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Existence of differentiable solutions of a system of functional equations of first order

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Abstract. A theorem on the existence of a differentiable solution of the system of functional equations

$$\varphi(f(x)) = g(x, \varphi(x)),$$

where $\varphi \subset R \times R^N$ is the unknown function, is given under the hypothesis that the symetric difference of the sets $\Omega_{f(x)}$ and Γ_x (definition, Section 3) is of N-dimensional Lebesgue measure zero.

1. In the present note we are concerned with the system of functional equations

$$\varphi(f(x)) = g(x, \varphi(x)), \tag{1}$$

where $\varphi \subset R \times R^N$ is the unknown function, and the functions $g \subset R^{N+1} \times R^N$ and $f \subset R \times R$ are given. This note is connected with author's paper [3] dealing with the problem of the existence of solutions of system (1) which are of class C^r in an open interval (a, b), in the case where the symmetric difference of the sets Γ_x and $\Omega_{f(x)}$ (see the definition below in Section 3) is at most countable.

In the note we assume that this symmetric difference is of N-dimensional Lebesgue measure zero.

In the case $\Gamma_x \div \Omega_{f(x)} = \emptyset$ the theory of solution of equation (1) is known (cf. for example [1], [2], [4], p. 67–105).

2. The investigation of C^r-solutions of system (1) is based on a certain theorem which we are going to prove with the aid of the Sard theorem [6], [5]. To this end we make the following assumptions:

(i) The functions $u_n \colon F_n \to R^N$, $n = 1, \ldots$, are defined in sets F_n contained in $\langle 0, 1\rangle \times R^{q-1}$, where $q < N$.

(ii) $u_n \in C^1(F_n)$ for every $n = 1, 2, \ldots$ (1).

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(1) In the whole of this paper we understand the C^r-class of a function in the global sense.

(iii) There exists a sphere $S(0, \varrho)$ in R^N such that

$$u_n(F_n) \subset S(0, \varrho) \quad \text{for every } n = 1, 2, \ldots$$

We shall prove the following

THEOREM 1. *If hypotheses* (i)–(iii) *are fulfilled, then for every fixed positive integer $r > 0$, and for every two elements $a_0 \in S(0, \varrho)$, $a_1 \in S(0, \varrho)$ such that $a_0 \notin u_n[(\{0\} \times R^{q-1}) \cap F_n]$, $a_1 \notin u_n[(\{1\} \times R^{q-1}) \cap F_n]$, for every $n = 1, 2, \ldots$, and for any $l_0^k \in R^N$, $l_1^k \in R^N$, $k = 1, \ldots, r$, there exists a function u such that*

$$u\colon \langle 0, 1\rangle \to R^N, \quad u \in C^r(\langle 0, 1\rangle), \quad u(0) = a_0, \quad u(1) = a_1,$$
$$u^{(k)}(0) = l_0^k, \quad u^{(k)}(1) = l_1^k, \quad k = 1, \ldots, r,$$
$$u(\langle 0, 1\rangle) \subset S(0, \varrho),$$
$$u(x) \neq u_n(x, t) \quad \textit{for every } (x, t) \in F_n, \ n = 1, 2, \ldots$$

Proof. Let us take a function $v\colon \langle 0, 1\rangle \to R^N$, $v \in C^r(\langle 0, 1\rangle)$, such that: $v(0) = a_0$, $v(1) = a_1$, $v^{(k)}(0) = l_0^k$, $v^{(k)}(1) = l_1^k$, $|v(x)| \leqslant \varrho_0 = \max(|a_0|, |a_1|)$, for every $x \in \langle 0, 1\rangle$, and a function $\lambda\colon \langle 0, 1\rangle \to R$, $\lambda \in C^r(\langle 0, 1\rangle)$ such that: $\lambda(0) = \lambda(1) = 0$, $\lambda^{(k)}(0) = \lambda^{(k)}(1) = 0$, $k = 1, \ldots, r$, $0 < \lambda(x) \leqslant (\varrho - \varrho_0)/\varrho$ for every $x \in (0, 1)$.

For any $t \in S(0, \varrho)$ let u_t denote the function $u_t\colon \langle 0, 1\rangle \to R^N$ given by: $u_t(x) = v(x) + \lambda(x) t$. Denote by E the family of all functions u_t, $t \in S(0, \varrho)$. Then each $u_t \in E$ has all the properties asserted on u in the theorem except the last one.

The set E endowed with the usual supremum metric

$$d(u_t, u_{\tilde{t}}) = \sup_{x \in \langle 0, 1\rangle} \lambda(x) |t - \tilde{t}|$$

is a metric space homeomorphic with the sphere $S(0, \varrho)$. The homeomorphism h is given by $S(0, \varrho) \ni t \mapsto u_t \in E$.

We put

$$F_{n,s} = F_n \cap \left\langle \frac{1}{s}, 1 - \frac{1}{s} \right\rangle \times R^{q-1}, \quad n = 1, \ldots, \ s = 3, \ldots$$

and define the sets:

$$E_{n,s} = \{u_t \in E\colon \text{there exists } (x, y) \in F_{n,s} \text{ such that } u_t(x) = u_n(x, y)\}.$$

We shall prove that the set

$$E_Z = \bigcup_{s=3}^{\infty} \bigcup_{n=1}^{\infty} E_{n,s}$$

has no interior in the space E. It suffices to show that the set

$$Z = \bigcup_{s=3}^{\infty} \bigcup_{n=1}^{\infty} Z_{n,s},$$

where

$$Z_{n,s} = h^{-1}(E_{n,s}) = \{t \in S(0, \varrho) \colon u_t \in E_{n,s}\}$$

has no interior in the sphere $S(0, \varrho)$.

Writing

$$\gamma_{n,s}(x, y) = \frac{u_n(x, y) - v(x)}{\lambda(x)}, \quad (x, y) \in F_{n,s},$$

we observe that

$$(*) \qquad Z_{n,s} \subset \gamma_{n,s}(F_{n,s}).$$

The function $\gamma_{n,s}$ is of class C^1 in the set $F_{n,s} \subset R^q$, $q < N$, hence on account of the theorem of Sard, we have

$$m_N\big(\gamma_{n,s}(F_{n,s})\big) = 0,$$

where m_N denotes the N-dimensional Lebesgue measure.

Because of $(*)$ we have also $m_N(Z_{n,s}) = 0$ for every $n \geqslant 1$ and $s \geqslant 3$; thus $m_N(Z) = 0$, and the set Z has no interior in $S(0, \varrho)$. Consequently E_Z has no interior in the space E, so there exists a function $u \in E \setminus E_Z$. The function u has all the desired properties because it belongs to E and does not possess common values with any of the functions u_n (according to the definition of E_Z). This completes the proof.

Remark. Since the set $E \setminus E_Z$ is dense in the space E, there exist infinitely many functions fulfilling the assertion of the theorem.

3. Now we are going to formulate a theorem on C^r-solutions of system (1). For this purpose we impose some hypotheses regarding the given functions f and g.

Let

$$\Omega \subset R^{N+1}$$

be some set, and suppose that we are given a function

$$g \colon \Omega \to R^N.$$

For an arbitrary x we shall denote by Ω_x the x-section of the set Ω, i.e.

$$\Omega_x = \{y \colon (x, y) \in \Omega\}.$$

We assume that the interval $\langle a, b\rangle$ is contained in the set

$$\{x \colon \Omega_x \neq \emptyset\}$$

and we write

$$\Gamma_x = g(x, \Omega_x), \qquad \Gamma = \bigcup_{x \in \langle a, b\rangle} \{x\} \times \Gamma_x.$$

Thus we have $\Gamma_x \subset R^N$ and $\Gamma \subset R^{N+1}$.

Let r be a fixed positive integer. We assume that:

(I) $f\colon \langle a, b\rangle \to \langle a, b\rangle$, $f(a) = a$, $f(b) = b$, $f(x) > x$ for $x \in (a, b)$, $f \in C^r(\langle a, b\rangle)$ and $f'(x) > 0$ in $\langle a, b\rangle$.

(II) $g \in C^r(\Omega)$ ([2]), and for every $x \in \langle a, b\rangle$ the function $y \mapsto g(x, y)$ is invertible.

(III) $h \in C^r(\Gamma)$, where h denotes the function inverse to the function $y \mapsto g(x, y)$.

(IV) There exist sets $A_i \subset R^{q-1}$, $i = 1, 2, \ldots$, $B_j \subset R^{q-1}$, $j = 1, 2, \ldots$, where $q < N$, and functions

$$u_i\colon \langle a, b\rangle \times A_i \to R^N, \quad u_i \in C^1(\langle a, b\rangle \times A_i), \quad i = 1, 2, \ldots,$$
$$v_j\colon \langle a, b\rangle \times B_j \to R^N, \quad v_j \in C^1(\langle a, b\rangle \times B_j), \quad j = 1, 2, \ldots$$

such that

$$\Gamma_x - \Omega_{f(x)} = \bigcup_{i=1}^{\infty} u_i(x, A_i), \qquad \Omega_{f(x)} - \Gamma_x = \bigcup_{j=1}^{\infty} v_j(x, B_j).$$

(V) There exist a point $(x_0, \eta) \in \Omega$, $x_0 \in (a, b)$ such that $\eta = g(x_0, \eta)$ and a $\varrho_0 > 0$ such that

$$\langle x_0, f(x_0)\rangle \times S_0 \subset \Omega \cup \Omega \cup \Gamma,$$

where S_0 is the sphere:

$$S_0 = \{y \in R^N, |y - \eta| \leqslant \varrho_0\}.$$

We introduce functions $g_k \subset (\Omega \times R^{kN}) \times R^N$ by the recurrent formulae

$$(2)\quad \begin{aligned} g_1(x, y, y_1) &= [f'(x)]^{-1}\left[\frac{\partial g}{\partial x}(x, y) + \frac{\partial g}{\partial y}(x, y) y_1\right], \\ g_{k+1}(x, y, y_1, \ldots, y_{k+1}) &= [f'(x)]^{-1}\left[\frac{\partial g_k}{\partial x} + \frac{\partial g_k}{\partial y} y_1 + \ldots + \frac{\partial g_k}{\partial y_k} y_{k+1}\right], \end{aligned}$$
$$(x, y) \in \Omega,\ y_i \in R^N,\ i = 1, \ldots, k;\ k = 1, \ldots, r-1.$$

Similarly we define functions $h_k \subset (\Gamma \times R^{kN}) \times R^N$ by the relations

$$h_1(x, y, y_1) = \frac{\partial h}{\partial x}(x, y) + f'(x)\frac{\partial h}{\partial y}(x, y) y_1,$$
$$h_{k+1}(x, y, y_1, \ldots, y_{k+1}) = \frac{\partial h_k}{\partial x} + f'(x)\left[\frac{\partial h_k}{\partial y} y_1 + \ldots + \frac{\partial h_k}{\partial y_k} y_{k+1}\right],$$
$$(x, y) \in \Gamma,\ y_i \in R^N,\ i = 1, \ldots, k;\ k = 1, \ldots, r-1.$$

([2]) Compare footnote ([1]), p. 119.

From assumptions (I) and (II) it follows that

$$\boldsymbol{g}_k \in C^{r-k}(\Omega \times R^{kN}), \quad k = 1, \ldots, r.$$

Similarly, by assumptions (I) and (III) we have $\boldsymbol{h}_k \in C^{r-k}(\Gamma \times R^{kN})$, $k = 1, \ldots, r$.

It can easily be verified that if hypotheses (I), (II), and (III) are fulfilled and $\boldsymbol{\varphi}$ is a C^r-solution of system (1) or, equivalently, of the system

$$\boldsymbol{\varphi}(x) = \boldsymbol{h}\big(x, \boldsymbol{\varphi}(f(x))\big), \tag{3}$$

then the derivatives $\boldsymbol{\varphi}^{(k)}$ satisfy the equations (cf. [2]; also [4], p. 85)

$$\boldsymbol{\varphi}^{(k)}\big(f(x)\big) = \boldsymbol{g}_k\big(x, \boldsymbol{\varphi}(x), \boldsymbol{\varphi}'(x), \ldots, \boldsymbol{\varphi}^{(k)}(x)\big), \quad k = 1, \ldots, r, \tag{4}$$

$$\boldsymbol{\varphi}^{(k)}(x) = \boldsymbol{h}_k\Big(x, \boldsymbol{\varphi}\big(f(x)\big), \boldsymbol{\varphi}'\big(f(x)\big), \ldots, \boldsymbol{\varphi}^{(k)}\big(f(x)\big)\Big), \quad k = 1, \ldots, r. \tag{4'}$$

We aim at proving the following

THEOREM 2. *If hypotheses* (I)–(V) *are fulfilled, then for every* ϱ *with* $0 < \varrho \leqslant \varrho_0$ *and for every system of elements* $\boldsymbol{l}_0^k \in R^N$, $k = 1, \ldots, r$, *there exists a function* $\boldsymbol{\varphi}$ *with the following properties*:

$$\boldsymbol{\varphi} \in C^r\big((a, b)\big), \tag{5}$$

$$\boldsymbol{\varphi} \textit{ satisfies system } (1) \textit{ in } (a, b), \tag{6}$$

$$|\boldsymbol{\varphi}(x) - \boldsymbol{\eta}| \leqslant \varrho \quad \textit{for every } x \in \langle x_0, f(x_0)\rangle, \tag{7}$$

$$\boldsymbol{\varphi}^{(k)}(x_0) = \boldsymbol{l}_0^k, \quad k = 1, \ldots, r. \tag{8}$$

Proof. We denote $x_1 = f(x_0)$, x_0 being defined in hypothesis (V), and we consider the interval $\langle x_0, x_1\rangle \subset (a, b)$. We also fix a ϱ, $0 < \varrho \leqslant \varrho_0$, and we denote

$$S = \{\boldsymbol{y} \in R^N\colon |\boldsymbol{y} - \boldsymbol{\eta}| \leqslant \varrho\}, \quad S \subset S_0.$$

With the aid of the given functions $\boldsymbol{u}_i$, $\boldsymbol{v}_j$, f and $\boldsymbol{g}$ occurring in hypotheses (I)–(IV) we now construct sequences $\{\boldsymbol{u}_{i,m}\}$ and $\{\boldsymbol{v}_{j,m}\}$ of functions (defined in some subsets of sets $\langle a, b\rangle \times A_i$ and $\langle a, b\rangle \times B_j$, respectively) by the formulae

$$\begin{aligned} \boldsymbol{u}_{i,0}(x, \boldsymbol{t}) &= \boldsymbol{u}_i\big(f^{-1}(x), \boldsymbol{t}\big), \\ \boldsymbol{u}_{i,m+1}(x, \boldsymbol{t}) &= \boldsymbol{h}\Big(x, \boldsymbol{u}_{i,m}\big(f(x), \boldsymbol{t}\big)\Big), \end{aligned} \tag{9}$$

$$\begin{aligned} \boldsymbol{v}_{j,0}(x, \boldsymbol{t}) &= \boldsymbol{v}_j\big(f^{-1}(x), \boldsymbol{t}\big), \\ \boldsymbol{v}_{j,m+1}(x, \boldsymbol{t}) &= \boldsymbol{g}\Big(f^{-1}(x), \boldsymbol{v}_{j,m}\big(f^{-1}(x), \boldsymbol{t}\big)\Big). \end{aligned} \tag{9'}$$

We shall examine the following sequences of sets:

$$A_{i,m} = \{(x, \boldsymbol{t}) \in \langle x_0, x_1\rangle \times A_i\colon \boldsymbol{u}_{i,m}(x, \boldsymbol{t}) \text{ exists and belongs to } S\}_{m=0,1,\ldots}^{i=1,\ldots}, \tag{10}$$

$$(10') \quad B_{j,m} = \{(x, t) \in \langle x_0, x_1\rangle \times B_j\colon v_{j,m}(x, t) \text{ exists and belongs to } S\}_{m=0,1,\ldots}^{j=1,\ldots}$$

Now we rearrange the functions $u_{i,m}$ and $v_{j,m}$ into a single sequence, and we denote by w_n the elements of this sequence. By F_n we denote the domain of definition of the function w_n (each F_n is equal to some $A_{i,m}$ or $B_{j,m}$. Thus we have

$$w_n\colon F_n \to R^N, \quad F_n \subset \langle x_0, x_1\rangle \times R^{q-1},$$
$$w_n(F_n) \subset S, \quad n = 1, \ldots$$

From hypotheses (I)–(IV) and from (9) and (9′) it follows that $w_n \in C^1(F_n)$, for $n = 1, 2, \ldots$

Let us put

$$P_{0,n} = (\{x_0\} \times R^{q-1}) \cap F_n; \quad P_{1,n} = (\{x_1\} \times R^{q-1}) \cap F_n.$$

We now try to find points a_0 and a_1, such that $a_0 \in S$, $a_1 \in S$, $a_0 \notin w_n(P_{0,n})$, $a_1 \notin w_n(P_{1,n})$ and $a_1 = g(x_0, a_0)$.

The point η is a fixed point of the transformation

$$y \mapsto g(x_0, y),$$

which maps the set Ω_{x_0} into R^N (cf. assumption (V)). Since the function g is continuous at the point (x_0, η), there exists a neighbourhood $U_\eta \subset S$ of η such that $g(x_0, U_\eta \cap \Omega_{x_0}) \subset S$.

Let C denote the set of all $\tilde{\eta} \in U_\eta$ for which there exist either a positive integer m and $t \in P_{0,n}$ such that $w_m(x_0, t) = \tilde{\eta}$ or a positive integer n and $s \in P_{1,n}$ such that $w_n(x_1, s) = g(x_0, \tilde{\eta})$. On account of assumption (II) we have the equivalence

$$g(x_0, \tilde{\eta}) = w_n(x_1, s) \Leftrightarrow \tilde{\eta} = h\big(x_0, w_n(x_1, s)\big).$$

Let us write

$$z_n(x_1, s) = h\big(x_0, w_n(x_1, s)\big), \quad n = 1, \ldots, \quad s \in P_{1,n}.$$

We notice the following facts:

$$C \subset \bigcup_{n=1}^{\infty} w_n(P_{0,n}) \cup \bigcup_{n=1}^{\infty} z_n(P_{1,n}),$$

$$w_n\colon F_n \to R^N, \quad F_n \subset R^q, \quad q < N, \quad w_n \in C^1(F_n), \quad P_{0,n} \subset F_n, \quad n = 1, \ldots,$$

$$z_n\colon P_{1,n} \to R^N, \quad P_{1,n} \subset F_n, \quad z_n \in C^1(P_{1,n}).$$

Hence, by the Sard theorem, we have

$$m_N\big(w_n(P_{0,n})\big) = 0 \quad \text{and} \quad m_N\big(z_n(P_{1,n})\big) = 0, \quad n = 1, \ldots,$$

which imply $m_N(C) = 0$. This means that the set C has now interior in $U_\eta \cap \Omega_{x_0}$; so there exists an η^* such that

$$\eta^* \in U_\eta \cap \Omega_{x_0} \quad \text{and} \quad \eta^* \notin C.$$

Therefore we may take

$$a_0 = \eta^* \quad \text{and} \quad a_1 = g(x_0, \eta^*).$$

Now we can make use of Theorem 1. Take an arbitrary system of points $l_0^k \in R^N$, and put $l_1^k = g_k(x_0, \eta^*, l_0^1, \ldots, l_0^k)$, $k = 1, \ldots, r$, where g_k are defined by (2). According to Theorem 1 there exists a function u with the properties

$$u \in C^r(\langle x_0, x_1\rangle), \tag{11}$$

$$u(x_0) = \eta^*, \quad u(x_1) = g(x_0, \eta^*), \tag{12}$$

$$u^{(k)}(x_0) = l_0^k, \quad u^{(k)}(x_1) = g_k(x_0, \eta^*, l_0^1, \ldots, l_0^k); \quad k = 1, \ldots, r, \tag{13}$$

$$u(x) \neq w_n(x, t) \quad \text{for every } (x, t) \in F_n,\ n = 1, \ldots, \tag{14}$$

$$u(\langle x_0, x_1\rangle) \subset S. \tag{15}$$

This means that besides (11) and (15) the function u satisfies also:

$$u(x_1) = g\big(x_0, u(x_0)\big), \tag{12'}$$

$$u^{(k)}(x_1) = g_k\big(x_0, u(x_0), u'(x_0), \ldots, u^{(k)}(x_0)\big), \quad k = 1, \ldots, r, \tag{13'}$$

$$u(x) \neq u_{i,m}(x, t) \quad \text{for } (x, t) \in A_{i,m},\ i = 1, \ldots;\ m = 0, 1, \ldots, \tag{14'}$$

$$u(x) \neq v_{j,m}(x, t) \quad \text{for } (x, t) \in B_{j,m},\ j = 1, \ldots;\ m = 0, 1 \ldots, \tag{14''}$$

where $u_{i,m}$ and $v_{j,m}$ are defined by formulas (9) and (9′), and $A_{i,m}$ and $B_{j,m}$ by formulas (10) and (10′).

Now we can define the function φ whose existence is asserted by our theorem. We put

$$\varphi(x) = \begin{cases} u(x) & \text{for } x \in \langle x_0, x_1), \\ g\big[f^{-1}(x), \varphi\big(f^{-1}(x)\big)\big] & \text{for } x \in \langle x_n, x_{n+1}),\ n = 1, \ldots, \\ h\big[x, \varphi\big(f(x)\big)\big] & \text{for } x \in \langle x_{-n}, x_{-n+1}),\ n = 1, \ldots, \end{cases} \tag{16}$$

where $x_n = f^n(x_0)$, $x_{-n} = f^{-n}(x_0)$ and $f^n(x_0)$ denotes the n-th iterate of the function f.

First we prove that formula (16) actually defines a function φ for every $x \in (a, b)$. From (14′), (10), and (9) it follows, in particular, that for $n = 0$

$$u(x) \notin \bigcup_{i=1}^{\infty} u_i\big(f^{-1}(x), A_i\big), \quad x \in \langle x_0, x_1\rangle.$$

Thus, on account of assumption (V), we conclude that

$$(17) \qquad \boldsymbol{u}(x) \in \Omega_x \quad \text{for every } x \in \langle x_0, x_1\rangle .$$

Now we consider the interval $\langle x_1, x_2)$. If $x \in \langle x_1, x_2)$, then, by assumption (I), $f^{-1}(x) \in \langle x_0, x_1)$, and by (17) we get $\boldsymbol{u}\big(f^{-1}(x)\big) \in \Omega_{f^{-1}(x)}$; thus formula (16) defines the function $\boldsymbol{\varphi}$ at the point x.

Assume that the function $\boldsymbol{\varphi}$ is well defined by formula (16) in the interval $\langle x_1, x_n)$.

We take an arbitrary $x \in \langle x_n, x_{n+1})$, so that $f^{-1}(x) \in \langle x_{n-1}, x_n)$. From the inductive assumption it follows that the function $\boldsymbol{\varphi}$ is defined at $f^{-1}(x)$. The value $\boldsymbol{g}\big[f^{-1}(x), \boldsymbol{\varphi}\big(f^{-1}(x)\big)\big]$ will make sense if

$$(18) \qquad \boldsymbol{\varphi}\big(f^{-1}(x)\big) \in \Omega_{f^{-1}(x)}.$$

We know that

$$(19) \qquad \boldsymbol{\varphi}\big(f^{-1}(x)\big) = \boldsymbol{g}\big[f^{-2}(x), \boldsymbol{\varphi}\big(f^{-2}(x)\big)\big],$$

so that $\boldsymbol{\varphi}\big(f^{-1}(x)\big) \in \Gamma_{f^{-2}(x)}$. In view of assumption (IV), condition (18) will be fulfilled when

$$\boldsymbol{\varphi}\big(f^{-1}(x)\big) \notin \Gamma_{f^{-2}(x)} \setminus \Omega_{f^{-1}(x)} = \bigcup_{i=1}^{\infty} \boldsymbol{u}_i\big(f^{-2}(x), A_i\big).$$

This is equivalent to the condition

$$(20) \qquad \boldsymbol{\varphi}\big(f^{-1}(x)\big) \neq \boldsymbol{u}_i\big(f^{-2}(x), \boldsymbol{t}\big) \quad \text{for every } \boldsymbol{t} \in A_i,\ i = 1, \ldots$$

Suppose that (20) is not fulfilled, i.e., there exist a positive integer i and a $\boldsymbol{t} \in A_i$ such that

$$\boldsymbol{\varphi}\big(f^{-1}(x)\big) = \boldsymbol{u}_i\big(f^{-2}(x), \boldsymbol{t}\big).$$

By (19) we have

$$\boldsymbol{g}\big[f^{-2}(x), \boldsymbol{\varphi}\big(f^{-2}(x)\big)\big] = \boldsymbol{u}_i\big(f^{-2}(x), \boldsymbol{t}\big)$$

and using the inverse function (cf. assumption (II)) we obtain

$$\boldsymbol{\varphi}\big(f^{-2}(x)\big) = \boldsymbol{h}\big[f^{-2}(x), \boldsymbol{u}_i\big(f^{-2}(x)\big)\big] = \boldsymbol{u}_{i,1}\big(f^{-2}(x), \boldsymbol{t}\big),$$

where $\boldsymbol{u}_{i,1}$ is defined by (9). After n steps we get

$$\boldsymbol{\varphi}\big(f^{-n}(x)\big) = \boldsymbol{u}_{i,n-1}\big(f^{-n}(x), \boldsymbol{t}\big);$$

but

$$f^{-n}(x) \in \langle x_0, f(x_0)\rangle,$$

and so by (16) we get

$$\boldsymbol{\varphi}\big(f^{-n}(x)\big) = \boldsymbol{u}\big(f^{-n}(x)\big) = \boldsymbol{u}_{i,n-1}\big(f^{-n}(x), \boldsymbol{t}\big), \qquad \big(f^{-n}(x), \boldsymbol{t}\big) \in A_{i,n-1},$$

which is impossible by (14′). Thus, if $x \in \langle x_n, x_{n+1})$, then condition (18) is fulfilled and formula (16) defines the value $\boldsymbol{\varphi}(x)$.

Finally, it follows, by the induction principle, from assumption (I) and from the theorem proved in [4], p. 21, that the function φ is defined for every $x \in (x_0, b)$.

We omit an analogous proof of the fact that formula (16) defines the function φ for every $x \in (a, x_0)$.

This part of the proof is analogous to the proof of the corresponding theorem contained in [1] (cf. also [3], p. 127–128). We notice that (11) and assumptions (I)–(III) imply that φ is a C^r-class in every interval $\langle x_n, x_{n+1})$, $n = 0, 1, \ldots$ Thus it suffices to prove that

$$\lim_{x\to x_n^-} \varphi^{(k)}(x) = \varphi_+^{(k)}(x_n), \qquad k = 0, \ldots, r, \;\; n = 0, 1, \ldots,$$

where $\varphi_+^{(k)}(x_n)$ denotes the k-th right derivative of the function φ at the point x_n.

First, by induction we show that

$$\lim_{x\to x_n} \varphi(x) = \varphi(x_n). \tag{21}$$

For $n = 1$, by (16) and (11) we have

$$\lim_{x\to x_1^-} \varphi(x) = \lim_{x\to x_1^-} u(x) = u(x_1).$$

On the other hand, it follows from (16), (I), (11), (III), and (12′) that

$$\begin{aligned}\lim_{x\to x_1^+} \varphi(x) &= \lim_{x\to x_1} g\left[f^{-1}(x), \varphi\big(f^{-1}(x)\big)\right] \\ &= \lim_{x\to x_1^+} g\left[f^{-1}(x), u\big(f^{-1}(x)\big)\right] = g\big(x_0, u(x_0)\big) = u(x_1);\end{aligned}$$

thus φ is continuous at the point x_1. Now, we suppose that

$$\lim_{x\to x_n} \varphi(x) = \varphi(x_n)$$

for certain $n > 1$ and we investigate $\lim\limits_{x\to x_{n+1}} \varphi(x)$. On account of (16), assumptions (I), (II) and the inductive assumption we may write

$$\begin{aligned}\lim_{x\to x_{n+1}} \varphi(x) &= \lim_{x\to x_n} \varphi\big(f(x)\big) = \lim_{x\to x_n} g\big(x, \varphi(x)\big) \\ &= g\big(x_n, \varphi(x_n)\big) = \varphi(x_{n+1});\end{aligned}$$

thus condition (21) is fulfilled for every natural n, which means that $\varphi \in C^0\big(\langle x_0, b)\big)$. Similarly we prove that $\varphi \in C^0\big((a, x_1\rangle\big)$. We omit this proof.

In the sequel we suppose that the solution (16) is C^{k-1} in the interval $\langle x_0, b)$, $k \geqslant 1$, and we show by induction that it is C^k in $\langle x_0, x_n)$ for every natural n. According to (16) and (11), this is true for $n = 1$. Now we suppose that $\varphi \in C^k(\langle x_0, x_n))$ for certain $n > 1$. Let us take an $x \in \langle x_n, x_{n+1})$; then $f^{-1}(x) \in \langle x_0, x_n)$, and so by (16), (I), (II) and by the inductive assumption, we have

$$\varphi \in [C^k(\langle x_0, x_n)) \cap C^k(\langle x_n, x_{n+1})]. \tag{22}$$

On account of (4) we may write

$$\varphi^{(k)}(x) = g_k[f^{-1}(x), \varphi(f^{-1}(x)), \varphi'(f^{-1}(x)), \ldots, \varphi^{(k)}(f^{-1}(x))] \tag{23}$$

for any $x \in (x_1, x_n)$.

Further, if $x \to x_n^-$, then $f^{-1}(x) \to x_{n-1}^-$; thus by (22), (23) and by assumptions (I)–(III) we get

$$\lim_{x \to x_n^-} \varphi^{(k)}(x) = g_k(x_{n-1}, \varphi(x_{n-1}), \varphi'(x_{n-1}), \ldots, \varphi^{(k)}(x_{n-1})).$$

On the other hand, because $\varphi \in C^k(\langle x_n, x_{n+1}))$, we have

$$\varphi^{(k)}(x_n) = g_k(x_{n-1}, \varphi(x_{n-1}), \varphi'(x_{n-1}), \ldots, \varphi^{(k)}(x_{n-1}))$$

and consequently

$$\lim_{x \to x_n^-} \varphi^{(k)}(x) = \varphi^{(k)}(x_n),$$

so that $\varphi \in C^k(\langle x_0, x_{n+1}))$. By induction (on n) we conclude that $\varphi \in C^k(\langle x_0, x_n))$ for every natural n, in view of Theorem 0.4 in [4] and, by induction on k we get the conclusion that $\varphi \in C^r(\langle x_0, b))$.

Similarly we prove that $\varphi \in C^r((a, x_1))$. Using (16), (11), (4′), and hypotheses (I), (III), we show that φ is C^r in every interval $\langle x_{-n}, x_1)$, $n = 1, \ldots$ We omit the numerical details. Finally, we conclude that $\varphi \in C^r((a, b))$. Properties (8) and (9) result immediately from (16), (15), and (13).

This completes the proof.

Remark 2. According to Remark 1, for any ϱ, $0 < \varrho \leqslant \varrho_0$, and for arbitrarily chosen $l_0^k \in R^N$, $k = 1, \ldots, r$, there exist infinitely many solutions of system (1), which enjoy properties (5)–(8).

From Theorem 2 we also obtain the following

Remark 3. If the assumptions of Theorem 2 are fulfilled, then the solution φ of system (1) defined by formula (16) satisfies the conditions:

$$\varphi(x) \notin \bigcup_{i=1}^{\infty} u_i(f^{-1}(x), A_i) \quad \text{for every } x \in \langle x_0, b),$$

and

$$\varphi(x) \notin \bigcup_{j=1}^{\infty} v_j(f^{-1}(x), B_j) \quad \text{for every } x \in (a, x_1\rangle.$$

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