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Fibre bundles associated with fields of geometric objects and the structure tensor*

by J. J. KONDERAK (Salerno)

Abstract. Fields of geometric objects and natural bundles are considered. With a given field we associate a sequence of vector bundles together with the Spencer cohomology morphisms. We show that the induced cohomology spaces are vector bundles. We demonstrate a relationship with the cohomology of the Lie algebra generated by the fields of geometric objects. A k-structure tensor is then constructed using these bundles. It is then shown that vanishing of the kth structure tensor is a necessary and sufficient condition for the field to be $(k+1)$-flat.

Introduction. A structure tensor of a G-structure is an invariant which measures the degree of flatness of the structure. If the G-structure is k-flat (k a non-negative integer) then there exists a kth structure tensor c^k. The tensor c^0 was introduced by Ehresmann [5] and was defined as the torsion associated with the elements of the first order prolongations. It has been also considered by Bauer [3], Bernard [4], Kolář [8], and Matsushima [11]. Singer and Sternberg [16] gave another definition of that tensor. Suppose that: E is a G-structure, $\mathcal{G}$ is a Lie algebra of G, $\mathbf{R}^{n^*}\otimes\mathcal{G}$ is embedded canonically in $\mathbf{R}^{n^*}\otimes\mathbf{R}^{n^*}\otimes\mathbf{R}^n$, $\mathcal{G}\subset \mathrm{gl}(\mathbf{R}^n)$ and ∂ is the skew symmetrization operator. The tensor c^0 is a function

$$c^0\colon\ E\to\bigwedge{}^2\mathbf{R}^{n^*}\otimes\mathbf{R}^n/\partial(\mathbf{R}^{n^*}\otimes\mathcal{G})$$

that can be defined as the composite $c^0 = q\circ T$ where $T\colon\ E\to\bigwedge{}^2\mathbf{R}^{n^*}\otimes\mathbf{R}^n$ is the torsion form associated with any connection and q is the quotient map. Structure tensors of higher order were introduced by Singer and Sternberg [16]. They first defined the successive prolongations of the first order G-structures and then defined c^k as the c^0 tensor of the kth prolongation. Guillemin [7] has redefined the structure tensor c^k using the notion of kth order structure preserving holonomic jets. The main theorem concerning the invariant is that the vanishing of c^k is a necessary and sufficient condition for a G-structure to be $(k+1)$-flat. Guillemin proved that in the case of finite order G-structures, i.e. such that $g^k = 0$ for some k, the structure is flat iff it is

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k-flat. This is of great importance since many Lie groups are of finite order. Later Molino proved in his equivalence theorem that flatness of infinite order of a G-structure implies its flatness ([2], [12]). Hence vanishing of all structure tensors is a sufficient condition for the structure to be flat.

Our main aim is to construct a structure tensor for a field of geometric objects. We construct it on a vector bundle generated by the given field. There is a correspondence between certain types of fields of geometric objects and G-structures. It is natural to expect that there exists an invariant of a field of geometric objects which is responsible for k-flatness of the field. An attempt to construct such a tensor was made by Zajtz who did it locally using Lie equations associated with the given field [17]. The main application of our structure tensor is an analogue of Guillemin's theorem: a kth structure tensor of a field of geometric objects vanishes iff the field is $(k+1)$-flat.

In the first part of this paper we recall some properties of natural bundles and structures associated with them. The concept of natural bundle was introduced by Nijenhuis [13] as a modern approach to the classical theory of geometric objects (see J. Aczél and S. Gołąb [1]). We give some basic properties of fields of geometric objects. Each 0-flat field of geometric objects generates a G-structure. We show how some properties of fields can be translated into the language of fibre bundles.

In the second part of this paper we construct a vector bundle $H^{k,2}(\sigma)$. A given k-flat field σ ($k \geqslant 0$) generates the vector bundle $R^1(\sigma)$ (cf. I.4). We prove that the prolongations $R^{1(0)}(\sigma), \ldots, R^{1(k)}(\sigma)$ are vector bundles. The symbols of these vector bundles generate Spencer's complex $g^m \otimes \bigwedge^l T^*M$ with the operator of antisymmetrization $\partial^{m,l}$ where m, l are non-negative integers. We prove that the morphisms $\partial^{k,2}$, $\partial^{k+1,1}$ are of constant rank. This leads to the conclusion that the induced cohomology space $H^{k,2}(\sigma)$ is a smooth vector bundle.

In the third part of this paper we construct, for a given $x \in M$, a linear mapping $\lambda\colon T_xM \to R^{1(k)}(\sigma)_x$ which satisfies some additional conditions. The section determines an element τ_λ^k belonging to $g_x^{k-1} \otimes \bigwedge^l T_x^*M$ such that for $v, w \in T_xM$, $\tau_\lambda^k = \{\lambda(v), \lambda(w)\}$. This element is a cocycle and hence generates an element of the space $H^{k,2}(\sigma)_x$. The mapping $\tau^k\colon M \to H^{k,2}(\sigma)$ constructed in this way we call the kth structure tensor of a field of geometric objects. It turns out to be non-trivial to show that τ^k is well-defined and a smooth section of the bundle $H^{k,2}(\sigma)$. Then we show a relation between τ^k and the kth structure tensor of the G-structure generated by σ. That correspondence gives immediately the theorem that under the assumption of k-flatness of σ, $\tau^k \equiv 0$ iff σ is $(k+1)$-flat.

All objects considered in this paper, i.e. manifolds, vector fields, bundles, etc. are smooth, that is, of class C^∞.

I. Preliminaries

I.1. *Natural bundles.* Let $\mathscr{M}_n$ denote the category of n-dimensional manifolds with smooth embeddings as morphisms. Let Λ denote the category whose objects are smooth bundle maps (i.e. a morphism of $\pi_1\colon E_1 \to M_1$ to

π_2: $E_2 \to M_2$ is a map H: $E_1 \to E_2$ such that for each $x \in M_1$ the fibre $E_{1x} = \pi_1^{-1}(x)$ is mapped diffeomorphically onto the fibre E_{2y}, over some point $y = h(x) \in M_2$; the induced map h: $M_1 \to M_2$ is automatically smooth and we say that H covers h).

DEFINITION (1.1) (Palais–Terng [14]). A *natural bundle* on $\mathcal{M}_n$ is a covariant functor F: $\mathcal{M}_n \to \Lambda$ such that

(1) for each $M \in \mathcal{M}_n$, $F(M)$ is a fibre bundle over M;
(2) for each embedding ϕ: $M \to N$ the diagram

$$\begin{tikzcd} F(M) \arrow[r, "F(\phi)"] \arrow[d] & F(N) \arrow[d] \\ M \arrow[r, "\phi"] & N \end{tikzcd}$$

commutes;

(3) if U is an open subset in $\mathbf{R}^m$, where m is a non-negative integer and f: $U \times \mathbf{R}^n \to \mathbf{R}^n$ is a map such that for each $t \in U$, $x \to f_t(x) = f(t, x)$ is an embedding then the map $\tilde{f}$: $U \times F(\mathbf{R}^n) \to F(\mathbf{R}^n)$ sending (t, v) to $F(f_t)v$ is smooth.

In [6] Epstein and Thurston showed that if $F(\mathbf{R}^n)$ has a countable basis then (3) is a consequence of (1) and (2). We assume that F is *of order* r; this means that any two embeddings ϕ, ψ: $M \to N$ with their r-jets equal at a point $x \in M$ have to satisfy $F(\phi)v = F(\psi)v$ for any point v from the fibre $F(M)_x$, and r is the smallest number which has this property. Palais and Terng showed that any natural bundle is of finite order [14].

EXAMPLE (1.2). The functor T which associates with each manifold M its tangent bundle TM is a natural bundle of order one. If ϕ: $M \to N$ is an embedding then $T(\phi) := d\phi$ is the tangent map induced by ϕ. This natural bundle has an additional structure of a vector bundle and the induced morphisms are morphisms of vector bundles. If there are given two natural bundles with the additional structure of vector bundles then their tensor, symmetric and skew symmetric products are also natural bundles.

EXAMPLE (1.3). Let $M \in \mathcal{M}_n$ and let $H^r(M)$ denote the principal fibre bundle of r-jets of diffeomorphisms of an open neighbourhood of 0 in $\mathbf{R}^n$ into M. The structure group of $H^r(M)$ is the set L_n^r consisting of all r-jets of diffeomorphisms defined on an open neighbourhood of 0 in $\mathbf{R}^n$ into $\mathbf{R}^n$ sending 0 to 0. If ϕ: $M \to N$ is an embedding then

$$H^r(\phi)\colon\ H^r(M) \to H^r(N)$$

is defined by $H^r(\phi)(j_0^r f) := j_0^r \phi \circ f$ for $j_0^r f \in H^r(M)$. It is easy to check that H^r is a natural bundle of order r. For any non-negative integers m, l such that $m \geqslant l$ there is the canonical projection π_l^m: $H^r(M) \to H^l(M)$.

Let F be a natural bundle of order r defined on the category $\mathcal{M}_n$. Then for each non-negative integer m there is defined the mth *prolongation* $J^m F$ of the bundle F; if $M \in \mathcal{M}_n$ then

$$J^m F(M) := \{j_x^m S \mid x \in M \text{ and } S \text{ is a local section of the bundle } F(M)\}.$$

There is a canonical structure of a fibre bundle on $J^m F(M)$ with the canonical projection π^m: $J^m F(M) \to M$. If ϕ: $M \to N$ is an embedding then

$$J^m F(\phi)(j_x^m S) := j_{\phi(x)}^m (F(\phi) \circ S \circ \phi^{-1})$$

where $j_x^m S \in J^m F(M)$. One can easily check that $J^m F$ is a natural bundle of order $m+r$.

I.2. *Fields of geometric objects.* We assume that F is a natural bundle of order r defined on the category $\mathcal{M}_n$. Suppose that $M \in \mathcal{M}_n$ and let σ be a *field of geometric objects* on a manifold M; this means that σ is a smooth section of the bundle π: $F(M) \to M$. Let F_0 denote the *standard fibre of* F, i.e. the fibre over 0 of the bundle π: $F(\mathbf{R}^n) \to \mathbf{R}^n$. Suppose that $v \in F_0$; then it defines a standard field σ_v on $\mathbf{R}^n$, namely $\sigma_v(x) := F(t_x)v$ where t_x denotes the translation by the vector x in $\mathbf{R}^n$ (cf. [9]).

Any embedding ϕ: $M \to N$ transports fields of geometric objects from M to $\phi(M)$. We denote that operator by ϕ_* and it acts as follows:

$$\phi_*(\sigma) := F(\phi) \circ \sigma \circ \phi^{-1}. \tag{1}$$

DEFINITION (1.4). A field σ is *flat* if it is locally of the form $\phi_*(\sigma_v)$.

In other words, σ is flat iff for each point $x \in M$ there exists a local diffeomorphism ϕ: $(\mathbf{R}^n, 0) \to (M, x)$ (this notation means that ϕ is a diffeomorphism of an open neighbourhood of 0 in $\mathbf{R}^n$ onto an open neighbourhood of x in M with $\phi(0) = x$) such that (1) holds for some standard field σ_v.

DEFINITION (1.5). A field σ is said to be *k-flat* if there exists a standard field σ_0 on $\mathbf{R}^n$ such that for each point $x \in M$ there is a local diffeomorphism ϕ: $(\mathbf{R}^n, 0) \to (M, x)$ such that $j_x^k \phi_* \sigma_0 = j_x^k \sigma$ where k is a non-negative integer.

Each 0-flat field σ generates a principal fibre bundle of order r which is defined as follows:

$$E(\sigma) := \{j_0^r f \in H^r(M) \mid F(f)\sigma_0(0) = \sigma(f(0))\} \tag{2}$$

(cf. [17]). Its structure group has to be of the form

$$G = \{j_0^r h \in L_n^r \mid F(f)\sigma_0(0) = \sigma_0(0)\}.$$

Hence there is a correspondence between 0-flat fields of geometric objects and principal subbundles of $H^r(M)$. The properties of flatness and k-flatness are well preserved with respect to this relationship. We describe that fact in the following proposition.

PROPOSITION (1.6). *Suppose that F is a natural bundle of order r defined on $\mathcal{M}_n$. Let σ_1, σ_2 be 0-flat fields of geometric objects and let*

$$E_i = \{j_0^r f \in H^r(M) \mid F(f)(\sigma_0(0)) = \sigma_i(0)\}$$

be principal bundles associated with σ_i for $i = 1, 2$. Then for each $x \in M$ and for each non-negative integer k the following two conditions are equivalent:

(1) $j_x^k\sigma_1 = j_x^k\sigma_2$;

(2) $E_{1x} = E_{2x}$ *and E_1, E_2 are k-tangent at each point of the fibre over x.*

For a proof see [2] (Proposition VI.4).

I.3. *The vector bundle $J^m TM$ and liftings of vector fields.* Let $M \in \mathcal{M}_n$. By $J^m TM$ we shall denote the vector bundle consisting of all m-jets of smooth vector fields on M where m is a non-negative integer. For technical reasons we also define $J^{-1}TM$ as the null subbundle of TM. For each pair of integers m, l such that $m \geqslant l \geqslant -1$ we have the canonical projection

$$p_l^m\colon\ J^m TM \to J^l TM.$$

We also define the subbundle of jets of vector fields which vanish at their sources; namely $\mathring{J}^m TM := \ker p_0^m$.

On the fibres of the bundle $J^m TM$ we define the *algebraic bracket*

$$\{\ ,\ \}\colon\ J_x^m TM \times J_x^m TM \to J_x^{m-1} TM \tag{3}$$

by $\{j_x^m X, j_x^m Y\} := j_x^{m-1}[X, Y]$ where $x \in M$, $j_x^m X$, $j_x^m Y \in J^m TM$ and $[X, Y]$ denotes the Lie bracket of two vector fields. For convenience we define this bracket also for $m = 0$; namely $\{v, w\} := 0$ for v, $w \in T_x M$.

One can easily see that the restriction of the algebraic bracket to the set $\mathring{J}^m TM$ is a well defined inner product which equips this set with the Lie algebra structure.

From I.1 we see that $J^m T$ is a natural bundle of order $m+1$ as the mth prolongation of the bundle T. Each local diffeomorphism ϕ: $(M, x) \to (N, y)$ defines the isomorphism of vector spaces:

$$J^m T(\phi)_x\colon\ J_x^m TM \to J_y^m TN.$$

Since $J^m T$ is of order $m+1$, $J^m T(\phi)_x$ depends only on the $(m+1)$-jet of ϕ at the point x.

Let X be a vector field on M; it induces a flow φ_t. From the definition of the natural bundle we easily find that $F(\varphi_t)$ is a flow on $F(M)$. Hence X generates a vector field on $F(M)$. This induced vector field is called the *complete lift of X to the bundle $F(M)$* and we shall denote it by $\mathscr{F}(X)$ (cf. [15]).

Let us consider liftings of vector fields in the particular case of the bundle H^m. Let X be a vector field on M and let $\mathscr{H}^m(X)$ denote its lifting to the bundle $H^m(M)$. For each $x \in M$ and $z_m \in H^m(M)_x$ there is a canonically defined function

$$\Phi_{z_m}\colon\ J_x^m TM \to T_{z_m} H^m(M)$$

such that $\Phi_{z_m}(j_x^m X) := \mathscr{H}^m(X)_{z_m}$. Libermann showed that Φ_{z_m} is an isomorphism of vector spaces (cf. [10]).

I.4. *Vector bundles associated with fields of geometric objects.* Assume that E is a principal subbundle of the bundle $H^m(M)$. With each $x \in M$ we associate the vector space

$$W(E)_x := \Phi_{z_m}^{-1}(T_{z_m}E)$$

where $z_m \in E_x$. The definition of $W(E)_x$ does not depend on the choice of the element z_m from the fibre E_x. Then we define

$$W(E) = \bigcup_{x \in M} W(E)_x.$$

The set $W(E)$ is a smooth vector subbundle of $J^m TM$.

Suppose that F is of order r and σ is a 0-flat field of geometric objects on M. Then σ generates a principal fibre bundle

$$E(\sigma) = \{j_0^r f \in H^r(M) \mid F(f)(\sigma_0(0)) = \sigma_0(f(0))\}$$

(cf. I.2). With $E(\sigma)$ we can associate a vector bundle as above. We shall denote that bundle by $R^r(\sigma) = W(E(\sigma))$. It will be useful to describe this vector bundle not using $E(\sigma)$. We give such a description using the Lie derivative of a field of geometric objects. This derivative is a generalization of the Lie derivative of tensor fields. A very beautiful exposition of that subject can be found in Salvioli's paper ([15]). We shall recall briefly this notion.

Let $V(F(M))$ denote the subbundle of the vector bundle $TF(M) \to M$ consisting of vectors tangent to the fibres of the bundle π: $F(M) \to M$. Let us remark that $V(F(M))$ is a natural bundle of order $r+1$. Suppose that X is a vector field on M. Then the *Lie derivative of σ in the direction of X* will be denoted by $L_X\sigma$ and is a section of the bundle $V(F(M))$ such that

$$(L_X\sigma)_x := d_x\sigma(X_x) - \mathscr{F}(X)_{\sigma(x)}$$

where $x \in M$.

We would like to point out one property of the Lie derivative of a field of geometric objects. Let $j^k\sigma$ denote the field on the bundle $J^m F(M)$ such that for each $x \in M$, $(j^k\sigma)(x) := j_x^k\sigma$. It is well known that the spaces $J^k V(F(M))$ and $V(J^k F(M))$ are canonically isomorphic. Then under this canonical isomorphism we can identify $j^k(L_X\sigma)$ with $L_X(j^k\sigma)$ (cf. [15], [17]).

II. The vector bundle $H^{k,2}(\sigma)$. Throughout this section we assume that F is a natural bundle of order one defined on $\mathscr{M}_n$. We also assume that σ is a k-flat field of geometric objects on $M \in \mathscr{M}_n$ ($k \geqslant 0$). We denote by σ_0 a standard field on $\mathbf{R}^n$ such that for each $x \in M$ there exists a local diffeomorphism

$$f\colon (\mathbf{R}^n, 0) \to (M, x)$$

satisfying the equality $j_x^k f_* \sigma_0 = j_x^k \sigma$.

II.1. *Prolongations of the vector bundle $R^1(\sigma)$.* For each non-negative integer m we define

$$E^m(\sigma) := \{j_0^{m+1} f \in H^{m+1}(M) \mid j_{f(0)}^m f_* \sigma_0 = j_{f(0)}^m \sigma\}.$$

We also put $E^{-1}(\sigma) := M$. For each $m \geqslant l \geqslant -1$ there is a canonical projection $\pi_l^m \colon E^m(\sigma) \to E^l(M)$.

PROPOSITION (2.1). *If σ is k-flat and if $m \in \{0, \ldots, k\}$ then the set $E^m(\sigma)$ is a principal fibre bundle over M.*

Proof. Let $m \in \{0, \ldots, k\}$. Then $j^m \sigma$ is a field of geometric objects on $J^m F(M)$. It is 0-flat because σ is k-flat. Hence the set

$$E(j^m \sigma) = \{j_0^{m+1} f \in H^{m+1}(M) \mid J^m F(f)(j^m \sigma_0)(0) = j^m \sigma(f(0))\}$$

is a principal subbundle of $H^{m+1}(M)$ (cf. (2)).

To end the proof it is enough to notice that the set above coincides with $E^m(\sigma)$ because $j_{f(0)}^m f_* \sigma_0 = J^m F(f)(j^m \sigma_0)(0)$ for each $j_0^{m+1} f \in H^{m+1}(M)$.

Let us remark that the structure group of $E^m(\sigma)$ is

$$G^m := \{j_0^{m+1} f \in L_n^m \mid j_0^m f_* \sigma_0 = j_0^m \sigma_0\},$$

where $m \in \{0, \ldots, k\}$. By Proposition (2.1), $E^0(\sigma), \ldots, E^k(\sigma)$ are principal fibre bundles. Moreover, they are generated by the fields $\sigma, j^1 \sigma, \ldots, j^k \sigma$, respectively.

With the field σ we also associate the vector bundle $R^1(\sigma)$ (cf. I.4). In the standard way we define the mth *prolongation* of $R^1(\sigma)$, namely

$$R^{1(m)}(\sigma) := J^m R^1(\sigma) \cap J^{m+1} TM$$

where m is a non-negative integer. For technical reasons we also put $R^{1(-1)}(\sigma) := TM$ and $R^{1(-2)}(\sigma)$ is the null subbundle of TM. For each $m \geqslant l \geqslant -2$ there is a natural projection

$$p_{l+1}^{m+1} \colon R^{1(m)}(\sigma) \to R^{1(l)}(\sigma).$$

The *symbol* of $R^{1(m)}(\sigma)$ is defined as

$$g^m := \ker\{p_m^{m+1} \colon R^{1(m)}(\sigma) \to R^{1(m-1)}(\sigma)\}.$$

Also for technical reasons we put $g^{-1} := TM$ and $g^{-2} := R^{1(-2)}(\sigma)$. The sets $R^{1(0)}(\sigma)$, $R^{1(1)}(\sigma), \ldots$ and $g^0, g^1, \ldots$ are not generally all vector bundles. However, we shall find that some of them, in fact, are. To do this we shall explore the relationship between the prolongations of $R^1(\sigma)$ and the sets $E^0(\sigma), E^1(\sigma), \ldots$

LEMMA (2.2). *If σ is k-flat then $R^{1(m)}(\sigma)$ is a smooth vector bundle for $m \in \{0, \ldots, k\}$.*

Proof. Zajtz showed that

$$R^{1(m)}(\sigma) = R^{m+1}(j^m \sigma)$$

(cf. Lemma 2.12c of [17]). In fact this equality was shown for any field of geometric objects and an arbitrary m. Now our lemma is obvious since $R^{m+1}(j^m\sigma)$ is associated with the field $j^m\sigma$ which is 0-flat for $m\in\{0,\dots,k\}$ (cf. I.4) and hence $R^{m+1}(j^m\sigma)$ is a vector bundle.

Our field σ generates the principal fibre bundles $E^0(\sigma),\dots,E^k(\sigma)$. Proposition (1.6) links k-flatness of σ with k-flatness of the G-structures associated with them.

It is easy to notice that the set

$$E^0(\sigma_0)=\{j_0^1 f\in H^1(\mathbf{R}^n)\mid F(f)(\sigma_0(0))=\sigma_0(0)\}$$

is the standard flat structure on $\mathbf{R}^n$.

Let us take a local diffeomorphism h: $(\mathbf{R}^n,0)\to(M,x)$. Then by Proposition (1.6) the following two conditions are equivalent:

(i) $j_x^m(h_*\sigma_0)=j_x^m\sigma$;

(ii) $H^m(h)E^0(\sigma_0)_x=E^0(\sigma)_x$ and $H^0(h)E^0(\sigma_0)$, $E^0(\sigma)$ are m-tangent at each point of the fibre above x.

The above equivalence provides some information about the set of jets which witness k-flatness of G-structures and the field of geometric objects. In fact, $E^m(\sigma)$ appears to be the set of all those $(m+1)$-jets which witness m-flatness of the G^0-structure $E^0(\sigma)$. Explicitly this means that

$$E^m(\sigma)=\{j_0^{m+1}f\in H^{m+1}(M)\mid H^m(h)E^0(\sigma_0),\ E^0(\sigma_1)\ \text{are } m\text{-tangent at the fibre above } x\}.$$

Hence $E^0(\sigma)$ is also k-flat . Moreover, the properties of these sets have already been investigated (cf. [3], [7]). From those papers we know that the canonical projections

$$\pi_m^{m+1}\colon E^m(\sigma)\to E^{m-1}(\sigma)$$

are surjective for $m\in\{0,\dots,k\}$. As a consequence we get the following property.

LEMMA (2.3). *If $m\in\{0,\dots,k\}$ then the canonical projections*

$$p_m^{m+1}\colon R^{1(m)}(\sigma)\to R^{1(m-1)}(\sigma)$$

are epimorphisms.

Proof. Take $x\in M$, $\xi\in E_x^m(\sigma)$, $\eta\in E_x^{m-1}(\sigma)$ such that $\pi_m^{m+1}(\xi)=\eta$. Then the diagram

$$\begin{tikzcd} R^{1(m)}(\sigma)_x \arrow[r, "\Phi_\xi"] \arrow[d, "p_m^{m+1}"'] & T_\xi E^m(\sigma) \arrow[d, "d\pi_m^{m+1}"] \\ R^{1(m-1)}(\sigma)_x \arrow[r, "\Phi_\eta"] & T_\eta E^{m-1}(\sigma) \end{tikzcd}$$

commutes. Now it is easy to see that p_m^{m+1} is an epimorphism since $d\pi_m^{m+1}$ is an epimorphism and Φ_ξ, Φ_η are isomorphisms.

As a corollary of the above lemma we find that g^m are vector bundles for $m \in \{-2, -1, 0, \ldots, k\}$.

We would like to connect the spaces $R^{1(m)}(\sigma_0)$ and $R^{1(m)}(\sigma)$ via the isomorphisms defined by the jets from the bundle $H^{k+1}(M)$; namely if $j_0^{m+1} f \in H^{m+1}(M)_x$ then we have the canonical isomorphism $J^m T(f)$: $J_0^m T\mathbf{R}^n \to J_x^m TM$ (cf. I.3).

LEMMA (2.4). *Let m be an arbitrary natural number and let $j_0^{m+2} f \in H^{m+2}(M)$. Suppose that $E^m(\sigma)$ is a principal fibre bundle over M such that $j_0^{m+1} f \in E^m(\sigma)$. Then the following conditions are equivalent*:

(1) $j_0^{m+2} f \in E^{m+1}(\sigma)$;

(2) $J^{m+1} T(f)[R^{1(m)}(\sigma_0)_0] \subset R^{1(m)}(\sigma)_{f(0)}$.

Proof. We regard $j^m\sigma_0$, $j^m\sigma$ as fields on $\mathbf{R}^n$ and M respectively. Let us point out that

$$j_x^{m+1} f_*(\sigma_0) = j_x^{m+1}\sigma \quad \text{iff} \quad j_x^1(j^m f_*\sigma_0) = j_x^1(j^m\sigma)$$

where $x = f(0)$. Hence $j_0^{m+2} f \in E^{m+1}(\sigma)$ iff $\operatorname{im}(j^m f_*\sigma)$ and $\operatorname{im}(j^m\sigma)$ are 1-tangent submanifolds of $J^m F(M)$ at the point $j_x^m\sigma$. But this is equivalent to the fact that

$$T_\vartheta \operatorname{im}(j^m f_*\sigma_0) = T_\vartheta \operatorname{im}(j^m\sigma) \tag{4}$$

where $\vartheta = j_x^m\sigma$. Each vector from the spaces in equation (4) is the complete lift of an element from $J_x^{m+1} TM$. Therefore equality (4) is equivalent to

$$R^{1(m)}[f_*(\sigma_0)_0] = R^{1(m)}(\sigma)_x.$$

To end the proof it is enough to notice that

$$J^{m+1} T(f)\big(R^{1(m)}(\sigma)_0\big) = R^{1(m)}(f_*\sigma_0)_x.$$

Let us notice that though $J^{m+1} T$ is of order $m+2$ the natural bundle $\mathring{J}^{m+1} T$ is of order $m+1$. Hence any element $j_0^{m+1} f \in H^{m+1}(M)$ generates a well-defined mapping

$$J^{m+1} T(f)\colon \mathring{J}_0^{m+1} T\mathbf{R}^n \to \mathring{J}_{f(0)}^{m+1} TM.$$

LEMMA (2.5). *Let m be a non-negative integer. If $E^m(\sigma)$ is a principal fibre bundle over M and $j_0^{m+1} f \in E^{1(m)}(\sigma)_x$ then*

$$J^{m+1} T(f)[R^{1(m)}(\sigma_0)_0 \cap \mathring{J}_0^{m+1} T\mathbf{R}^n] \subset R^{1(m)}(\sigma)_{f(0)}.$$

Proof. Let $j_0^{m+1} X \in R^{1(m)}(\sigma_0)_0 \cap \mathring{J}_0^{m+1} T\mathbf{R}^n$. This means that $(L_X j^m\sigma_0)(0) = 0$ and $X_x = 0$. Hence

$$0 = f_*(L_X j^m\sigma_0)(x) = (L_{f_*X} j^m f_*\sigma_0)(x).$$

Since the operator L_{f_*X} is of order zero at x we have

$$(L_{f_*X} j^m\sigma)(x) = (L_{f_*X} j^m f_*\sigma_0)(x) = 0.$$

Hence $j_x^{m+1} f_* X \in R^{m+1}(j^m \sigma)_x$. Our lemma is now obvious since $R^{m+1}(j^m \sigma) = R^{1(m)}(\sigma)$ (cf. [17]).

II.2. *The vector bundle* $H^{k,2}(\sigma)$. We shall now recall the definition of the Spencer cohomology spaces associated with σ.

Let m, l be non-negative integers. Then we have the following sequence of functions:

$$(5)\quad \to g^m \otimes \bigwedge^{l-1} T^*M \xrightarrow{\partial^{m,l}} g^{m-1} \otimes \bigwedge^{l} T^*M \xrightarrow{\partial^{m-1,l+1}} g^{m-2} \otimes \bigwedge^{l+1} T^*M \to$$

where if $x \in M$, $\xi \in g_x^{m-1} \otimes \bigwedge^l T_x^* M$ and $w_1, \ldots, w_{l+1} \in T_x M$ then

$$(6)\qquad (\partial_x^{m,l} \xi)(w_1, \ldots, w_{l+1}) := \sum_{i=1}^{l+1} (-1)^{i+1} w_i \bullet \xi(w_1, \ldots, \hat{w}_i, \ldots, w_{l+1}).$$

The symbol $\bullet$ in formula (6) needs some explanation: if $a \in g^m$ and $w \in T_x M$ then $w \bullet a := \{\tilde{w}, a\}$ where $\tilde{w}$ is an element of $J_x^m TM$ such that $p_0^m(\tilde{w}) = w$ (cf. I.3). Using standard manipulations and the Jacobi identity one can prove that (5) is a complex (cf. [17]). Let

$$H^{m,l}(\sigma) := \ker \partial^{m,l} / \operatorname{im} \partial^{m+1,l-1}$$

be the induced cohomology spaces. We are particularly interested in the space $H^{k,2}(\sigma)$. It plays an important role in deciding whether σ is $(k+1)$-flat.

The field σ_0 is flat. Hence we may construct the space $H^{k,2}(\sigma_0)$. We shall show that $H^{k,2}(\sigma_0)_0$ is a standard fibre for the space $H^{k,2}(\sigma)$.

Let $x \in M$ and let $j_0^{k+1} f \in E^k(\sigma)_x$ be a jet witnessing k-flatness of σ, i.e. $j_x^k f_* \sigma_0 = j_x^k \sigma$. From Lemmas (2.4) and (2.5) we see that $j_0^{k+1} f$ induces isomorphisms between the following spaces: $g_0^k \sim g_x^k$, $g_0^{k-1} \sim g_x^{k-1}$, $g_0^{k-2} \sim g_x^{k-2}$. Set $\beta = j_0^{k+1} f$. Hence β induces the following isomorphisms:

$$\begin{aligned}
&[\beta]_0 \colon g_0^k \otimes \mathbf{R}^{n*} \to g_x^k \otimes T_x^* M, \\
(7)\qquad &[\beta]_1 \colon g_0^{k-1} \otimes \textstyle\bigwedge^2 \mathbf{R}^{n*} \to g_x^{k-1} \otimes \bigwedge^2 T_x^* M, \\
&[\beta]_2 \colon g_0^{k-2} \otimes \textstyle\bigwedge^3 \mathbf{R}^{n*} \to g_x^{k-2} \otimes \bigwedge^3 T_x^* M.
\end{aligned}$$

We are interested in the following result concerning the maps $[\beta]_0$, $[\beta]_1$ and $[\beta]_2$.

LEMMA (2.6). *The diagram*

$$\begin{tikzcd}
g_0^k \otimes \mathbf{R}^{n*} \arrow[r, "{[\beta]_0}"] \arrow[d, "\partial_0^{k+1,1}"] & g_x^k \otimes T_x^* M \arrow[d, "\partial_x^{k+1,1}"] \\
g_0^{k-1} \otimes \bigwedge^2 \mathbf{R}^{n*} \arrow[r, "{[\beta]_1}"] \arrow[d, "\partial_0^{k,2}"] & g_x^{k-1} \otimes \bigwedge^2 T_x^* M \arrow[d, "\partial_x^{k,2}"] \\
g_0^{k-2} \otimes \bigwedge^3 \mathbf{R}^{n*} \arrow[r, "{[\beta]_2}"] & g_x^{k-2} \otimes \bigwedge^3 T_x^* M
\end{tikzcd}$$

commutes.

Proof. We shall show that the upper rectangle commutes. The commutativity of the lower one can be shown in a similar way. Let $\xi \in g_0^k \otimes \mathbf{R}^{n^*}$ and $v, w \in T_x M$. Then

$$\begin{aligned}
&([\beta]_1 \circ \partial_0^{k+1,1}\xi)(v, w) \\
&= J_0^{k+1}(f)[\partial_0^{k+1,1}\xi]\big(d_x f^{-1}(v), d_x f^{-1}(w)\big) \\
&= J_0^{k+1}(f)\big(d_x f^{-1}(v) \bullet \xi(d_x f^{-1}(w))\big) - J^{k+1}\big(d_x f^{-1}(w) \bullet \xi(d_x f^{-1}(v))\big) \\
&= v \bullet J^m T\xi\big(d_x f^{-1}(w)\big) - w \bullet J^m T\xi\big(d_x f^{-1}(v)\big) \\
&= v \bullet [\beta]_0(\xi) w - w \bullet [\beta]_0(\xi) v \\
&= (\partial_x^{k+1,1} \circ [\beta]_0 \xi)(v, w).
\end{aligned}$$

COROLLARY (2.7). *The morphisms $\partial^{k+1,1}$, $\partial^{k,2}$ are of constant rank, i.e.* $\dim \operatorname{im} \partial_x^{k+1,1} = \text{const}$ *and* $\dim \ker \partial_x^{k,2} = \text{const}$.

COROLLARY (2.8). *The set $H^{k,2}(\sigma)$ is a smooth vector bundle.*

COROLLARY (2.9). *Each $(k+1)$-jet from the bundle $E^k(\sigma)$ canonically induces an isomorphism between $H^{k,2}(\sigma_0)_0$ and the respective fibre of $H^{k,2}(\sigma)$. If $\beta \in E^k(\sigma)_x$ then we shall denote that induced isomorphism by*

$$(8) \qquad [\beta]_3\colon H^{k,2}(\sigma_0)_0 \to H^{k,2}(\sigma)_x.$$

III. Structure tensor. As in the previous section we assume that F is a natural bundle of order one and σ is a k-flat field of geometric objects on M where $k \geqslant 0$. We denote by σ_0 a standard field of geometric objects on $\mathbf{R}^n$ such that for each $x \in M$ there exists a local diffeomorphism $f\colon (\mathbf{R}^n, 0) \to (M, x)$ such that $j_x^k f_* \sigma_0 = j_x^k \sigma$. Hence there exist principal fibre bundles $E^m(\sigma)$ of $(m+1)$-jets witnessing m-flatness of σ for $m \in \{0, \ldots, k\}$. There is a canonical projection

$$\mathrm{pr}\colon J_0^k T\mathbf{R}^n \to \mathring{J}_0^k T\mathbf{R}^n$$

such that $\mathrm{pr}(j_0^k X) = j_0^k(X - \tilde{X})$ where $\tilde{X} \equiv X_0$. There also exists a canonical inclusion $u\colon \mathbf{R}^n \to R^{1(k-1)}(\sigma)_0$. The existence of these mappings is a consequence of the fact that all the above spaces can be expressed canonically as direct sums of their suitable subspaces.

III.1. *Structure tensor of a field of geometric objects.* For a given $x \in M$ and $j_0^{k+1} f \in E^k(\sigma)_x$ we shall construct an element of the space $H^{k,2}(\sigma)$.

LEMMA (3.1). *There exists a linear mapping $\lambda\colon T_x M \to R^{1(k)}(\sigma)_x$ such that*

$$(9) \qquad p_0^{k+1} \circ \lambda \equiv \mathrm{id}_{T_x M}; \quad \mathrm{pr} \circ J^k T(f^{-1}) \circ p_k^{k+1} \circ \lambda \equiv 0.$$

Proof. Let $\lambda_1 := J^k T(f^{-1}) \circ u \circ d_x f^{-1}$; it has its values in $R^{1(k-1)}(\sigma)_x$ because

$$J^k T(f)[R^{1(k-1)}(\sigma)_0] \subset R^{1(k-1)}(\sigma)_x.$$

It is easy to notice that λ_1 is a section of the projection p_0^k. On the other hand, from Lemma (2.3) we see that

$$p_k^{k+1}\colon\ R^{1(k)}(\sigma)_x \to R^{1(k-1)}(\sigma)_x$$

is an epimorphism. Suppose that λ_2 is a section of this epimorphism. Finally, we define $\lambda = \lambda_1 \circ \lambda_2$ which satisfies (9).

Let λ be a linear mapping which satisfies (9). Then it induces the mapping $\tau_\lambda^k\colon\ T_xM \times T_xM \to g_x^{k-1}$ such that

$$\tau_\lambda^k(v, w) := \{\lambda(v), \lambda(w)\}$$

for $v, w \in T_xM$. The explicit definition of τ_λ^k implies that it has its values in $R^{1(k-1)}(\sigma)_x$ but

$$\begin{aligned} J^{k-1}T(f^{-1})p_{k-1}^k\{\lambda(v), \lambda(w)\} &= \{J^{k-1}T(f^{-1})p_k^{k+1}\lambda(v), J^{k-1}T(f^{-1})p_k^{k+1}\lambda(w)\} \\ &= \{u \circ d_x f^{-1}(v), u \circ d_x f^{-1}(w)\} = 0. \end{aligned}$$

Hence $p_{k-1}^k \circ \lambda \equiv 0$ and τ_λ^k has its values in g_x^{k-1}.

From the Jacobi identity one can easily find that $\tau_\lambda^k \in \ker \partial_x^{k,2}$. Therefore τ_λ^k defines an element in $H^{k,2}(\sigma)_x$. We repeat this construction for each $x \in M$ so we obtain a section

$$\tau^k\colon\ M \to H^{k,2}(\sigma).$$

DEFINITION (3.2). The section τ^k will be called the *k-th structure tensor* of the field σ of geometric objects.

Let us stress that the kth structure tensor was defined only for fields which are k-flat. We are to show that τ^k is well-defined, i.e. that τ^k does not depend on the choice of a jet $j_0^{k+1}f$ and a section λ. The independence on the choice of the jet will appear as Corollary (3.8).

LEMMA (3.3). *If a jet $j_0^{k+1}f \in E^k(\sigma)_x$ is fixed then $\tau^k(x)$ does not depend on the choice of a section λ satisfying* (9).

Let λ, $\tilde{\lambda}$ be two sections satisfying (9). Hence the mapping $\eta := \tilde{\lambda} - \lambda$ belongs to $g_x^k \otimes T_x^*M$. Then for $v, w \in T_xM$ we have

$$\begin{aligned} \tau_{\tilde{\lambda}}^k(v, w) - \tau_\lambda^k(v, w) &= \{\tilde{\lambda}(v), \tilde{\lambda}(w)\} - \{\lambda(v), \lambda(w)\} \\ &= \{\tilde{\lambda}(v), \eta(w)\} - \{\lambda(w), \eta(v)\} \\ &= v \bullet \eta(w) - w \bullet \eta(v) \\ &= (\partial_x^{k+1,1}\eta)(v, w). \end{aligned}$$

This precisely means that $\tau_{\tilde{\lambda}}^k - \tau_\lambda^k \in \operatorname{im} \partial_x^{k+1,1}$.

To show that $\tau^k(x)$ is independent of the choice of a jet we shall first recall the definition of the kth structure tensor of a G-structure. Then we shall compare that tensor with the one defined above. As a side result of this comparison we shall obtain the independence of $\tau^k(x)$ from the choice of the jet defining it. The construction of the kth structure tensor of a G-structure is taken from [3].

III.2. *The structure tensor of a G-structure.* From Proposition (1.6) we know that $E^0(\sigma)$ is m-flat and $E^m(\sigma)$ is the set of $(m+1)$-jets which witness the m-flatness of this G^0-structure where $m \in \{0, \ldots, k\}$ (cf. (2.1)). Let $\mathscr{G}$ denote the Lie algebra of the structure group of the bundle $E^0(\sigma)$. Then there are defined the prolongations $\mathscr{G}^m$ of the Lie algebra $\mathscr{G}$. It is known that the Lie algebra of the structure group of the bundle $E^m(\sigma)$ is of the form

$$\bigoplus_{i=0}^{m} \mathscr{G}^i$$

where $m \in \{0, \ldots, k\}$. On the space $E^k(\sigma)$ there is defined the fundamental 1-form Θ^k; if $\eta \in E^k(\sigma)$, $\eta = j_0^{k+1} f$, $\zeta = j_0^k f$ and $X^{k+1} \in T_\eta E^k(\sigma)$ then

$$\Theta_\eta^k(X^{k+1}) := d_\zeta H^k(f^{-1}) \circ d_\eta \pi_k^{k+1}(X^{k+1}). \tag{10}$$

The form Θ^k has its values in the space $T_{\mathcal{O}_k}(E^{k-1}(\sigma_0))$ where $\mathcal{O}_k = j_0^k \mathbf{1}_{\mathbf{R}^n}$. The space $T_{\mathcal{O}_k} E^{k-1}(\sigma_0)$ is identified, via the isomorphism $\Phi_{\mathcal{O}_k}$, with a subspace of $J_0^k T\mathbf{R}^n$, namely

$$\mathbf{R}^n \oplus \Big(\bigoplus_{i=0}^{k-1} \mathscr{G}^i\Big).$$

Then Θ^k may be expressed as a direct sum $\Theta^k = \omega + \Omega^0 + \ldots + \Omega^{k-1}$ such that ω has its values in $\mathbf{R}^n$ and Ω^i has its values in $\mathscr{G}^i$.

The *k-th structure tensor of* $E^0(\sigma)$ is a mapping c^k: $E^k(\sigma) \to H^{k,2}(\mathscr{G})$ such that $c^k(\eta)$ is a cohomology element defined by the bilinear mapping

$$c_\delta^k(\eta) := d\Theta^k \circ (\delta \wedge \delta) \circ (d_0 f \wedge d_0 f) \tag{11}$$

where δ is a linear section of the projection $d\pi_0^{k+1}$: $T_\eta E^k \to T_x M$ and $\Omega^m \circ \delta \equiv 0$ for $k-1 \geqslant m$ and $m \geqslant 0$.

There is a canonical representation ϱ: $L_n^{k+1} \to \mathrm{Aut}(J_0^k TM)$ such that

$$\varrho(j_0^{k+1} h) j_0^k X := j_0^k h_*^{-1} X$$

where $j_0^{k+1} h \in L_n^{k+1}$ and $j_0^k X \in J_0^k T\mathbf{R}^n$. This representation induces the representation ϱ^k: $G^k \to H^{k,2}(\mathscr{G})$. The structure tensor c^k is of the ϱ^k-tensorial type, i.e. for each $\eta \in E^k(\sigma)$ and $\alpha \in G^k$ we have

$$c^k(\eta\alpha) = \varrho^k(\alpha) c^k(\eta). \tag{12}$$

Let now Θ_0^k denote the fundamental 1-form on $H^{k+1}(M)$. The form Θ_0^k has its values in $T_{\mathcal{O}_k}(H^k(\mathbf{R}^n))$.

LEMMA (3.4) (Maurer–Cartan equations). *Suppose that* $\eta \in H^{k+1}(M)$, $\eta = j_0^{k+1} f$ *and* $X^{k+1}, Y^{k+1} \in T_\eta H^{k+1}(M)$. *Then*

$$2d\Theta_0^k(X^{k+1}, Y^{k+1}) = J^k T(f^{-1})\{\Phi_\eta^{-1}(X^{k+1}), \Phi_\eta^{-1}(Y^{k+1})\}.$$

Proof. There exist vector fields X, Y such that $\mathscr{H}^{k+1}(X)_\eta = X^{k+1}$ and $\mathscr{H}^{k+1}(Y)_\eta = Y^{k+1}$. Set

$$\tilde{X} := \mathscr{H}^{k+1}(X) \quad \text{and} \quad \tilde{Y} := \mathscr{H}^{k+1}(Y).$$

Then

$$2\Theta_0^k(\tilde{X}, \tilde{Y}) = (L_{\tilde{X}}\Theta_0^k)\tilde{Y} - (L_{\tilde{Y}}\Theta_0^k)\tilde{X} + \Theta_0^k([\tilde{X}, \tilde{Y}]).$$

Since the form Θ_0^k is invariant with respect to the natural liftings of diffeomorphisms from M to $H^{k+1}(M)$ we have

$$(L_{\tilde{X}}\Theta_0^k)\tilde{Y} \equiv (L_{\tilde{Y}}\Theta_0^k)\tilde{X} \equiv 0.$$

Therefore

$$\begin{aligned} 2(d\Theta_0^k)_\eta(X^{k+1}, Y^{k+1}) &= \Theta_0^k([\tilde{X}, \tilde{Y}]_\eta) \\ &= d_9 H^k(f^{-1}) \circ d\pi_k^{k+1}([\tilde{X}, \tilde{Y}]_\eta) \\ &= J^k T(f^{-1})\{\Phi_\eta^{-1}(X^{k+1}), \Phi_\eta^{-1}(Y^{k+1})\}. \end{aligned}$$

We would like to stress that we identify here $T_{e_{k+1}} H^{k+1}(M)$ with $J_0^k TM$.

III.3. *Flatness of fields of geometric objects.* We shall now compare the kth structure tensor of a field of geometric objects with the kth structure tensor of $E^0(\sigma)$. For given points $x \in M$, $\eta \in E^m(\sigma)_x$ and a section λ: $T_x M \to R^{1(k)}(\sigma)$ satisfying (9) we defined the bilinear form τ_λ^k; let us now further define $\delta := \Phi_\eta \circ \lambda$. The mapping δ appears to be a linear section of the projection $d\pi_0^{k+1}$ and $\Omega^m \circ \delta \equiv 0$ for $m \geqslant 0$ and $k-1 \geqslant m$. Hence the bilinear mapping c_δ^k generates the value of c^k at the point η.

In the second part of this paper we constructed the mapping $[\eta]_1$. We shall use this mapping to describe the relationship between c_δ^k and τ_λ^k.

LEMMA (3.5). *The following equality holds*: $2[\eta]_1 c_\delta^k = \tau_\lambda^k$.

Proof. Let $v, w \in T_x M$. Then

$$\begin{aligned} ([\eta]_1 c_\lambda^k)(v, w) &= J^k T(h) c_\delta^k(d_x h^{-1}(v), d_x h^{-1}(w)) \\ &= J^k T(h)[d\Theta^k(\delta(v), \delta(w))]. \end{aligned}$$

Now we apply the Maurer–Cartan equations, which hold for the form Θ^k too, and we get

$$\begin{aligned} 2J^k T(h)[d\Theta^k(\delta(v), \delta(w))] &= \{\Phi_\eta^{-1}(v), \Phi_\eta^{-1}(w)\} \\ &= \{\lambda(v), \lambda(w)\} = \tau_\lambda^k(v, w). \end{aligned}$$

COROLLARY (3.6). *The definition of $\tau^k(x)$ does not depend on the choice of a jet belonging to $E^k(\sigma)$.*

Proof. Let $\eta, \tilde{\eta} \in E^k(\sigma)_x$ and let $\tau^k(x)$, $\tilde{\tau}^k(x)$ denote the values of the structure tensors defined by η, $\tilde{\eta}$ respectively. From Lemma (3.5) we get

$$2[\eta]_3 c^k(\eta) = \tau^k(x) \quad \text{and} \quad 2[\tilde{\eta}]_3 c^k(\tilde{\eta}) = \tilde{\tau}^k(x)$$

where $[\eta]_3$, $[\tilde{\eta}]_3$ are the mappings generated by the jets η, $\tilde{\eta}$ (cf. (8)). There exists $\xi \in G^k$ such that $\tilde{\eta} = \eta\xi$. Now our statement is obvious since

$$\tilde{\tau}^k(x) = 2[\tilde{\eta}]_3 c^k(\tilde{\eta}) = 2[\eta\xi]_3 \varrho^k(\xi) c^k(\eta) = 2[\eta]_3 c^k(\eta) = \tau^k(x).$$

COROLLARY (3.7). *The tensor c^k vanishes at each point of the fibre $E^k(\sigma)_x$ iff $\tau^k(x) = 0$.*

COROLLARY (3.8). *The function τ^k is smooth.*

Proof. Let $\mathscr{S}$: $U \to E^k(\sigma)$ be a local section of the bundle $E^k(\sigma)$ defined on an open set U. Then for every $y \in U$ we have $\tau^k(y) = 2[\mathscr{S}(y)]_3 c^k(\mathscr{S}(y))$. Since c^k is smooth we see that τ^k is smooth.

THEOREM (3.9). *Assume that F is a natural bundle of order one and let σ be a k-flat field of geometric objects on M ($k \geqslant 0$). Then σ is $(k+1)$-flat iff $\tau^k \equiv 0$.*

Proof. From Proposition (1.6) we deduce that σ is k-flat iff $E^0(\sigma)$ is k-flat. By the main theorem from Guillemin's paper [7] we know that $E^0(\sigma)$ is $(k+1)$-flat iff c^k vanishes. Hence once again by Proposition (1.6) and by Corollary (3.7) we get our theorem.

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ISTITUTO DI MATEMATICA
UNIVERSITÀ DI SALERNO
84081 Baronissi (SA)
Italia

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