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Hille–Wintner type oscillation criteria for linear ordinary differential equations of second order

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Abstract. A new non-oscillation theorem is presented for linear ordinary differential equations of second order, implying, in particular, two classical results. An open question of Taam is answered in part as a result.

1. Introduction. The main purpose of this note is to present a new non-oscillation theorem, based on a known result due to Levin [3]. The theorem compares two linear ordinary differential equations of second order, and concludes that one of them is non-oscillatory if the other one is non-oscillatory and certain Hille–Wintner type conditions, relating the coefficients, hold on an interval. It is then shown that two classical results of Moore [4] and Leighton [2] follow as corollaries. In addition, an open question of Taam [6] regarding non-oscillation is partially settled as a corollary of the theorem.

2. Oscillation Theorems. We begin this section with a lemma which is the basis for all subsequent results. In this lemma, as well as throughout this section, the functions $a(x)$, $A(x)$, $c(x)$, and $C(x)$ are assumed to have the following properties: $A(x) \geqslant a(x) > 0$, $C(x)$ and $c(x)$ are continuous, and $A'(x)$ and $a'(x)$ are continuous, throughout the relevant intervals.

The following lemma is a trivial extension of a theorem of Levin [3]; [5], p. 34.

LEMMA. *Let u and v be non-trivial solutions of*

$$(au')' + cu = 0, \quad x \in [a, \beta], \tag{1}$$

$$(Av')' + Cv = 0, \quad x \in [a, \beta], \tag{2}$$

respectively, such that $u(x)$ does not vanish on $[a, \beta]$, and such that $v(a) \neq 0$. Moreover, let the inequality

$$-\frac{a(a)u'(a)}{u(a)} + \int_a^x c(t)\,dt \geqslant \left| -\frac{A(a)v'(a)}{v(a)} + \int_a^x C(t)\,dt \right| \tag{3}$$

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hold for all x on $[\alpha, \beta]$. Then $v(x)$ does not vanish on $[\alpha, \beta]$ and

$$-\frac{a(x)u'(x)}{u(x)} \geqslant \left|\frac{A(x)v'(x)}{v(x)}\right|, \quad x \in [\alpha, \beta]. \tag{4}$$

The preceding lemma leads to the following main theorem.

THEOREM 1. *Let* (1) *and* (2) *hold in the interval* $[a, \infty)$. *Moreover,*

(i) *Let v be a solution of* (2) *with* $v(a) \neq 0$, $v'(a) = 0$.

(ii) *Assume there exists a solution u of* (1) *such that* $u'(a) \leqslant 0$, *and* $u(x) > 0$ *for* $x \geqslant a$, *so that* (1) *is non-oscillatory.*

(iii) *Assume the inequality*

$$\int_a^x c(t)\,dt \geqslant \left|\int_a^x C(t)\,dt\right| \tag{5}$$

holds for $x \geqslant a$, where the integrals need not converge as $x \to \infty$.

Then $v(x)$ does not vanish for $x \geqslant a$, and we have

$$-\frac{a(x)u'(x)}{u(x)} \geqslant \left|\frac{A(x)v'(x)}{v(x)}\right|, \quad x \geqslant a. \tag{6}$$

Proof. We choose a solution of (2) such that $v'(a) = 0$, $v(a) \neq 0$. Condition (ii) of Theorem 1 guarantees that $-a(a)u'(a)/u(a) \geqslant 0$. Hence, using also (5), we see that (3) in the lemma is satisfied on every closed interval $[a, \beta]$. The conclusions of the lemma therefore hold on every such interval. In particular, (4) holds and hence also (6). Moreover, $v(a) \neq 0$ in every closed interval $[a, \beta]$ (and hence v is non-oscillatory). This completes the proof of Theorem 1.

We next show that Theorem 1 can be utilized to answer, in part, an open question in connection with a theorem of Taam [6], which is a generalization of Hille–Wintner's comparison theorem [1]; [7]; [5], Theorem 2.12.

THEOREM 2 (Taam). *Let $c(x)$ and $C(x)$ be integrable functions in $(0, \infty)$ such that*

$$\int_x^\infty c(t)\,dt \geqslant \left|\int_x^\infty C(t)\,dt\right|, \tag{7}$$

and both integrals converge for $x \geqslant a > 0$. Let $a(x)$ and $A(x)$ be as in the lemma and, moreover, let $a(x) \leqslant K$, a constant, on $(0, \infty)$. Then (2) *is non-oscillatory if* (1) *is non-oscillatory.*

In Taam's theorem, (7) replaces (5) of Theorem 1, and condition (ii) is replaced by the requirement $a(x) \leqslant K$. Swanson [5], p. 62, observes that it is an open question whether $a(x) \leqslant K$ is necessary for the conclusion of Taam's theorem. It is seen here that Theorem 1 does go without this condition. Moreover, it is easily seen that if $0 < \int_x^\infty C(t)\,dt$, the integral being convergent, then (7) implies (5).

For suppose that for each a_k there is an a_{k+1} such that

$$\int_{a_k}^{a_{k+1}} c(t)\,dt < \int_{a_k}^{a_{k+1}} C(t)\,dt, \tag{8}$$

where $a_0 = a$. Let $\gamma = \sup_k \{a_k\}$. If $\gamma < \infty$, then

$$\int_\gamma^x c(t)\,dt \geqslant \int_\gamma^x C(t)\,dt \tag{9}$$

for all $x > \gamma$, and we may take $a = \gamma$ in (5). On the other hand, if $\gamma = \infty$, we sum (8) on k and get

$$\int_a^\infty c(t)\,dt < \int_a^\infty C(t)\,dt, \tag{10}$$

contradicting (7). Hence (7) implies (5) under this circumstance. But note that Theorem 1 does not require the convergence of the integrals in (5).

Finally, we observe that condition (ii) is certainly not vacuous. For example, if $\int_a^\infty c(t)\,dt = \infty$, then $u'/u < 0$ for sufficiently large x [5], Theorem 2.40.

Next we show that Theorem 1 implies a slightly different version of the classical non-oscillation theorem of Moore [4]; [5], p. 73.

COROLLARY 1. *If*

$$\int_1^\infty \frac{dt}{A(t)} < \infty \quad \textit{and} \quad 0 < \limsup_{x\to\infty} \int_1^x C(t)\,dt < \infty, \tag{11}$$

then (2) *is non-oscillatory.*

To prove the corollary, we associate with (2) the equation

$$(Au')' + \frac{k^2}{A}u = 0, \tag{12}$$

with a solution

$$u(x) = \cos\left[k\int_a^x \frac{dt}{A(t)}\right]. \tag{13}$$

If we can find a and $k = k(a)$ such that

$$0 \leqslant k\int_a^x \frac{dt}{A(t)} < \frac{\pi}{2}, \tag{14}$$

as well as

$$k^2 \int_a^x \frac{dt}{A(t)} \geqslant \int_a^x C(t)\,dt \tag{15}$$

for all $x \geqslant a$, then u satisfies the conditions of Theorem 1, and hence the solution v of (2) is non-oscillatory. To this end, define

$$k(a) = \frac{\pi}{2\int_a^\infty \frac{dt}{A(t)}}. \tag{16}$$

Evidently, $k(a)$ is monotonically increasing with a and is unbounded. If no $k(a)$ satisfies (15), there is a sequence $\{a_n\}$ such that

$$k^2(a_n) \int_{a_n}^{a_{n+1}} \frac{dt}{A(t)} < \int_a^\infty C(t)\,dt. \tag{17}$$

If $\{a_n\}$ is bounded, then $\sup_n \{a_n\} = a_{n_0}$, and $k_0 = k(a_{n_0})$ satisfies (15). If $\{a_n\}$ is unbounded, we sum (17) on n, to get

$$\sum_{n=0}^\infty k^2(a_n) \int_{a_n}^{a_{n+1}} \frac{dt}{A(t)} < \int_a^{a_{n+1}} C(t)\,dt, \tag{18}$$

where $a_0 = a$. Since $k(a)$ is monotonic, (18) leads to

$$k^2(a) \int_a^\infty \frac{dt}{A(t)} < \int_a^\infty C(t)\,dt, \tag{19}$$

which is true for all a. Putting (16) in (19), we find that $k(a)$ is bounded, contradicting (16). This completes the proof of Corollary 1.

In a similar manner we show that Theorem 1 implies the classical oscillation theorem of Leighton [2]; [5], p. 70.

COROLLARY 2. *If*

$$\int_1^\infty \frac{dt}{a(t)} = \int_1^\infty c(t)\,dt = \infty, \tag{20}$$

then (1) *is oscillatory.*

To prove the corollary, consider

$$(av')' + \frac{k^2}{a} v = 0, \tag{21}$$

with a solution

$$v(x) = \cos\left[k \int_a^x \frac{dt}{a(t)}\right]. \tag{22}$$

We proceed indirectly. Suppose (1) is non-oscillatory. From (20) it follows, as before, that $u'/u < 0$ for sufficiently large x. Hence (ii) of Theorem 1 is fulfilled. If we can show that for a given a there is a β and a $k = k(a, \beta)$ such that, say,

$$k(a, \beta) \int_a^\beta \frac{dt}{a(t)} = \pi, \tag{23}$$

as well as

$$k^2(a, \beta) \int_a^x \frac{dt}{a(t)} \leqslant \int_a^x c(t)\,dt \tag{24}$$

for all $x \in [a, \beta]$, then Theorem 1 would show that $v(x) \neq 0$ for $x \in [a, \beta]$, because of (23), contradicting (22). Thus the non-oscillation of (1) would be contradicted, proving the corollary. To show that β and $k(a, \beta)$ can be found, define

$$k(a, \beta) \int_a^\beta \frac{dt}{a(t)} = \pi, \tag{25}$$

i.e. let (25) define $k(a, \beta)$ for a given a and each β. Suppose (24) cannot be realized, i.e. suppose for each given a there is an a_1 such that

$$k^2(a, \beta) \int_a^{a_1} \frac{dt}{a(t)} > \int_a^{a_1} c(t)\,dt \tag{26}$$

for some $a_1 \in [a, \beta]$. Then if we substitute (25) in (26) we obtain

$$k(a, \beta) > \frac{1}{\pi} \int_a^{a_1} c(t)\,dt, \tag{27}$$

where the right-hand side is independent of β. Letting $\beta \to \infty$ in (25) and using (20), we find that $k(a, \beta) \to 0$, contradicting (27). This completes the proof of Corollary 2.

References

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