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On a generalization of Hille's functional equation

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Abstract. The purpose of this paper is to solve a generalization of Hille's functional equation using a lemma which is obtained from Nevanlinna–Pólya Theorem.

1. Hille (see [5], [4], [1], p. 19 and p.31) solved the following functional equation:

$$|f(s+it)|^2 = |f(s)|^2 + |f(it)|^2, \tag{1}$$

where $f = f(z)$ is an entire function of a complex variable z and s, t are real variables.

A simple calculation shows that (1) implies the following functional equation:

$$|f(x+y)|^2 + |f(x-y)|^2 = |f(x+\bar{y})|^2 + |f(x-\bar{y})|^2, \tag{2}$$

where $f = f(z)$ is an entire function of z and x, y are complex variables.

The following theorem was proved in [1], p. 30–31, [2]:

THEOREM A. *If $f = f(z)$ is an entire function of z, then the only solutions of* (2) *are $f(z) = az + b$ and $f(z) = a\sin az + b\cos az$, where a, b are arbitrary complex constants and a is a real or purely imaginary constant.*

Now we consider the following functional equation:

$$\sum_{k=0}^{n-1} |f(x+\omega^k y)|^2 = \sum_{k=0}^{n-1} |f(x+\omega^k \bar{y})|^2, \tag{3}$$

where x, y are complex variables, $f = f(z)$ is an entire function of z and ω denotes the complex number $\exp(2\pi i/n)$. Here n is a positive integer greater than 1.

It is obvious that (2) results from (3) with $n = 2$.

The purpose of this note is to solve (3), i.e., to prove the following

THEOREM. *If $f = f(z)$ is an entire function of z, then the only solutions of* (3) *are $f(z) = \sum_{k=0}^{n-1} a_k z^k$ and $f(z) = \sum_{k=0}^{n-1} a_k \exp(\omega^k az)$, where for each $k = 0, 1, 2, \ldots, n-1$, a_k is an arbitrary complex constant and a is an arbitrary real constant or $a = c\exp(\pi i/n)$, where c is an arbitrary real constant.*

2. To prove the theorem in Section 1, we shall apply the following four lemmas.

LEMMA 1. *Let n (>1) be an arbitrary positive integer. If ω denotes the complex number* $\exp(2\pi i/n)$, *the sum* $\sum_{k=0}^{n-1} \omega^{km}$ *has the value n or 0 according as m is or is not a multiple of n.*

Proof. Since the proof is easy, we omit it (cf. [10], p. 924).

LEMMA 2. *If $f=f(z)$ is regular in a non-empty domain D, then $\Delta|f(z)|^2 = 4|f'(z)|^2$, where Δ stands for the Laplacian $\partial^2/\partial s^2+\partial^2/\partial t^2$ $(z=s+it,\ s,\ t$ real$)$ holds in D.*

Proof. See [7], p. 94.

Before we state Lemma 3, we state the following

THEOREM B (Nevanlinna–Pólya Theorem). *Let n (>1) be an arbitrarily fixed positive integer and let D be a non-empty domain. Suppose that $f_1=f_1(z)$, $f_2=f_2(z)$, $f_3=f_3(z)$, $\ldots$, $f_n=f_n(z)$; $g_1=g_1(z)$, $g_2=g_2(z)$, $g_3=g_3(z)$, $\ldots$, $g_n=g_n(z)$ are regular in D. Suppose further that $f_1=f_1(z)$, $f_2=f_2(z)$, $f_3=f_3(z)$, $\ldots$, $f_n=f_n(z)$ are linearly independent in D and that $f_1=f_1(z)$, $f_2=f_2(z)$, $f_3=f_3(z)$, $\ldots$, $f_n=f_n(z)$; $g_1=g_1(z)$, $g_2=g_2(z)$, $g_3=g_3(z)$, $\ldots$, $g_n=g_n(z)$ satisfy $\sum_{k=1}^{n}|f_k(z)|^2=\sum_{k=1}^{n}|g_k(z)|^2$ in D. Then there exists an $n\times n$ unitary matrix*

$$(a_{kl}), \quad k=1,2,3,\ldots,n;\ l=1,2,3,\ldots,n,$$

such that for each $k=1,2,3,\ldots,n$, $g_k(z)=\sum_{l=1}^{n}a_{kl}f_l(z)$ holds in D. Here, for each pair of $k=1,2,3,\ldots,n$, $l=1,2,3,\ldots,n$, a_{kl} is a complex constant.

Proof. See [3], [6], [8].

We may now prove the following

LEMMA 3. *Let n (>1) be an arbitrarily fixed positive integer and let D be a domain including 0. Suppose that for each $k=1,2,3,\ldots,n$, $f_k(z)$ and $g_k(z)$ are regular in D and that for each $k=1,2,3,\ldots,n$, $f_k(0)=g_k(0)=0$. Suppose further that $f_1=f_1(z)$, $f_2=f_2(z)$, $f_3=f_3(z)$, $\ldots$, $f_n=f_n(z)$ are linearly independent in D and that $f_1=f_1(z)$, $f_2=f_2(z)$, $f_3=f_3(z)$, $\ldots$, $f_n=f_n(z)$; $g_1=g_1(z)$, $g_2=g_2(z)$, $g_3=g_3(z)$, $\ldots$, $g_n=g_n(z)$ satisfy $\sum_{k=1}^{n}|f_k'(z)|^2=\sum_{k=1}^{n}|g_k'(z)|^2$ in D. Then $\sum_{k=1}^{n}|f_k(z)|^2=\sum_{k=1}^{n}|g_k(z)|^2$ holds in D.*

Proof. By hypothesis and by Theorem B there exists a unitary matrix

$$(a_{kl}),\ k=1,2,3,\ldots,n;\ l=1,2,3,\ldots,n,$$

such that for each $k = 1, 2, 3, \ldots, n$,

$$g_k'(z) = \sum_{l=1}^{n} a_{kl} f_l'(z) \tag{4}$$

holds in D. Here, for each pair of $k = 1, 2, 3, \ldots, n$, $l = 1, 2, 3, \ldots, n$, a_{kl} is a complex constant.

Integrating both sides of (4) from 0 to z along a contour and using the hypothesis that for each $k = 1, 2, 3, \ldots, n$, $f_k(0) = g_k(0) = 0$ yields, for each $k = 1, 2, 3, \ldots, n$, in D

$$g_k(z) = \sum_{l=1}^{n} a_{kl} f_l(z). \tag{5}$$

Since the matrix (a_{kl}) is unitary, by (5) we have $\sum_{k=1}^{n} |f_k(z)|^2 = \sum_{k=1}^{n} |g_k(z)|^2$ in D. Q.E.D.

LEMMA 4. *Let n be an arbitrary positive integer. If for each $k = 1, 2, 3, \ldots, n$, A_k is a point set in the finite complex plane which has no accumulation point, then the set $\bigcup_{k=1}^{n} A_k$ has no accumulation point.*

Proof. Since the proof is easy, we omit it.

3. We may now prove the theorem in Section 1.

First, we shall prove that an arbitrary polynomial of degree at most $n-1$ is a solution of (3). Let $f = f(z)$ be an arbitrary polynomial of degree at most $n-1$. Then we can put

$$f(x+y) = \sum_{j=0}^{n-1} A_j(x) y^j$$

for all complex x, y.

Hence, by Lemma 1 we have

$$\begin{aligned}
(6) \quad \sum_{k=0}^{n-1} |f(x+\omega^k y)|^2 &= \sum_{k=0}^{n-1} f(x+\omega^k y)\overline{f(x+\omega^k y)} \\
&= \sum_{k=0}^{n-1} \Big(\sum_{j=0}^{n-1} A_j(x)(\omega^k y)^j \sum_{l=0}^{n-1} \overline{A_l(x)(\omega^k y)^l} \Big) \\
&= \sum_{k=0}^{n-1} \Big(\sum_{j=0}^{n-1} \sum_{l=0}^{n-1} A_j(x)\overline{A_l(x)}\, \omega^{k(j-l)} y^j \bar{y}^l \Big) \\
&= \sum_{j=0}^{n-1} \sum_{l=0}^{n-1} \Big(A_j(x)\overline{A_l(x)}\, y^j \bar{y}^l \sum_{k=0}^{n-1} \omega^{k(j-l)} \Big) \\
&= n \sum_{j=0}^{n-1} |A_j(x)|^2 |y|^{2j}.
\end{aligned}$$

Replacing y by $\bar{y}$ in (6) yields that $f = f(z)$ is a solution of (3).

We may now assume that $f = f(z)$ is an entire function which is not a polynomial of degree at most $n-1$.

Keeping y arbitrarily fixed and taking the Laplacians $\partial^2/\partial s^2 + \partial^2/\partial t^2$ of both sides of (3) with respect to $x = s+it$ (s, t real), by Lemma 2 we have

$$4\sum_{k=0}^{n-1}|f'(x+\omega^k y)|^2 = 4\sum_{k=0}^{n-1}|f'(x+\omega^k\bar{y})|^2,$$

or

$$\sum_{k=0}^{n-1}|f'(x+\omega^k y)|^2 = \sum_{k=0}^{n-1}|\overline{f'(x+\omega^k\bar{y})}|^2. \tag{7}$$

When x is arbitrarily fixed, for each $k = 0, 1, 2, \ldots, n-1$, $f(x+\omega^k y) - f(x)$, $\overline{f(x+\omega^k\bar{y})-f(x)}$ are entire functions of y with $(f(x+\omega^k y)-f(x))_{y=0} = (\overline{f(x+\omega^k\bar{y})-f(x)})_{y=0} = 0$. Moreover, by (7) we have in $|y| < +\infty$

$$\sum_{k=0}^{n-1}|(\partial/\partial y)(f(x+\omega^k y)-f(x))|^2 = \sum_{k=0}^{n-1}|(\partial/\partial y)\overline{(f(x+\omega^k\bar{y})-f(x))}|^2,$$

since for each $k = 0, 1, 2, \ldots, n-1$, $|\omega^k| = \overline{|\omega^k|} = 1$.

Further, we shall prove that when x is arbitrarily fixed, $f(x+y)-f(x)$, $f(x+\omega y)-f(x)$, $f(x+\omega^2 y)-f(x), \ldots, f(x+\omega^{n-1}y)-f(x)$ are linearly independent in $|y| < +\infty$. To this end, we assume that

$$\sum_{k=0}^{n-1}C_k(x)(f(x+\omega^k y)-f(x)) = 0, \tag{8}$$

where for each $k = 0, 1, 2, \ldots, n-1$, $C_k(x)$ does not depend on y and is a function of x only.

Differentiating both sides of (8) l times with respect to y and putting $y = 0$ in the resulting equality yields for each $l = 1, 2, 3, \ldots, n$

$$f^{(l)}(x)\sum_{k=0}^{n-1}\omega^{kl}C_k(x) = 0. \tag{9}$$

Let

$$D = \Big\{x \mid \prod_{l=1}^{n} f^{(l)}(x) \neq 0\Big\}. \tag{10}$$

Since, by our assumption, f is not a polynomial of degree at most $n-1$, we have $\prod_{l=1}^{n} f^{(l)}(x) \not\equiv 0$. Hence, by (10) D is a non-empty domain.

By (9), (10) we have for each x belonging to D and for each $l = 1, 2, 3, \ldots, n$

$$\sum_{k=0}^{n-1}\omega^{kl}C_k(x) = 0. \tag{11}$$

Since, by Vandermonde's Identity in the theory of determinants (see [9], p. 102), the determinant of system (11) of linear equations for $C_0(x), C_1(x), C_2(x), \ldots, C_{n-1}(x)$ does not vanish, we have $C_0(x) = 0$, $C_1(x) = 0, C_2(x) = 0, \ldots, C_{n-1}(x) = 0$ for each x belonging to D. Hence, when x is arbitrarily fixed in $D, f(x+y)-f(x), f(x+\omega y)-f(x), f(x+\omega^2 y)-$ $-f(x), \ldots, f(x+\omega^{n-1}y)-f(x)$ are linearly independent in $|y| < +\infty$.

Since all hypotheses of Lemma 3 are satisfied, by Lemma 3 we have

$$\sum_{k=0}^{n-1} |f(x+\omega^k y)-f(x)|^2 = \sum_{k=0}^{n-1} |f(x+\omega^k \bar{y})-f(x)|^2 \tag{12}$$

in $|y| < +\infty$ and for each x belonging to D. Taking into account the fact that, by our assumption, $f = f(z)$ is not a polynomial of degree at most $n-1$, by a famous theorem in analytic function theory for each $l = 1, 2, 3, \ldots, n$, the set $\{x | f^{(l)}(x) = 0\}$ has no accumulation point. Hence, by Lemma 4 the set $S = \bigcup_{l=1}^{n} \{x | f^{(l)}(x) = 0\}$ has no accumulation point. Hence each point of S is an isolated point. Therefore, by the continuity of f we see that (12) holds not only for each x belonging to D, but also holds for each x belonging to $\bigcup_{l=1}^{n} \{x | f^{(l)}(x) = 0\}$. Thus, (12) holds for all complex x, y.

Subtracting (12) from (3) side by side and using the formula $|a-b|^2 = |a|^2 + |b|^2 - 2\,\mathrm{Re}(a\bar{b})$ (a, b complex) yields

$$2\sum_{k=0}^{n-1} \mathrm{Re}\left(f(x+\omega^k y)\overline{f(x)}\right) = 2\sum_{k=0}^{n-1} \mathrm{Re}\left(f(x+\omega^k \bar{y})\overline{f(x)}\right). \tag{13}$$

Using the formula $\mathrm{Re}(\gamma) = \mathrm{Re}(\bar{\gamma})$ (γ complex) in (13) yields

$$\mathrm{Re}\left(\sum_{k=0}^{n-1} \left(f(x+\omega^k y)\overline{f(x)} - \overline{f(x+\omega^k \bar{y})} f(x)\right)\right) = 0. \tag{14}$$

When x is arbitrarily fixed, for each $k = 0, 1, 2, \ldots, n-1$, $f(x+\omega^k y)\,\overline{f(x)} - \overline{f(x+\omega^k \bar{y})} f(x)$ is an entire function of y. Hence, by (14) and by a famous theorem in analytic function theory we have

$$\sum_{k=0}^{n-1} \left(f(x+\omega^k y)\overline{f(x)} - \overline{f(x+\omega^k \bar{y})} f(x)\right) = A(x), \tag{15}$$

where $A(x)$ is a function of x only.

Putting $y = 0$ in (15) yields $A(x) = 0$ for each complex x. Hence, by (15) we have for all complex x, y

$$\overline{f(x)} \sum_{k=0}^{n-1} f(x+\omega^k y) = f(x) \sum_{k=0}^{n-1} \overline{f(x+\omega^k \bar{y})}. \tag{16}$$

Differentiating both sides of (16) n times with respect to y, using the formula $\omega^n = 1$ and putting $y = 0$ in the resulting equality yields for all complex x

$$nf^{(n)}(x)\overline{f(x)} = nf(x)\overline{f^{(n)}(x)},$$

or

$$f^{(n)}(x)\overline{f(x)} = f(x)\overline{f^{(n)}(x)}. \tag{17}$$

By our assumption we have $f(x) \not\equiv 0$. If we put $E = \{x \mid f(x) \neq 0\}$, then E is a non-empty domain. By (17) we have in E

$$f^{(n)}(x)/f(x) = \overline{f^{(n)}(x)/f(x)}.$$

Hence the regular function $f^{(n)}(x)/f(x)$ in E is real-valued. Hence, by a famous theorem in analytic function theory we have in E

$$f^{(n)}(x)/f(x) = A,$$

where A is a real constant, and so

$$f^{(n)}(x) = Af(x). \tag{18}$$

Since $f = f(z)$ is an entire function which is not a polynomial of degree at most $n-1$, by the Identity Theorem (18) holds for all complex x, A being a non-zero real constant.

Solving (18), we have

$$f(z) = \sum_{k=0}^{n-1} a_k \exp(\omega^k \alpha z), \tag{19}$$

where for each $k = 0, 1, 2, \ldots, n-1$, a_k is a complex constant and α is a real constant or $\alpha = c\exp(\pi i/n)$, where c is a real constant.

We shall prove that (19) is a solution of (3). By (19) we have

$$\begin{aligned}\sum_{k=0}^{n-1} |f(x+\omega^k y)|^2 &= \sum_{k=0}^{n-1} f(x+\omega^k y)\overline{f(x+\omega^k y)} \\ &= \sum_{k=0}^{n-1}\left(\sum_{j=0}^{n-1} a_j \exp(\omega^j \alpha(x+\omega^k y))\sum_{l=0}^{n-1} \bar{a}_l \overline{\exp(\omega^l \alpha(x+\omega^k y))}\right) \\ &= \sum_{j=0}^{n-1}\sum_{l=0}^{n-1}\left(a_j \bar{a}_l \sum_{k=0}^{n-1} \exp(\omega^j \alpha x + \omega^{-l}\bar{\alpha}\bar{x} + \omega^{j+k}\alpha y + \omega^{-l-k}\bar{\alpha}\bar{y})\right).\end{aligned} \tag{20}$$

We discuss two cases.

Case 1. α is a real constant.

Replacing y by $\bar{y}$ in (20) and observing that $\bar{a} = a$ together with

$$\sum_{k=0}^{n-1} \exp(\omega^j ax + \omega^{-l} a\bar{x} + \omega^{j+k} ay + \omega^{-l-k} a\bar{y})$$

$$= \sum_{k=0}^{n-1} \exp(\omega^j ax + \omega^{-l} a\bar{x} + \omega^{j+k} a\bar{y} + \omega^{-l-k} ay)$$

$$(\{\omega^{j+k} \mid k \in Z, 0 \leqslant k \leqslant n-1\} = \{\omega^{-l-k} \mid k \in Z, 0 \leqslant k \leqslant n-1\}$$

$$= \{1, \omega, \omega^2, \ldots, \omega^{n-1}\})$$

yields that (19) is a solution of (3).

Case 2. $a = c\exp(\pi i/n)$, where c is a real constant.

Since $\bar{a} = c\exp(-\pi i/n) = a/\omega$, by a similar calculation to that in Case 1 (19) is a solution of (3). Q.E.D.

Remark. It is obvious that Theorem A in Section 1 results from the theorem proved in this section with $n = 2$.

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