty} |f(x)| = M < \infty$,

3) $\overline{\lim}_{x\to\infty} \int_a^{\infty} |K(x, t)|\,dt = \mu < 1$ *and* $\lim_{x\to\infty} \int_a^{t_0} |K(x, t)|\,dt = 0$ *for each* $t_0 \geqslant a$,

4) *the integral equation*

$$y(x) = f(x) + \int_a^{\infty} K(x, t)y(t)dt \qquad (x \geqslant a)$$

has a solution $\bar{y}(x)$ *such that* $|\bar{y}(x)| \leqslant L$ *for* $x \geqslant a$.

Then the following inequality holds:

$$\overline{\lim_{x\to\infty}} |\bar{y}(x)| \leqslant \frac{\mu}{1-\mu}.$$

For given $\varepsilon \in (0, 1-\mu)$ we choose $x_0 \geqslant a$ and then $x_1 \geqslant x_0$, so that

$$\int_a^{x_0} |K(x, t)|dt \leqslant \varepsilon, \qquad \int_{x_0}^{\infty} |K(x, t)|dt \leqslant \mu + \varepsilon,$$

$$|f(x)| \leqslant M + \varepsilon \quad \text{and} \quad |\bar{y}(x)| \leqslant L_1 + \varepsilon \quad (x \geqslant x_1),$$

where $L_1 = \overline{\lim_{x\to\infty}} |\bar{y}(x)|$. We obtain

$$\begin{aligned} |\bar{y}(x)| &\leqslant M + \varepsilon + L\int_a^{x_0} |K(x, t)|dt + (L_1+\varepsilon)\int_{x_0}^{\infty} |K(x, t)|dt \\ &\leqslant M + \varepsilon + \varepsilon L + (L_1+\varepsilon)(\mu+\varepsilon), \end{aligned}$$

$$L_1 \leqslant \frac{M + \varepsilon(1 + L + \mu + \varepsilon)}{1-\mu-\varepsilon}.$$

THEOREM 1. *Let* $f(t)$, $F(t)$ *and* $\psi(t)$ *be defined for* $t \geqslant x_0$ *and let* $N(x, t)$ *be defined for* $t \geqslant x \geqslant x_0$. *Suppose that for* $t \geqslant x \geqslant x_0$

1) *there exist integrals* $\int_x^t f(s)ds$, $\int_x^t \psi(s)ds$, $\int_x^t N(x, s)ds$, $\int_x^t N(s, t)ds$ *and*

$$\int_x^t |N(x, s)|\,|N(s, t)|^a ds \leqslant \lambda |N(x, t)|^a$$

with some $a \in \langle 0, 1\rangle$ *and fixed* $\lambda < 1$,

2) *we have* $|N(x, t)| \leqslant F(t)$ *and* $F(t)$ *is almost uniformly bounded,*

3) $\int_x^{\infty} |\psi(t)|dt < \infty$, $\psi(t)$ *is almost uniformly bounded,*

furthermore

4) $\lim\limits_{x\to\infty} \sup\limits_{x_0 \leqslant \xi \leqslant x} \int_x^{\infty} F^{1-a}(t)|N(\xi, t)|^a dt = 0$,

5a) $\overline{\lim_{x\to\infty}} |f(x)| = M < \infty$, *or*

5b) $\lim\limits_{x\to\infty} f(x) = s$ $(|s| < \infty)$.

Then the integral equation

$$y(x) = f(x) + \int_x^\infty K_0(x, t)\, y(t)\, dt, \tag{1}$$

where $K_0(x, t) = N(x, t) + \psi(t)$ *for* $t \geq x \geq x_0$, *has for large* x ($\geq x_0$) *exactly one solution* $\bar{y}(x)$ *bounded for* $x \to \infty$. *We have* $\overline{\lim_{x\to\infty}} |\bar{y}(x)| = M$ *in case* 5a), *resp.* $\lim_{x\to\infty} \bar{y}(x) = s$ *in case* 5b). *If* $f(x)$ *is continuous and* $N(x, t)$ *is an almost uniformly continuous function of the variable* x *for* $t \geq x \geq x_0$, *then* $\bar{y}(x)$ *is continuous for large* x.

Proof. We choose a number a satisfying the condition $\lambda < a < 1$. Then with some $x_1 \geq x_0$ there exists an integral $\int_x^t F^{1-a}(s)|N(x, s)|^a ds$ for $t \geq x \geq x_1$ and we have $\int_x^\infty |K_0(x, t)|\, dt \leq a$ for $x \geq x_1$.

Let $K_n(x, t) = \int_x^t K_0(x, s) K_{n-1}(s, t)\, ds$ for $t \geq x \geq x_0$ and $n = 1, 2, \ldots$ We shall prove by induction the inequality

$$|K_n(x, t)| \leq a^n F^{1-a}(t)|N(x, t)|^a + n a^{n-1} \psi_1(x, t) + a^n |\psi(t)| \tag{2}$$

for $t \geq x \geq x_1$ and $n = 0, 1, 2, \ldots$, where $\psi_1(x, t) = F^{1-a}(t) \int_x^t |N(s, t)|^a |\psi(s)|\, ds$.

We immediately verify that (2) is true for $n = 0$. Suppose now that it is true for the index $n-1$ ($n \geq 1$). Then, observing that $\psi_1(x, t)$ is a decreasing function of the variable x for $t \geq x \geq x_1$, we have

$$\begin{aligned} &|K_n(x, t)| \\ &\leq \int_x^t |N(x, s) + \psi(s)|\, \{a^{n-1} F^{1-a}(t)|N(s, t)|^a + (n-1) a^{n-2} \psi_1(s, t) + a^{n-1}|\psi(t)|\}\, ds \\ &\leq a^n F^{1-a}(t)|N(x, t)|^a + a^{n-1} F^{1-a}(t) \int_x^t |N(s, t)|^a |\psi(s)|\, ds + \\ &\qquad + \{(n-1) a^{n-2} \psi_1(x, t) + a^{n-1}|\psi(t)|\} \int_x^t |N(x, s) + \psi(s)|\, ds. \end{aligned}$$

Hence follows (2).

Therefore the series $\sum_{n=0}^\infty K_n(x, t)$ is almost uniformly convergent for $t \geq x \geq x_1$. Taking $\sum_{n=0}^\infty K_n(x, t) = R(x, t)$ we obtain from (2) for $t \geq x \geq x_1$

$$|R(x, t)| \leq \frac{1}{1-a} F^{1-a}(t)|N(x, t)|^a + \frac{1}{(1-a)^2} \psi_1(x, t) + \frac{1}{1-a} |\psi(t)|.$$

We have

$$\overline{\lim_{x\to\infty}} \int_x^\infty \psi_1(x, t)\,dt = \overline{\lim_{x\to\infty}} \int_x^\infty F^{1-a}(t) \int_x^t |N(s, t)|^a |\psi(s)|\,ds\,dt$$

$$= \overline{\lim_{x\to\infty}} \int_x^\infty |\psi(s)| \int_s^\infty F^{1-a}(t)|N(s, t)|^a\,dt\,ds$$

$$\leqslant \lim_{x\to\infty} \int_x^\infty F^{1-a}(t)|N(x, t)|^a dt \cdot \lim_{x\to\infty} \int_x^\infty |\psi(s)|\,ds = 0\,.$$

We choose $x_2 \geqslant x_1$ so that the functions $\int_x^\infty F^{1-a}(t)|N(x, t)|^a dt$, $\int_x^\infty \psi_1(x, t)\,dt$ and $f(x)$ are bounded for $x \geqslant x_2$ and we find that the integral $\int_x^\infty |R(x, t)|\,dt$ is convergent and uniformly bounded for $x \geqslant x_2$. Then the functions $J(x) = \int_x^\infty R(x, t)\,f(t)\,dt$ and $\bar{y}(x) = f(x) + J(x)$ remain bounded for $x \geqslant x_2$.

We shall prove the uniform convergence of the integral $\int_t^\infty R(t_0, s)f(s)\,ds$ for $t_1 \leqslant t_0 \leqslant t_2$, $t_1 \geqslant x_2$. We have for $t \geqslant x_2$

$$\int_t^\infty |R(t_0, s)f(s)|\,ds \leqslant N\Big\{\frac{1}{1-a}\int_t^\infty F^{1-a}(s)|N(t_0, s)|^a ds +$$

$$+\frac{1}{(1-a)^2}\int_t^\infty F^{1-a}(s)\int_{t_0}^s |N(u, s)|^a|\psi(u)|\,du\,ds + \frac{1}{1-a}\int_t^\infty |\psi(s)|\,ds\Big\},$$

where $N = \sup_{x \geqslant x_2} |f(x)|$.

For given $\varepsilon > 0$ we choose $x_3 \geqslant t_2$ so that for $t \geqslant x_3$ we get

$$\sup_{t_1 \leqslant u \leqslant t} \int_t^\infty F^{1-a}(s)|N(u, s)|^a ds \leqslant \varepsilon \quad \text{and} \quad \int_t^\infty |\psi(u)|\,du \leqslant \varepsilon\,.$$

We have for $t \geqslant x_3$, $t_1 \leqslant t_0 \leqslant t_2$:

$$\int_t^\infty F^{1-a}(s)\int_{t_0}^s |N(u, s)|^a|\psi(u)|\,du\,ds = \int_t^\infty \int_{t_0}^t + \int_t^\infty \int_t^s$$

$$= \int_{t_0}^t |\psi(u)| \int_t^\infty F^{1-a}(s)|N(u, s)|^a ds\,du +$$

$$+\int_t^\infty |\psi(u)| \int_u^\infty F^{1-a}(s)|N(u, s)|^a ds\,du \leqslant \varepsilon A + \varepsilon^2,$$

where $A = \int_{t_1}^\infty |\psi(u)|\,du$.

Finally, for $t \geqslant x_3$, $t_1 \leqslant t_0 \leqslant t_2$ we obtain:

$$\int_t^\infty |R(t_0, s) f(s)|\, ds \leqslant \frac{\varepsilon N}{1-a} + \frac{\varepsilon A N}{(1-a)^2} + \frac{\varepsilon^2 N}{(1-a)^2} + \frac{\varepsilon N}{1-a}.$$

It follows that $J(x)$ is integrable in every finite interval $\langle a, b\rangle$ $(x_2 \leqslant a < b < \infty)$. Since the integral $\int_x^\infty |K_0(x,t)| \int_t^\infty |R(t,s)|\, ds\, dt$ converges for $x \geqslant x_2$, we have for $x \geqslant x_2$:

$$\int_x^\infty K_0(x,t) J(t)\, dt = \int_x^\infty K_0(x,t) \int_t^\infty R(t,s) f(s)\, ds\, dt$$

$$= \int_x^\infty f(s) \int_x^s K_0(x,t) R(t,s)\, dt\, ds = \int_x^\infty f(s) \sum_{n=0}^\infty K_{n+1}(x,s)\, ds$$

and

$$\int_x^\infty K_0(x,t)\bar{y}(t)\, dt = \int_x^\infty K_0(x,t)\{f(t) + J(t)\}\, dt = J(x).$$

Hence it follows that $\bar{y}(x)$ satisfies (1) for $x \geqslant x_2$. Next, by (2), the equality $\lim_{n\to\infty} \int_x^\infty |K_n(x,t)|\, dt = 0$ holds for $x \geqslant x_2$. Since every solution $y(x)$ of (1) for $f(x) = 0$ satisfies the relation $y(x) = \int_x^\infty K_n(x,t) y(t)\, dt$, we find that the unique solution of (1) for $x \geqslant x_2$, bounded for $x \to \infty$, is in this case the function $y(x) = 0$. We infer hence that in the general case there exists for $x \geqslant x_2$ exactly one solution of (1) bounded for $x \to \infty$.

We have $\bar{y}(x) - f(x) = \int_x^\infty K_0(x,t)\bar{y}(t)\, dt \to 0$ with $x \to \infty$ by 4). It follows that $\overline{\lim}_{x\to\infty} |\bar{y}(x)| = M$ in case 5a), and $\lim_{x\to\infty} \bar{y}(x) = s$ in case 5b).

If $f(x)$ is continuous and $N(x,t)$ is an almost uniformly continuous function of the variable x for $t \geqslant x \geqslant x_0$, then it is easy to prove by 4) that the functions $\int_x^\infty K_0(x,t)\bar{y}(t)\, dt$ and $\bar{y}(x)$ are continuous for $x \geqslant x_2$.

THEOREM 1′. *Let $f(t)$, $F(t)$ and $\psi(t)$ be defined and almost uniformly bounded for $t \geqslant x_0$ and let $N(x,t)$ be defined for $x \geqslant t \geqslant x_0$. Suppose that*

1) *there exist integrals* $\int_t^x f(s)\, ds$, $\int_t^x \psi(s)\, ds$, $\int_t^x N(x,s)\, ds$, $\int_t^x N(s,t)\, ds$ *and that*

$$\int_t^x |N(x,s)||N(s,t)|^a\, ds \leqslant \lambda |N(x,t)|^a$$

with some $a \in \langle 0,1\rangle$ *and some* $\lambda < 1$, *for* $x \geqslant t \geqslant x_0$,

2) $|N(x,t)| \leqslant F(t)$ $(x \geqslant t \geqslant x_0)$,

3) $\overline{\lim}_{x\to\infty} \int_{x_0}^{x} F^{1-a}(t)|N(x,t)|^a dt < \infty$ *and* $\overline{\lim}_{x\to\infty} \int_{x_0}^{x} |N(x,t)|\,dt + \int_{x_0}^{\infty} |\psi(t)|\,dt = \mu < 1$,

4a) $\overline{\lim}_{x\to\infty} |f(x)| = M < \infty$, *or*

4b) $\lim_{x\to\infty} f(x) = s$ $(|s| < \infty)$, $\lim_{x\to\infty} \int_{x_0}^{x} N(x,t)\,dt = \sigma$ *and* $\lim_{x\to\infty} \int_{x_0}^{t_0} |N(x,t)|\,dt = 0$ *for fixed* $t_0 \geqslant x_0$.

Then the unique solution $\bar{y}(x)$, *bounded for finite* $x \geqslant x_0$, *of the integral equation*

$$y(x) = f(x) + \int_{x_0}^{x} K_0(x,t)y(t)\,dt \qquad (x \geqslant x_0)\,, \tag{3}$$

where $K_0(x,t) = N(x,t) + \psi(t)$, *remains bounded for* $x \to \infty$. *In case* 4b) *it is convergent for* $x \to \infty$. *If* $f(x)$ *is continuous and* $N(x,t)$ *is a continuous function of the variable* x *for* $x \geqslant t \geqslant x_0$, *then* $\bar{y}(x)$ *is continuous for* $x \geqslant x_0$.

Proof. Let

$$K_n(x,t) = \int_{t}^{x} K_0(x,s)K_{n-1}(s,t)\,ds \qquad \text{for} \qquad x \geqslant t \geqslant x_0 \text{ and } n = 1, 2, \ldots$$

Now, we proceed as in the proof of Theorem 1: we define the functions $R(x,t) = \sum_{n=0}^{\infty} K_n(x,t)$, $J(x) = \int_{x_0}^{x} R(x,t)f(t)\,dt$ and $\bar{y}(x) = f(x) + J(x)$ for $x \geqslant t \geqslant x_0$ and state that $\bar{y}(x)$ satisfies (3). Next, we choose a number a satisfying the inequality $\max(\lambda, \mu) < a < 1$. Then with some $x_1 \geqslant x_0$ we have

$$\int_{x_0}^{x} |K_0(x,t)|\,dt \leqslant a \qquad \text{for} \qquad x \geqslant x_1\,.$$

We prove the inequality

$$|K_n(x,t)| \leqslant a^n F^{1-a}(t)|N(x,t)|^a + na^{n-1}\psi_1(x,t) + a^n|\psi(t)|$$

for $x \geqslant x_1$, $x \geqslant t \geqslant x_0$ and $n = 0, 1, 2, \ldots$ where

$$\psi_1(x,t) = F^{1-a}(t) \int_{t}^{x} |N(s,t)|^a |\psi(s)|\,ds$$

and then we get $\overline{\lim}_{x\to\infty} |\bar{y}(x)| < \infty$.

In case 4b) it is easy to verify that the function

$$y_1(x) = \bar{y}(x) - \frac{s_1}{1-\sigma}, \quad \text{where} \quad s_1 = s + \int_{x_0}^{\infty} \psi(x)\bar{y}(x)\,dx \ \ (x \geqslant x_0)$$

satisfies the integral equation

$$y_1(x) = f_1(x) + \int_{x_0}^{x} N(x,t)\,y_1(t)\,dt \qquad (x \geqslant x_0),$$

where

$$f_1(x) = f(x) + \int_{x_0}^{x} \psi(t)\bar{y}(t)\,dt + \frac{s_1}{1-\sigma}\Big(\int_{x_0}^{x} N(x,t)\,dt - 1\Big).$$

Since $\lim_{x\to\infty} f_1(x) = 0$, we obtain by 1.3 that $\lim_{x\to\infty} y_1(x) = 0$ and it follows that $\lim_{x\to\infty} y(x) = s_1$. The continuity of $\bar{y}(x)$ for $x \geqslant x_0$ follows as in the proof of Theorem 1.

THEOREM 2. *Suppose that*

1) *the function* $g^p(x)$ *has the property* H *at the point* $\xi = \infty$ *with the constant* K (cf. p. 170) *uniformly for* $p \in (0, 1\rangle$, $g(x) \neq 0$, $|g(x)|$ *is monotone and* $g'(x)$ *is almost uniformly bounded for* $x \geqslant x_0$,

2) $f(x)$, $\varphi(x)$ *and* $\psi(x)$ *are bounded and integrable in every finite interval* $\subset \langle x_0, \infty)$,

3a) $\overline{\lim_{x\to\infty}} |f(x)| = M < \infty$, *or*

3b) $\lim_{x\to\infty} f(x) = s$ $(|s| < \infty)$,

4) $\int_{x_0}^{\infty} |\psi(t)|\,dt < \infty$,

5) $\lim_{x\to\infty} \varphi(x) = 0$.

Let

$$K(x,t) = \frac{g'(t)}{g(x)}\varphi(t) + \psi(t) \qquad \textit{for} \qquad x \geqslant x_0,\ t \geqslant x_0.$$

Then in the case of $\lim_{x\to\infty} g(x) = 0$ *the integral equation*

$$y(x) = f(x) + \int_{x}^{\infty} K(x,t)\,y(t)\,dt \tag{4}$$

has for large x $(\geqslant x_0)$ *exactly one solution* $\bar{y}(x)$ *bounded for* $x \to \infty$. *We have* $\overline{\lim_{x\to\infty}} |\bar{y}(x)| = M$ *in case* 3a), *resp.* $\lim_{x\to\infty} \bar{y}(x) = s$ *in case* 3b).

In the case of $\lim_{x\to\infty}|g(x)| = \infty$ *the unique solution* $\bar{y}_1(x)$, *bounded for finite* x $(\geqslant x_0)$, *of the integral equation*

$$(5)\qquad y(x) = f(x) + \int_{x_1}^{x} K(x,t)y(t)dt \qquad (x \geqslant x_1)$$

($x_1 \geqslant x_0$ *is chosen so large that* $\int_{x_1}^{\infty}|\psi(t)|dt < 1$) *remains bounded for* $x\to\infty$. *In case* 3b) $\bar{y}_1(x)$ *is convergent as* $x\to\infty$.

If $f(x)$ *is continuous, then* $\bar{y}(x)$ *for large* x *and* $\bar{y}_1(x)$ *for* $x \geqslant x_1$ *are also continuous.*

Proof. In the case of $\lim_{x\to\infty} g(x) = 0$ we choose a fixed $a \in (0,1)$ and for given $\varepsilon > 0$ a small $\delta > 0$ such that the inequalities $K\delta/(1-a) < 1$ and $K\delta/a \leqslant \varepsilon$ are true. Next, we choose $x_2 \geqslant x_0$ such that $|\varphi(x)| \leqslant \delta$ holds for $x \geqslant x_2$ and that

$$p\int_{x}^{\infty} |g(t)|^{p-1}|g'(t)|dt \leqslant K|g(x)|^p$$

is satisfied for $x \geqslant x_2$ and every $p \in (0,1\rangle$. We obtain by 1) for $t \geqslant x \geqslant x_2$:

$$\int_{x}^{t} |N(x,s)||N(s,t)|^a ds$$

$$= |g'(t)\varphi(t)|^a \frac{1}{|g(x)|}\int_{x}^{t} |\varphi(s)||g(s)|^{-a}|g'(s)|ds \leqslant K\frac{\delta}{1-a}|N(x,t)|^a .$$

Then the inequality in hypothesis 1) of Theorem 1 is satisfied with $\lambda = K\delta/(1-a)$. Next, we state that hypothesis 2) of Theorem 1 is satisfied for

$$F(t) = \left|\frac{g'(t)}{g(t)}\varphi(t)\right| \quad \text{for} \quad t \geqslant x_2 .$$

We shall show that hypothesis 4) of Theorem 1 is also satisfied. We have

$$\sup_{x_0\leqslant\xi\leqslant x}\int_{x}^{\infty} F^{1-a}(t)|N(\xi,t)|^a dt = \sup_{x_0\leqslant\xi\leqslant x}\frac{1}{|g(\xi)|^a}\int_{x}^{\infty} |g(t)|^{a-1}|g'(t)\varphi(t)|dt$$

$$\leqslant \sup_{x_0\leqslant\xi\leqslant x}\frac{K\delta|g(x)|^a}{a|g(\xi)|^a} \leqslant \varepsilon \quad \text{for} \quad x \geqslant x_2 .$$

To prove this part of Theorem 2 we now use Theorem 1.

In the case of $\lim_{x\to\infty} |g(x)| = \infty$ we choose as above $x_2 \geqslant x_1$. As in the case of $\lim_{x\to\infty} g(x) = 0$, we prove the inequality in hypothesis 1), hypothesis 2) and the first hypothesis in 3) of Theorem 1'. Furthermore, we have

$$\overline{\lim_{x\to\infty}} \int_{x_1}^{x} |K(x,t)|\,dt \leqslant \lim_{x\to\infty} \frac{1}{|g(x)|} \int_{x_1}^{x} |g'(t)\varphi(t)|\,dt + \int_{x_1}^{\infty} |\psi(t)|\,dt = A\,,$$

and similarly

$$\lim_{x\to\infty} \int_{x_1}^{x} K(x,t)\,dt = A_1\,,$$

where $A = \int_{x_1}^{\infty} |\psi(t)|\,dt$, $A_1 = \int_{x_1}^{\infty} \psi(t)\,dt$. The second hypothesis in 3) and hypothesis 4b) of Theorem 1' are then satisfied for $x_0 = x_1$, $\mu = A$ and $\sigma = A_1$. To prove the second part of Theorem 2 we use Theorem 1'.

2. In this section we shall prove three theorems about asymptotic properties of integrals of some differential equations of the second order.

2.1. *Suppose that in the interval $\langle x_0, \bar{x})$, $\bar{x} \leqslant \infty$ there exist $A'''(x)$, $a_2''(x)$, $a_1''(x)$, $a_0'(x)$ and $B'(x)$, and that $A(x) \neq 0$, $a_2(x) \neq 0$. Moreover suppose that there exists a solution $\bar{y}(x)$ of the integral equation*

$$y(x) = f(x) + \int_{\xi}^{x} K(x,t)y(t)\,dt \qquad (x_0 \leqslant x \leqslant \bar{x})\,,$$

where $f(x) = c + \dfrac{c_1}{g(x)} + \dfrac{1}{g(x)} \int_{\xi}^{x} g'(t)B(t)\,dt$, $g(x) = \exp\left(\int_{a}^{x} \dfrac{dt}{A(t)}\right)$, $K(x,t) = \dfrac{g'(t)}{g(x)}\varphi(t) - \psi(t)$, $\varphi(x) = A'(x) + A(x)\left\{\psi(x) - \dfrac{a_1(x)}{a_2(x)}\right\} + 1$, $\psi(x) = A''(x) - \left[\dfrac{a_1(x)}{a_2(x)}A(x)\right]' + \dfrac{a_0(x)}{a_2(x)}A(x)$, *with* $\xi, a \in \langle x_0, \bar{x}\rangle$; c *and* c_1 *are constants.*

Then $\bar{y}(x)$ satisfies the differential equation

$$a_2 y'' + a_1 y' + a_0 y = b(x) \qquad (x_0 \leqslant x \leqslant \bar{x})\,, \tag{6}$$

where $b(x) = \dfrac{a_2(x)}{A(x)} B'(x)$.

To prove 2.1 we multiply by $g(x)$ the above integral equation and obtain by differentiation the differential equation of the second order:

$$Ay'' + (1 + A' - \varphi + A\psi)y' + [\psi - \varphi' + (A\psi)']y = \frac{A}{a_2} b\,,$$

in which we substitute the values for $\varphi(x)$ and $\psi(x)$.

THEOREM 3. *Suppose that for* $x \geqslant x_0$

1) *there exist continuous* A''', a_2'', a_1'', a_0' *and* b $(x \geqslant x_0)$,

2) $A \neq 0$, $a_2 \neq 0$ *and* $|A| \leqslant N|\mathrm{re}\,A|$ *with some constant* $N \geqslant 1$ $(x \geqslant x_0)$; *furthermore we have*

3) $\int\limits_{x_0}^{\infty} \frac{dx}{|A(x)|} = \infty$,

4) $\lim\limits_{x\to\infty} \varphi(x) = 0$,

5a) $\overline{\lim\limits_{x\to\infty}} \int\limits_{x_0}^{x} \left|\frac{A(t)b(t)}{a_2(t)}\right| dt = m < \infty$, *or*

5b) $\int\limits_{x_0}^{\infty} \frac{A(t)b(t)}{a_2(t)}\, dt = s$ $(|s| < \infty)$,

6) $\int\limits_{x_0}^{\infty} |\psi(x)|\, dx < \infty$, *where* $\varphi(x)$ *and* $\psi(x)$ *are defined as in* 2.1.

Then the differential equation (6) *for* $\bar{x} = \infty$ *has an integral* $\bar{y}(x)$ *bounded for* $x \to \infty$ *in case* 5a), *and convergent in case* 5b). *If in addition we have*

7a) $\underline{\lim\limits_{x\to\infty}} |A(x)| > 0$, $\overline{\lim\limits_{x\to\infty}} |\psi(x)| < \infty$,

then in case 5a) *we have* $\overline{\lim\limits_{x\to\infty}} |\bar{y}'(x)| < \infty$; *if*

7b) $\lim\limits_{x\to\infty} A(x) = L$ $(0 < |L| \leqslant \infty)$, $\lim\limits_{x\to\infty} \psi(x) = 0$,

then in case 5b) *we have* $\lim\limits_{x\to\infty} \bar{y}'(x) = 0$.

Under the hypothesis $\mathrm{re}\,A > 0$ *every integral of* (6) *and its derivative have analogous asymptotic properties as* $\bar{y}(x)$ *and* $\bar{y}'(x)$, *respectively.*

Proof. Let us observe that from 1) and 2) it follows that we always have $\mathrm{re}\,A < 0$ or $\mathrm{re}\,A > 0$ for $x \geqslant x_0$. In the case of $\mathrm{re}\,A < 0$ we consider the integral equation (4) with $f(x)$ and $g(x)$ defined as in 2.1 for $B(x) = \mathrm{const} + \int\limits_{x_0}^{x} \frac{A(t)b(t)}{a_2(t)}\, dt$ (with some $c, c_1 = 0$, $a = x_0$, $\xi = \infty$ and with $-\varphi(x)$ instead of $\varphi(x)$). We obtain after 1.2 the relation $\overline{\lim\limits_{x\to\infty}} |f(x)| < \infty$ in case 5a), and using l'Hopital's rule in the formulation of Theorem C ([2]) [1], p. 20, we find that $f(x)$ is convergent in case 5b). Then under our hypothesis, in virtue of Theorem 2, the integral equation (4) has for large x a continuous solution $\bar{y}(x)$ bounded as $x \to \infty$ in case 5a), and

([2]) In the formulation of this rule the hypothesis of continuity of $f'(x)$ and $g'(x)$ must be added.

convergent in case 5b). Multiplying by $g(x)$ the integral equation (4) we obtain for large x by differentiation:

$$(7)\qquad \bar{y}'(x) = \frac{1}{A(x)}\Big\{B(x)+[\varphi(x)-1]\bar{y}(x)+\int\limits_{x}^{\infty}\psi(t)\bar{y}(t)dt\Big\}-\psi(x)\bar{y}(x).$$

It follows that there exist $\bar{y}'(x)$ and $\bar{y}''(x)$, and by 2.1 the function $\bar{y}(x)$ satisfies for large x the differential equation (6) with $\bar{x}=\infty$. From (7) follow the asymptotic properties of $\bar{y}'(x)$ for $\operatorname{re} A<0$ under hypothesis 7a) or 7b).

In the case of $\operatorname{re} A>0$ it is easy to see in virtue of 2.1 that every integral of the differential equation (6) with $\bar{x}=\infty$ satisfies the integral equation (5) with $f(x)$ and $g(x)$ defined as in 2.1 for $\xi = a = x_1$ and for some c and c_1. It follows by Theorem 2 that in the case of $\operatorname{re} A>0$ every integral $y(x)$ of (6) (with $\bar{x}=\infty$) remains bounded as $x\to\infty$ in case 5a) and is convergent in case 5b). Under hypothesis 7a) or 7b) we prove the asymptotic properties of $y'(x)$ as in the case of $\operatorname{re} A<0$.

COROLLARY 1. *Under the hypothesis $a_2(x)=1$ the assertion of Theorem 3 is true if $y=A(x)$ is a solution of the adjoint differential equation*

$$y''-(a_1y)'+a_0y=0,$$

satisfying with $a_1(x)$, $a_0(x)$ and $b(x)$ the hypothesis of Theorem 3.

COROLLARY 2. *In the case of $A(x)=\text{const}$, $\operatorname{re} A\neq 0$ and $a_2(x)=1$ we obtain the following theorem: If*

1) *there exist continuous $a_1''(x)$, $a_0'(x)$ and $b(x)$ for $x\geqslant x_0$,*

2a) $\overline{\lim\limits_{x\to\infty}}\left|\int\limits_{x_0}^{x}b(t)dt\right|<\infty$, *or*

2b) $\int\limits_{x_0}^{\infty}b(x)dx=s\ \ (|s|<\infty)$,

3) $\lim\limits_{x\to\infty}a_1(x)=s_1\ \ (|s_1|<\infty,\ \operatorname{re}s_1\neq 0)$,

4) $\int\limits_{x_0}^{\infty}|a_0(x)-a_1(x)|dx<\infty$ *and* $\lim\limits_{x\to\infty}\{a_0(x)-a_1'(x)\}=0$,

then the differential equation

$$(8)\qquad y''+a_1(x)y'+a_0(x)y=b(x)\qquad (x\geqslant x_0)$$

has an integral $\bar{y}(x)$ such that $\bar{y}(x)$ and $\bar{y}'(x)$ remain bounded as $x\to\infty$ in case 2a) *and $\bar{y}(x)$ is convergent and $\lim\limits_{x\to\infty}\bar{y}'(x)=0$ in case* 2b). *Under the hypothesis $\operatorname{re}s_1>0$ every solution of* (8) *and its derivative have analogous asymptotic properties as $\bar{y}(x)$ and $\bar{y}'(x)$.*

In the case of $A(x) = a_2(x)$ and $a_1(x) = 1$ we obtain from Theorem 3 the following

THEOREM 4. *Suppose that*

1) $a_2'''(x)$, $a_0'(x)$ *and* $b(x)$ *are continuous for* $x \geqslant x_0$,

2a) $\overline{\lim_{x\to\infty}} \left| \int_{x_0}^{x} b(t)dt \right| < \infty$, *or*

2b) $\int_{x_0}^{\infty} b(x)dx = s$ $(|s| < \infty)$,

3) $a_2(x) \neq 0$ *and* $|a_2(x)| \leqslant N|\operatorname{re} a_2(x)|$ *for* $x \geqslant x_0$,

4) $\int_{x_0}^{\infty} |a_2''(x) + a_0(x)|dx < \infty$ *and* $\int_{x_0}^{\infty} \frac{dx}{|a_2(x)|} = \infty$,

5) $\lim_{x\to\infty} (a_2 a_2'' + a_2' + a_0 a_2) = 0$.

Then the differential equation

$$a_2(x)y'' + y' + a_0(x)y = b(x) \qquad (x \geqslant x_0) \tag{9}$$

has an integral $\bar{y}(x)$ *bounded as* $x\to\infty$ *in case* 2a) *and convergent in case* 2b).

If in addition we have

6a) $\underline{\lim_{x\to\infty}} |a_2(x)| > 0$ *and* $\overline{\lim_{x\to\infty}} |a_2''(x) + a_0(x)| < \infty$,

then in case 2a) *we have* $\overline{\lim_{x\to\infty}} |\bar{y}'(x)| < \infty$.

If

6b) $\lim_{x\to\infty} a_2(x) = L$ $(0 < |L| \leqslant \infty)$ *and* $\lim_{x\to\infty} \{a_2''(x) + a_0(x)\} = 0$,

then in case 2b) *we have* $\lim_{x\to\infty} \bar{y}'(x) = 0$.

Under the hypothesis $\operatorname{re} a_2(x) > 0$ *every integral of* (9) *and its derivative have analogous asymptotic properties for* $x\to\infty$ *as* $\bar{y}(x)$ *and* $\bar{y}'(x)$, *respectively.*

Let us remark that hypotheses 4) and 5) are fulfilled if

$$a_0(x) = O(x^{-1-\varepsilon}), \quad a_2(x) = O(x^{1-\varepsilon}), \quad a_2'(x) = O(x^{-\varepsilon}), \quad a_2''(x) = O(x^{-1-\varepsilon})$$

as $x\to\infty$, with some $\varepsilon > 0$.

THEOREM 5. *Suppose that*

1) *there exist continuous* $a_2'''(x)$, $a_0'(x)$ *and* $b(x)$ *for* $x \geqslant x_0$,

2) $a_2(x) \neq 0$ *and* $|a_2(x)| \leqslant N|\operatorname{re} a_2(x)|$ *for* $x \geqslant x_0$,

3a) $\overline{\lim_{x\to\infty}} \left| \int_{x_0}^{x} b(t) \left(\exp \int_{x_0}^{t} a_0(s)ds\right) dt \right| < \infty$, *or*

3b) $\int\limits_{x_0}^{\infty} b(t)\big(\exp\int\limits_{x_0}^{t} a_0(s)\,ds\big)\,dt = L \quad (|L| < \infty)$,

4) $a_0(x) = O(x^{\alpha})$, $a_0'(x) = O(x^{\alpha-1})$, $a_2(x) = O(x^{\beta})$, $a_2'(x) = O(x^{\beta-1})$ *and* $a_2''(x) = O(x^{\beta-2})$ $(x\to\infty)$, *for* $\operatorname{re}\beta < 1$, $\operatorname{re}(2\alpha+\beta) < -1$.

Then there exists an integral $\bar{y}(x)$ *of the differential equation* (9) *such that the function* $\bar{y}(x)\exp\int\limits_{x_0}^{x} a_0(t)\,dt$ *remains bounded for* $x\to\infty$ *in case* 3a), *and is convergent in case* 3b). *Under the hypothesis* $\operatorname{re} a_2(x) > 0$ *every integral of* (9) *has analogous asymptotic properties for* $x\to\infty$ *as* $\bar{y}(x)$.

Proof. Substituting into (9) $y(x) = z(x)\exp\big(-\int\limits_{x_0}^{x} a_0(t)\,dt\big)$ for $x \geqslant x_0$ we obtain the differential equation

$$(10) \qquad a_2 z'' + (1-2a_0a_2)z' + a_2(a_0^2 - a_0')z = b\exp\int\limits_{x_0}^{x} a_0(t)\,dt \qquad (x \geqslant x_0).$$

In this case the functions $\varphi(x)$ and $\psi(x)$ (see 2.1) have for $A(x) = a_2(x)$ the following form:

$$\varphi = a_2' + a_2(\psi + 2a_0), \qquad \psi = a_2'' + 2a_0a_2' + a_2(a_0^2 + a_0').$$

We get by 4): $\lim\limits_{x\to\infty}\varphi(x) = 0$ and $\psi(x) = O(x^{-1-\varepsilon})$ for $x\to\infty$ with some $\varepsilon > 0$. Using Theorem 3 for $1-2a_0a_2$, $a_2(a_0^2 - a_0')$ and $b\exp\int\limits_{x_0}^{x} a_0(t)\,dt$ instead of a_1, a_0 and b we find that there exists a solution $\bar{z}(x)$ of (10) bounded for $x\to\infty$ in case 3a) and convergent in case 3b). Under the hypothesis $\operatorname{re} a_2(x) > 0$ every integral of (10) has analogous asymptotic properties for $x\to\infty$ as $\bar{z}(x)$.

Reference

[1] Z. Polniakowski, *Polynomial Hausdorff transformations; I. Mercerian theorems*, Ann. Polon. Math. 5 (1958), pp. 1-24.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 7. 1. 1963