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Paley–Wiener theorems for the Mellin transformation

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Abstract. The paper is devoted to establishing theorems on the Mellin transforms of Mellin distributions analogous to the Paley–Wiener–Schwartz theorems on the Fourier transforms of distributions with bounded supports and of functions in C_0^∞. The role of C_0^∞-functions is being played by a subset, defined by boundary conditions (6), of the space $\mathfrak{M}_{[-1]}$ of Mellin multipliers. Section 4 contains a characterization of the set of Mellin distributions.

The paper is strictly connected with papers [5] and [6] by B. Ziemian, and Section 3 contains some of his unpublished results.

1. Notation and basic facts on the Mellin transformation. Throughout the paper, we use the following vector notation: if $a, b \in \boldsymbol{R}^n$, $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$, then $a < b$ (resp. $a \leqslant b$) denotes $a_j < b_j$ (resp. $a_j \leqslant b_j$) for $j = 1, \ldots, n$.

We denote $\boldsymbol{R}^n_+ = \{x \in \boldsymbol{R}^n\colon 0 < x\}$, $J = (0, r] = \{x \in \boldsymbol{R}^n_+\colon x \leqslant r\}$ for some fixed $r = (r_1, \ldots, r_n) \in \boldsymbol{R}^n_+$. $\boldsymbol{N}_0$ is the set of non-negative integers, $|\alpha| = \alpha_1 + \ldots + \alpha_n$ for $\alpha \in \boldsymbol{N}_0^n$. $\mathbf{1} = (1, \ldots, 1) \in \boldsymbol{R}^n_+$.

If $x \in \boldsymbol{R}^n_+$, $z = (z_1, \ldots, z_n) \in \boldsymbol{C}^n$ we write

$$x^z = x_1^{z_1} \ldots x_n^{z_n}.$$

Vector notation is also used for differentiation, namely, for $\nu \in \boldsymbol{N}_0^n$,

$$D^\nu = D_1^{\nu_1} \ldots D_n^{\nu_n} = \frac{\partial^{\nu_1}}{\partial x_1^{\nu_1}} \ldots \frac{\partial^{\nu_n}}{\partial x_n^{\nu_n}},$$

$$\tilde{D}^\nu = \tilde{D}_1^{\nu_1} \ldots \tilde{D}_n^{\nu_n} = \left(x_1 \frac{\partial}{\partial x_1}\right)^{\nu_1} \ldots \left(x_n \frac{\partial}{\partial x_n}\right)^{\nu_n}.$$

We use the notation commonly used in the theory of generalized functions. The value of a generalized function u on a test function φ is denoted by $u[\varphi]$.

Following [5] we recall the definition of the Mellin transform of a distribution and some of its properties which will be useful in the sequel.

Let $a \in \boldsymbol{R}^n$, $J = (0, r]$. By $\mathfrak{M}_a = \mathfrak{M}_a(J)$ we denote the complex vector space of functions $\varphi \in C^{\infty}(J)$ such that [1]

$$\varrho_{a\alpha}(\varphi) = \sup_{x \in J} \left| x^{a+\alpha+1} \frac{\partial^{\alpha}}{\partial x^{\alpha}} \varphi(x) \right| < \infty \quad \text{for } \alpha \in N_0^n$$

with the topology given by the seminorms $\varrho_{a\alpha}$, $\alpha \in N_0^n$. For any $\omega \in (R \cup \{+\infty\})^n$ we define the inductive limit

$$\mathfrak{M}_{(\omega)} = \bigcup_{\overrightarrow{a < \omega}} \mathfrak{M}_a.$$

By $\mathfrak{M}_a^m(J)$ we denote the complex vector space of functions $\varphi \in C^m(J)$ such that $\varrho_{a\alpha}(\varphi) < \infty$ for $|\alpha| \leqslant m$ with the norm $\sum_{|\alpha| \leqslant m} \varrho_{a\alpha}(\varphi)$.

The space $\mathfrak{M}'_a(J)$ denotes the dual of $\mathfrak{M}_a(J)$ [2]. $u \in \mathfrak{M}'_a$ if and only if there exist $m \in N_0$ and $C < \infty$ such that

$$|u[\varphi]| \leqslant C \sum_{|\alpha| \leqslant m} \varrho_{a\alpha}(\varphi) \quad \text{for } \varphi \in \mathfrak{M}_a(J). \tag{1}$$

We say that a generalized function $u \in \mathfrak{M}'_a$ is *of Mellin order* $\leqslant m$ if (1) holds. The dual of $\mathfrak{M}_a^m$ denoted by $(\mathfrak{M}_a^m)'$ is isomorphic to the subspace of $\mathfrak{M}'_a$ formed by generalized functions of Mellin order $\leqslant m$.

Denote by $C_0^{\infty}(J)$ the linear space of restrictions to J of $C_0^{\infty}(R_+^n)$ functions, by $\mathfrak{D}(J)$ the space $C_0^{\infty}(J)$ with the topology induced by $\mathfrak{D}(R_+^n)$.

Denote by $\mathfrak{D}'(J)$ and $\mathfrak{M}'_{(\omega)}(J)$ the dual spaces of $\mathfrak{D}(J)$ and $\mathfrak{M}_{(\omega)}$, respectively.

From the definitions given above, the following topological [3] inclusions follow easily

$$\mathfrak{D}(J) \subset \mathfrak{M}_{(a)} \subset \mathfrak{M}_a \subset \mathfrak{M}_{(\omega)}, \quad \mathfrak{M}'_{(\omega)} \subset \mathfrak{M}'_a \subset \mathfrak{M}'_{(a)} \subset \mathfrak{D}'(J) \quad \text{for} \quad a < \omega. \tag{2}$$

It turns out that for every ω, $C_0^{\infty}(J)$ is a dense subset of $\mathfrak{M}_{(\omega)}$, thus $\mathfrak{M}'_{(\omega)}$ is a subspace of $\mathfrak{D}'(J)$.

Let

$$\mathfrak{D}'_J(R_+^n) = \{u \in \mathfrak{D}'(R_+^n) : \operatorname{supp} u \subset J\}$$

and note the following proposition:

PROPOSITION 1. *There exists a linear isomorphism* L

$$L\colon \mathfrak{D}'_J(R_+^n) \ni u \mapsto Lu \in \mathfrak{D}'(J)$$

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[1] The space $\mathfrak{M}_a$ can be equivalently defined by the system $\{\tilde{\varrho}_{a\alpha}\}_{\alpha \in N_0^n}$ of seminorms, where $\tilde{\varrho}_{a\alpha}(\varphi) = \sup_{x \in J} |x^{a+1} \tilde{D}^{\alpha} \varphi(x)|$, $\varphi \in C^{\infty}(J)$.

[2] In the sequel we often omit the symbol J of a basic cube, because it remains fixed throughout the paper.

[3] This means that the convergence of a sequence in the smaller space implies the convergence in the bigger space.

defined by: $(Lu)[\varphi] = u[\tilde{\varphi}]$, *where* $\varphi \in C_0^\infty(J)$ *and* $\tilde{\varphi} \in C_0^\infty(\boldsymbol{R}_+^n)$ *is an extension of* φ.

Proof. To see that this formal definition defines correctly $Lu \in \mathfrak{D}'(J)$ it is necessary to verify the implication: $\Psi \in C_0^\infty(\boldsymbol{R}_+^n)$, $\Psi = 0$ on J implies $u[\Psi] = 0$. To this end, take $\sigma \in C_0^\infty(\boldsymbol{R}_+^n)$, $\sigma = 1$ on $\operatorname{supp}\Psi$. Then $\Psi = \sigma\Psi$, $u[\Psi] = (\sigma u)[\Psi]$ and Ψ is k-flat on $\operatorname{supp}\sigma u$ for every $k \in \boldsymbol{N}$. Thus, by Theorem 7.4 of [3], $(\sigma u)[\Psi] = 0$, and hence $u[\Psi] = 0$. The end of the proof of Proposition 1 is now clear.

In view of the isomorphism given by Proposition 1 one often writes u instead of Lu.

COROLLARY 1. *For every* $\omega \in (\boldsymbol{R}_+ \cup \{+\infty\})^n$, $\mathfrak{M}'_{(\omega)}$ *is a subspace of the space of distributions* $\mathfrak{D}'_J$.

We say that $u \in \mathfrak{M}'_{(\omega)}$ is *of Mellin order* $\leqslant m$, $m \in \boldsymbol{N}_0$ if for every $a < \omega$ there exists a constant $C = C_a < +\infty$ such that (1) holds. The space

$$\mathfrak{M}'(J) = \bigcup_{\omega \in (\boldsymbol{R} \cup \{+\infty\})^n} \mathfrak{M}'_{(\omega)}(J)$$

is called the space of *Mellin distributions*(4).

By Corollary 1 every Mellin distribution is a distribution. It is important to note that the notion of Mellin order $\leqslant m$ differs from the analogous classical notion from the theory of distributions (see Section 4, Remark 5).

LEMMA 1. *Let* $f \in \mathfrak{M}_b(J)$, $b \in \boldsymbol{R}^n$. *Then the functional* u_f *defined by*

$$\mathfrak{M}_{(-b-1)} \ni \Psi \mapsto u_f[\Psi] = \int_J f(x)\Psi(x)\,dx$$

belongs to the space $\mathfrak{M}'_{(-b-1)}$.

Proof. Let $f \in \mathfrak{M}_b$. Take an arbitrary sequence $\Psi_\nu \to 0$ in $\mathfrak{M}_{(-b-1)}$ and let $c < -b-\boldsymbol{1}$ be such that $\Psi_\nu \to 0$ in $\mathfrak{M}_c$. We get easily the estimation

$$|u_f[\Psi_\nu]| \leqslant \varrho_{c0}(\Psi_\nu) \sup_{x \in J} |x^{b+1} f(x)| \int_J x^{\varepsilon - 1}\,dx$$

where $\varepsilon = -c-b-\boldsymbol{1} > 0$.

Thus $u_f[\Psi_\nu] \to 0$; hence $u_f \in \mathfrak{M}'_{(-b-1)}$.

Remark 1. By Lemma 1 we write shortly $\mathfrak{M}_b \subset \mathfrak{M}'_{(-b-1)}$ identifying the functional $u_f \in \mathfrak{M}'_{(-b-1)}$ with the function $f \in \mathfrak{M}_b$. Hence by (2) we get

$$\mathfrak{M}_b \subset \mathfrak{M}'_{(-b-1)} \subset \mathfrak{M}'_c \quad \text{for } c < -b-\boldsymbol{1}.$$

The operations of multiplication and differentiation

$$(3) \qquad \begin{array}{lll} x^\beta\colon \mathfrak{M}_a \to \mathfrak{M}_{a-\mathrm{Re}\beta}, & x^\beta\colon \mathfrak{M}_{(\omega)} \to \mathfrak{M}_{(\omega-\mathrm{Re}\beta)} & \text{for } \beta \in \boldsymbol{C}^n, \\ D^\alpha\colon \mathfrak{M}_a \to \mathfrak{M}_{a+\alpha}, & D^\alpha\colon \mathfrak{M}_{(\omega)} \to \mathfrak{M}_{(\omega+\alpha)} & \text{for } \alpha \in \boldsymbol{N}_0^n \end{array}$$

(4) We have also $\mathfrak{M}'(J) = \bigcup_{a \in \boldsymbol{R}^n} \mathfrak{M}'_a(J)$.

are continuous. By duality the operations,

$$x^{\beta}\colon \mathfrak{M}'_a \to \mathfrak{M}'_{a+\operatorname{Re}\beta}, \quad x^{\beta}\colon \mathfrak{M}'_{(\omega)} \to \mathfrak{M}'_{(\omega+\operatorname{Re}\beta)} \quad \text{for } \beta \in \boldsymbol{C}^n,$$

$$D^{\alpha}\colon \mathfrak{M}'_a \to \mathfrak{M}'_{a-\alpha}, \quad D^{\alpha}\colon \mathfrak{M}'_{(\omega)} \to \mathfrak{M}'_{(\omega-\alpha)} \quad \text{for } \alpha \in N_0^n$$

are continuous. Hence follows the continuity of the operations

$$\begin{array}{ll} \tilde{D}^{\nu}\colon \mathfrak{M}_a \to \mathfrak{M}_a, & \tilde{D}^{\nu}\colon \mathfrak{M}_{(\omega)} \to \mathfrak{M}_{(\omega)} \\ \tilde{D}^{\nu}\colon \mathfrak{M}'_a \to \mathfrak{M}'_a, & \tilde{D}^{\nu}\colon \mathfrak{M}'_{(\omega)} \to \mathfrak{M}'_{(\omega)} \end{array} \quad \text{for } \nu \in N_0^n. \tag{4}$$

The Mellin transform of $u \in \mathfrak{M}'_{(\omega)}$ was defined in [5] by the formula

$$(Mu)(z) = u[x^{-z-1}] \quad \text{for } z \in \boldsymbol{C}^n,\ \operatorname{Re} z < \omega.$$

Mu is holomorphic for $\operatorname{Re} z < \omega$ and has the following properties:

$$\begin{aligned} & M(x^{\alpha}u)(z) = Mu(z-\alpha) \quad \text{for } \alpha \in \boldsymbol{C}^n \text{ and } \operatorname{Re} z < \omega + \operatorname{Re}\alpha, \\ & M(D_j u)(z) = (z_j+1)Mu(z_1, \ldots, z_j+1, \ldots, z_n) \\ & \qquad\qquad \text{for } \operatorname{Re} z_k < \omega_k\ (k \neq j),\ \operatorname{Re} z_j < \omega_j - 1 \quad (j = 1, \ldots, n), \\ & M(\tilde{D}_j u)(z) = z_j (Mu)(z) \quad \text{for } \operatorname{Re} z < \omega\ (j = 1, \ldots, n). \end{aligned} \tag{5}$$

2. Paley–Wiener theorems for the Mellin transformation. The following Paley–Wiener theorem for the Mellin transform of an n-dimensional Mellin distribution $u \in \mathfrak{M}'_{(\omega)}$ was stated by B. Ziemian:

THEOREM 1(5). *In order that a function $F(z) = F(z_1, \ldots, z_n)$ be the Mellin transform of a Mellin distribution $u \in \mathfrak{M}'_{(\omega)}$ it is necessary and sufficient that F be holomorphic in the set $\{z \in \boldsymbol{C}^n\colon \operatorname{Re} z < \omega\}$ and that for every $b < \omega$ there exists a polynomial P such that*(6)

$$|F(z)| < r^{-\operatorname{Re} z} |P(z)| \quad \textit{for } \operatorname{Re} z \leqslant b.$$

This theorem is analogous to the Paley–Wiener–Schwartz theorem for the Fourier–Laplace transforms of distributions with compact support(7). The aim of this paper is to give for the Mellin transform a theorem analogous to the Paley–Wiener–Schwartz theorem for the Fourier–Laplace transforms of $\boldsymbol{C}_0^{\infty}(\boldsymbol{R}^n)$ functions.

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(5) This is a precise statement of Theorem 1, p. 278 in [5]. The same formulation in dimension 1 was given in [6], Theorem 3.

(6) In accordance with the notation introduced in Section 1:

$$r^{-\operatorname{Re} z} = r_1^{-\operatorname{Re} z_1} \ldots r_n^{-\operatorname{Re} z_n}.$$

(7) See [1], Theorem 7.3.1.

The set which corresponds to $C_0^\infty(\boldsymbol{R}^n)$ forms in this case a subset of $\bigcap_{b<1} \mathfrak{M}_{-b}$. For short we shall write

$$\mathfrak{M}_{[-1]} = \bigcap_{b<1} \mathfrak{M}_{-b} = \bigcap_{\varepsilon>0} \mathfrak{M}_{-1+\varepsilon}.$$

Observe that $\bigcap_{b>a} \mathfrak{M}_b = \bigcap_{b>a} \mathfrak{M}_{(b)}$ for every $a \in \boldsymbol{R}^n$. So $\mathfrak{M}_{[-1]} = \bigcap_{\varepsilon>0} \mathfrak{M}_{(-1+\varepsilon)}$.

THEOREM 2. *In order that a function $G(z)$ be the Mellin transform of a function $\varphi \in \mathfrak{M}_{[-1]}$ satisfying the conditions*

$$\left.\frac{\partial^p \varphi(x)}{\partial x_j^p}\right|_{x_j=r_j} = 0 \quad (j = 1, \ldots, n,\ p = 0, 1, 2, \ldots), \tag{6}$$

it is necessary and sufficient that G be holomorphic in the set $\{z \in \boldsymbol{C}^n\colon \operatorname{Re} z < 0\}$ and that for every $m \in \boldsymbol{N}_0$ and $b < 0$ there exist constants $C_{mb} < \infty$ such that

$$|G(z)| \leqslant C_{mb} \frac{1}{1 + \sum_{j=1}^{n} |z_j|^m} r^{-\operatorname{Re} z} \quad \textit{for } \operatorname{Re} z \leqslant b. \tag{7}$$

The proof of Theorem 2 will be based on a part of Theorem 1 and on Lemma 2 which B. Ziemian used in the proof of the sufficiency condition in his Theorem 1. For convenience we write them down as Theorem 3 and Lemma 2, but we omit their proofs.

THEOREM 3(8). *Let $u \in \mathfrak{M}'_{(\omega)}$ and $F(z) = (Mu)(z)$ for $z \in \Omega = \{z\colon \operatorname{Re} z < \omega\}$. Then F is holomorphic in Ω and for every $b < \omega$ there exist $m(b) \in \boldsymbol{N}_0$ and $C(b) < \infty$ such that $u \in (\mathfrak{M}_b^{m(b)})'$ and*

$$|F(z)| \leqslant C(b) \sum_{|\alpha| \leqslant m(b)} |p_\alpha(z)| r^{-\operatorname{Re} z} \quad \textit{for } \operatorname{Re} z \leqslant b,$$

where $p_\alpha(z) = (z_1+1) \ldots (z_1+\alpha_1) \ldots (z_n+1) \ldots (z_n+\alpha_n)$, $|\alpha| \leqslant m(b)$. Consequently, there exists a polynomial P such that

$$|F(z)| \leqslant |P(z)| r^{-\operatorname{Re} z} \quad \textit{for } \operatorname{Re} z \leqslant b.$$

LEMMA 2(9). *Let $b \in \boldsymbol{R}^n$. Suppose that G is holomorphic on the set $\{z \in \boldsymbol{C}^n\colon \operatorname{Re} z \leqslant b\}$ and that*

$$|G(z)| \leqslant \frac{C}{\langle z_1 \rangle^2 \ldots \langle z_n \rangle^2} r^{-\operatorname{Re} z} \quad \textit{for} \quad \operatorname{Re} z \leqslant b,$$

where $C < \infty$, $\langle z_i \rangle^2 = |z_i|^2 + 1$ $(i = 1, \ldots, n)$, $r \in \boldsymbol{R}_+^n$.

Then the formal definition

$$g(x) = (2\pi i)^{-n} \int_{c+i\boldsymbol{R}^n} G(z) x^z dz \quad \textit{for } x > 0 \ (c \leqslant b)$$

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(8) From Theorem 3 follows at once the necessity condition in Theorem 1.

(9) For $n = 1$, see [4], Theorem 4.3.1.

defines correctly a function g continuous for $x > 0$ which does not depend on the choice of $c \leqslant b$. Moreover, $\operatorname{supp} g \subset J$, *the function $\boldsymbol{R}^n_+ \ni x \mapsto x^{-b} g(x)$ is bounded, $g \in \mathfrak{M}'_{(b)}(J)$ and $G(z) = (Mg)(z)$ for* $\operatorname{Re} z \leqslant b$.

LEMMA 3. *Let $\varphi \in \mathfrak{M}_b(J)$, $b \in \boldsymbol{R}^n$. Write*

$$\varphi_{jp}(x) = \left(x_j \frac{\partial}{\partial x_j}\right)^p \varphi(x) \quad \text{for } x \in J. \tag{8}$$

Then for every $m \in \boldsymbol{N}$ and $j = 1, 2, \ldots, n$ the boundary conditions

$$\varphi_{jp}(x_1, \ldots, x_{j-1}, r_j, x_{j+1}, \ldots, x_n) = 0 \quad (p = 0, 1, \ldots, m-1) \tag{9}$$

hold if and only if

$$(\tilde{D}_j)^p u\varphi = u_{\varphi_{jp}} \quad \text{in } \mathfrak{M}'_{(-b-1)} \ (p = 1, \ldots, m). \tag{10}$$

Proof. We shall prove Lemma 3 for $m = 1$, leaving the induction proof to the reader. For simplicity of notation we suppose that $j = 1$ and write $x = (x_1, x')$, $x' = (x_2, \ldots, x_n)$. Take $\varphi \in \mathfrak{M}_b(J)$, $b \in \boldsymbol{R}^n$. From Lemma 1 $u_\varphi \in \mathfrak{M}'_{(-b-1)}$. Let $\sigma \in \mathfrak{M}_{(-b-1)}$. Then, integrating by parts, we get

$$\begin{aligned} \tilde{D}_1 u_\varphi[\sigma] &= -\int_J \varphi(x) x_1 \frac{\partial \sigma}{\partial x_1} dx - \int_J \varphi(x)\sigma(x)dx \\ &= -r_1 \int_{J'} \varphi(r_1, x')\sigma(r_1, x')dx' + \int_J \sigma(x) x_1 \frac{\partial \varphi}{\partial x_1} dx, \end{aligned}$$

where $J' = \{x' \in \boldsymbol{R}^{n-1}_+ : x' < r'\}$.

Hence

$$\tilde{D}_1 u_\varphi = u_{\varphi_{11}}$$

if and only if

$$\int_{J'} \varphi(r_1, x')\sigma(r_1, x')dx' = 0 \quad \text{for every } \sigma \in \mathfrak{M}_{(-b-1)}(J). \tag{11}$$

Therefore from (8), (9) with $p = 0$ we get (10) with $p = 1$. Assuming now (10) with $p = 1$ we get (11) and it suffices to prove that $\varphi(r_1, x') = 0$ for $x' \in J'$. Suppose to the contrary that there exists $\tilde{x}'$ such that $\varphi(r_1, \tilde{x}') \neq 0$. Without loss of generality we may assume that $\varphi(r_1, \tilde{x}') = \varepsilon > 0$.

Let $\varrho > 0$ be such that $\varphi(r_1, x') > \frac{1}{2}\varepsilon$ for $x' \in B(\tilde{x}', \varrho) = \{|x - \tilde{x}'| < \varrho\}$. Choose $\sigma \in \mathfrak{M}_{(-b-1)}$ such that $\sigma(r_1, x') = 1$ for $x' \in B(\tilde{x}', \frac{1}{2}\varrho)$ and $\sigma(r_1, x') = 0$ for $|x' - \tilde{x}'| > \varrho$. Then $\int_{J'} \varphi(r_1, x')\sigma(r_1, x')dx' > \frac{1}{2}\varepsilon |B(\tilde{x}', \frac{1}{2}\varrho)| > 0$ contrary to (11).

In the sequel we shall use only the part (9) $\Rightarrow$ (10) of Lemma 3.

LEMMA 4. *Let $\varphi \in \mathfrak{M}_{[-1]}$ satisfy conditions (6). Then $u_\varphi \in \mathfrak{M}'_{(0)}$, $(\tilde{D}_j)^m u_\varphi \in \mathfrak{M}'_{(0)}$ $(m = 1, 2, \ldots)$ are Mellin distributions of Mellin order $\leqslant 0$ and* (see (8))

(12) $\quad (\tilde{D}_j)^m u_\varphi = u_{\varphi_{jm}} \quad in\ \mathfrak{M}'_{(0)}\ (j = 1, \ldots, n,\ m = 1, 2, \ldots).$

Proof. Let $\varphi \in \mathfrak{M}_{[-1]}$ and assume that (6) holds. Hence, by (8), (4) and Lemma 1, $\varphi_{jm} \in \mathfrak{M}_{-1+\varepsilon}$, $u_\varphi \in \bigcap_{\varepsilon > 0} \mathfrak{M}'_{-\varepsilon} = \mathfrak{M}'_{(0)}$ and $(\tilde{D}_j)^m u_\varphi \in \mathfrak{M}'_{(0)}$ $(j = 1, \ldots, n, m = 1, 2, \ldots)$. From Lemma 3 we get (12). We shall show now that $(\tilde{D}_j)^m u_\varphi$ are distributions of Mellin order $\leqslant 0$. To this end choose arbitrarily $\eta > 0$ and $\lambda \in \mathfrak{M}_{-\eta}$. By (10) we know that

$$\begin{aligned} |(\tilde{D}_j)^m u_\varphi[\lambda]| &= \Big|\int_J \varphi_{jm}(x)\lambda(x)\,dx\Big| \\ &\leqslant \sup_{x\in J} |x^{-\eta+1}\lambda(x)| \sup_{x\in J} |x^{\eta/2}\varphi_{jm}(x)| \cdot \int_J x^{\eta/2-1}\,dx \\ &\leqslant C_\eta \varrho_{-\eta,0}(\lambda), \end{aligned}$$

where
$$C_\eta = \varrho_{-1+\eta/2,0}(\varphi_{jm}) \int_J x^{\eta/2-1}\,dx < \infty.$$

Proof of Theorem 2. (i) Suppose first that $\varphi \in \mathfrak{M}_{[-1]}$ satisfies conditions (6). By Lemma 4, $u_\varphi \in \mathfrak{M}'_{(0)}$ and $(\tilde{D}_j)^m u_\varphi \in \mathfrak{M}'_{(0)}$ are Mellin distributions of Mellin order $\leqslant 0$. Therefore $G(z) = (Mu_\varphi)(z)$ is holomorphic for $\operatorname{Re} z < 0$ and by (5)

(13) $\quad M\big((\tilde{D}_j)^m u_\varphi\big)(z) = z_j^m (Mu_\varphi)(z) \quad \text{for } \operatorname{Re} z < 0\ (j = 1, 2, \ldots, n).$

By Theorem 3 for every $b < 0$: $(\tilde{D}_j)^m u_\varphi \in (\mathfrak{M}_b^0)'$ (because $m(b) = 0$, since $(\tilde{D}_j)^m u_\varphi$ are of Mellin order $\leqslant 0$) and there exist constants C_{jmb}, C_b such that

(14) $$\left.\begin{aligned} \big|\big(M(\tilde{D}_j)^m u_\varphi\big)\big)(z)\big| &< C_{jmb} r^{-\operatorname{Re} z} \\ |(M(u_\varphi))(z)| &< C_b r^{-\operatorname{Re} z} \end{aligned}\right. \quad \text{for } \operatorname{Re} z \leqslant b.$$

Putting

$$C_{mb} = C_b + \sum_{j=1}^n C_{jmb},$$

we get from (13) and (14) the estimation

$$\Big(1 + \sum_{j=1}^n |z_j|^m\Big)|(Mu_\varphi)(z)| \leqslant C_{mb} r^{-\operatorname{Re} z} \quad \text{for } \operatorname{Re} z \leqslant b.$$

Hence

$$|(Mu_\varphi)(z)| \leqslant C_{mb} \frac{1}{1 + \sum_{j=1}^n |z_j|^m} r^{-\operatorname{Re} z} \quad \text{for } \operatorname{Re} z \leqslant b.$$

(ii) Suppose now that G is a holomorphic function in the set $\{z \in C^n$:

$\operatorname{Re} z < 0\}$ such that for every $m \in N_0$ and $b < 0$ there exist constants $C_{mb} < \infty$ satisfying (7). If $m \geqslant 2n$ then there exists a constant C such that

$$\langle z_1 \rangle^2 \ldots \langle z_n \rangle^2 \leqslant C(1+|z_1|^m + \ldots + |z_n|^m).$$

By (7) we get the estimation

$$|G(z)| \leqslant C \cdot C_{mb} \frac{1}{\langle z_1 \rangle^2 \ldots \langle z_n \rangle^2} r^{-\operatorname{Re} z} \qquad \text{for } \operatorname{Re} z \leqslant b.$$

By Lemma 2 the function

$$g(x) = (2\pi)^{-n} \int_{R^n} G(c+iy) x^{c+iy} dy \qquad \text{for } x > 0 \ (c \leqslant b) \tag{15}$$

is continuous for $x > 0$, and its definition does not depend on the choice of $c \leqslant b$, $\operatorname{supp} g \subset J$, $g \in \mathfrak{M}'_{(b)}(J)$, $G(z) = Mg(z)$ for $\operatorname{Re} z \leqslant b$. We shall show that $g \in \mathfrak{M}_{[-1]}$. To this end it is enough to show that $g \in \mathfrak{M}_{-1-b}$, that is (see footnote ([1])),

$$\sup_{x \in J} |x^{-b} (\tilde{D})^\alpha g(x)| < \infty \qquad \text{for } \alpha \in N_0^n.$$

Fix arbitrarily $\alpha \in N_0^n$ and take m such that $m > \max(2n, |\alpha|+n)$. By (15) and (7) we get for $c = b$ the estimation

$$|(\tilde{D})^\alpha g(x)| \leqslant (2\pi)^{-n} C_{mb} r^{-b} x^b \int_{R^n} \frac{|(b+iy)^\alpha| dy}{1 + \sum_{j=1}^{n} |b_j + iy_j|^m} \qquad \text{for } x > 0;$$

hence $x^{-b}(\tilde{D})^\alpha g(x)$ is bounded in J. Since $\alpha \in N_0^n$ was arbitrary, it follows that $g \in \mathfrak{M}_{-1-b}$. Thus we have proved that $g \in \mathfrak{M}_{[-1]}$. Since $\operatorname{supp} g \subset J$, the function g satisfies conditions (6).

Remark 2. It is worth noticing that the set $\mathfrak{M}_{[-1]} = \bigcap_{\varepsilon > 0} \mathfrak{M}_{-1+\varepsilon}$ from Theorem 2 coincides with the set of the Mellin multipliers defined by B. Ziemian. The following section is devoted to this notion.

3. Mellin multipliers ([10]).

Definition 1. By a *Mellin multiplier* we mean a function $\mu \in C^\infty(J)$ such that the multiplication by μ

$$\mu \colon \mathfrak{M}_{(\omega)} \to \mathfrak{M}_{(\omega)}$$

is a continuous operation for every $\omega \in (\boldsymbol{R} \cup \{+\infty\})^n$.

([10]) This section contains some unpublished results of B. Ziemian.

It is easy to see that $\mu \in C^{\infty}(J)$ is a Mellin multiplier if and only if the operation

$$(16) \qquad \mu\colon \mathfrak{M}_{(0)} \to \mathfrak{M}_{(0)}$$

is continuous.

THEOREM 4. *$\mu \in C^{\infty}(J)$ is a Mellin multiplier if and only if $\mu \in \mathfrak{M}_{[-1]}$.*

Proof. Let $\mu \in \mathfrak{M}_{[-1]}$. To prove the continuity of the operation (16) we choose arbitrarily a sequence $\{\varphi_\nu\}$ convergent to zero in $\mathfrak{M}_{(0)}$. Thus there exists $a < 0$ such that $\varphi_\nu \to 0$ in $\mathfrak{M}_a$. Choose $a < c < 0$, $\alpha \in N_0^n$ and observe that there exist constants C_δ, $\delta \leqslant \alpha$ such that

$$\varrho_{c\alpha}(\mu\varphi_\nu) \leqslant \sum_{\delta \leqslant \alpha} C_\delta \varrho_{a\delta}(\varphi_\nu)\varrho_{-1+c-a,\alpha-\delta}(\mu).$$

As $c-a > 0$ and $\mu \in \mathfrak{M}_{[-1]}$ we see that $\mu \in \mathfrak{M}_{-1+c-a}$, therefore $\varrho_{-1+c-\alpha,\alpha-\delta}(\mu) < \infty$. Since $\varphi_\nu \to 0$ in $\mathfrak{M}_a$

$$\varrho_{a\delta}(\varphi_\nu) \to 0 \quad \text{for every } \delta \in N_0^n,$$

from the estimation proved above it follows that $\varrho_{c\alpha}(\mu\varphi_\nu) \to 0$. α being arbitrary, we get $\mu\varphi_\nu \to 0$ in $\mathfrak{M}_c$, so $\mu\varphi_\nu \to 0$ in $\mathfrak{M}_{(0)}$ since $c < 0$.

To prove the converse, take a Mellin multiplier μ and $\varepsilon \in R_+^n$. Observe that $J \ni x \mapsto \varphi(x) = x^{-1+\varepsilon}$ belongs to $\mathfrak{M}_b$ if $b \geqslant -\varepsilon$. In particular, $\varphi \in \mathfrak{M}_{-\varepsilon/2} \subset \mathfrak{M}_{(0)}$, and hence $x^{-1+\varepsilon}\mu \in \mathfrak{M}_{(0)}$ and, by (3), $\mu \in \mathfrak{M}_{(-1+\varepsilon)}$. Since $\varepsilon > 0$ was arbitrary, it follows that $\mu \in \mathfrak{M}_{[-1]}$ which ends the proof of Theorem 2.

Remark 3. Clearly, $\mathfrak{M}_{-1} \subset \bigcap_{\varepsilon > 0} \mathfrak{M}_{-1+\varepsilon}$ and the inclusion is proper. In fact, the function $J \ni x \mapsto (\ln x)^1 = \ln x_1 \ldots \ln x_n$ belongs to the space $\mathfrak{M}_{[-1]}$ and does not belong to $\mathfrak{M}_{-1}$.

EXAMPLES OF MELLIN MULTIPLIERS. It follows from Theorem 4 and Remark 3 that the function $(\ln x)^1$ is a Mellin multiplier. Another example gives the function

$$\mu(x) = \frac{x_1^{\alpha_1} \ldots x_n^{\alpha_n}}{\sum_{i=1}^{n} x_i^{\beta_i}} \quad \text{for } x \in J,\ \alpha_i \in C,\ \beta_i \in R\ (i = 1, \ldots, n)$$

satisfying the condition $\sup_{x \in J} |\mu(x)| < C < \infty$. It is not difficult to prove that such a function belongs to $\mathfrak{M}_{-1}$.

Remark 4. By duality it follows from Definition 1 that the multiplication by a Mellin multiplier μ

$$\mu\colon \mathfrak{M}'_{(\omega)} \to \mathfrak{M}'_{(\omega)} \quad \text{for } \omega \in (R \cup \{+\infty\})^n$$

is a continuous operation.

4. Characterization of Mellin distributions.

THEOREM 5[11]. *The space $\mathfrak{M}'(J)$ of Mellin distributions coincides with the space of restrictions to $\boldsymbol{R}^n_+$ of distributions on $\boldsymbol{R}^n$ with support in $\bar{J}$.*

Proof. If w is a distribution on $\boldsymbol{R}^n$ with support in $\bar{J}$, then there exist constants $C < \infty$ and $m \in \boldsymbol{N}_0$ such that [12]

$$(17)\qquad |w[\Psi]| \leqslant C \sum_{|\alpha| \leqslant m} \sup_{x \in J} |D^\alpha \Psi(x)| \quad \text{for } \Psi \in \boldsymbol{C}^m(\boldsymbol{R}^n).$$

Let $a = (-m-1, \ldots, -m-1)$ and choose arbitrarily a sequence $\{\varphi_\nu\}$ such that

$$(18)\qquad \boldsymbol{C}_0^\infty(J) \ni \varphi_\nu \to 0 \quad \text{in } \mathfrak{M}_{(a)}.$$

Hence, there exists $\varepsilon > 0$ such that

$$\varrho_{a-\varepsilon,\alpha}(\varphi_\nu) \to 0 \quad \text{for every } \alpha \in \boldsymbol{N}_0^n.$$

Let $u = w|_{\boldsymbol{R}^n_+}$. By Proposition 1, $Lu[\varphi_\nu] = w[\tilde{\varphi}_\nu]$, where $\tilde{\varphi}_\nu \in \boldsymbol{C}_0^\infty(\boldsymbol{R}^n)$ is an arbitrary extension of φ_ν $(\nu = 1, 2, \ldots)$. Thus by (17) we get

$$|(Lu)[\varphi_\nu]| \leqslant C \sum_{|\alpha| \leqslant m} \sup_{x \in J} |D^\alpha \varphi_\nu(x)| \leqslant C \varrho_{a-\varepsilon,\alpha}(\varphi_\nu) \sup_{x \in J} |x^{-a-\alpha-1+\varepsilon}|,$$

and therefore $(Lu)[\varphi_\nu] \to 0$. In Section 1 we observed that $\boldsymbol{C}_0^\infty$ is a dense subset of $\mathfrak{M}_{(a)}$ and $\mathfrak{M}'_{(a)}$ is a subspace of $\mathfrak{D}'(J)$. So, by arbitrariness of sequence (18), $Lu \in \mathfrak{M}'_{(a)}$. Taking into account Proposition 1, we can identify u with Lu and write $u \in \mathfrak{M}'_{(a)}$.

We begin the proof of the second part of Theorem 5 with the following lemma.

LEMMA 5. *Let $\Omega = \boldsymbol{R}^n \backslash (\bar{J} \backslash J)$ and let Ω_0 be a bounded open set $\Omega_0 \subset \Omega$. Suppose that $\boldsymbol{C}_0^\infty(\Omega_0) \ni \Psi_\nu \to 0$ in $\mathfrak{D}(\boldsymbol{R}^n)$ and let $\varphi_\nu = \Psi_\nu|_J$ $(\nu = 1, 2, \ldots)$. Then for every $a \in \boldsymbol{R}^n$*

$$\mathfrak{M}_a \ni \varphi_\nu \to 0 \quad \textit{in } \mathfrak{M}_a.$$

Proof. Take $\boldsymbol{C}_0^\infty(\Omega_0) \ni \Psi_\nu \to 0$ in $\mathfrak{D}(\boldsymbol{R}^n)$. Thus $\operatorname{supp} \Psi_\nu \subset \Omega_0$, $\Psi_\nu \in \boldsymbol{C}_0^\infty(\boldsymbol{R}^n)$ $(\nu = 1, 2, \ldots)$, $\sup_{x \in \boldsymbol{R}^n} |D^\beta \Psi_\nu(x)| \to 0$ for every $\beta \in \boldsymbol{N}_0^n$. Fix arbitrarily $a \in \boldsymbol{R}^n$, $\alpha \in \boldsymbol{N}_0^n$ and take $\boldsymbol{p} = (p, \ldots, p)$ such that $\boldsymbol{p} + a + \alpha + \boldsymbol{1} > 0$. Write $g_\nu(x_n)$

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[11] Theorem 5 is an extension of Proposition 5 from [6], see Theorem 6.

[12] See [1], Theorem 2.3.10.

$= D^\alpha \Psi_\nu(x_1, \ldots, x_{n-1}, x_n)$ for $x \in J$ ($\nu = 1, 2, \ldots$). By the Taylor formula there exists θ_n, $0 < \theta_n < 1$, such that

$$g_\nu(x_n) = \frac{x_n^p}{p!}\frac{d^p}{dx_n^p} g(\theta_n x_n).$$

Proceeding in the same way with respect to the variables $x_1, \ldots, x_{n-1}$, we arrive at the formula

$$D^\alpha \Psi_\nu(x) = \frac{x^p}{(p!)^n} D^{\alpha+p} \Psi_\nu(\theta_1 x_1, \ldots, \theta_n x_n),$$

where $0 < \theta_j < 1$ ($j = 1, \ldots, n$).

From the inequality

$$\varrho_{a\alpha}(\varphi_\nu) = \sup_{x \in J} |x^{a+\alpha+1} D^\alpha \Psi_\nu(x)| \leqslant \frac{1}{(p!)^n} \sup_{x \in J} |x^{a+\alpha+1+p}| \sup_{x \in J} |D^{\alpha+p} \Psi_\nu(x)|$$

it follows at once that $\varrho_{a\alpha}(\varphi_\nu) \to 0$.

Returning to the proof of Theorem 5, suppose that $u \in \mathfrak{M}'(J)$, that is, ([13]) $u \in \mathfrak{M}'_a(J)$ for some $a \in \boldsymbol{R}^n$. By (2) and Proposition 1 there exists a unique $v \in \mathfrak{D}'_J(\boldsymbol{R}^n_+)$ such that $u = Lv$. By Lemma 5 the formal definition

(19) $$T[\Psi] = u[\Psi|_J] \quad \text{for } \Psi \in C_0^\infty(\Omega)$$

defines correctly $T \in \mathfrak{D}'(\Omega)$. In fact, if $C_0^\infty(\Omega) \ni \Psi_\nu \to 0$ in $\mathfrak{D}(\Omega)$, then there exists a bounded open set $\Omega_0 \subset \Omega$ such that $\Psi_\nu \in C_0^\infty(\Omega_0)$ and if $\varphi_\nu = \Psi_\nu|_J$ then $\varphi_\nu \to 0$ in $\mathfrak{M}_a$. Hence $u[\varphi_\nu] \to 0$ and therefore $T[\Psi_\nu] \to 0$.

For the proof that T extends to a distribution on $\boldsymbol{R}^n$ take an arbitrary bounded open set $\Omega_0 \subset \Omega$. It suffices to prove([14]) that $T|_{\Omega_0}$ extends to $\boldsymbol{R}^n$. To this end take an arbitrary sequence $\{\Psi_\nu\}$ such that $C_0^\infty(\Omega_0) \ni \Psi_\nu \to 0$ in $\mathfrak{D}(\boldsymbol{R}^n)$. By Lemma 5, $\Psi_\nu|_J \to 0$ in $\mathfrak{M}_a$ and hence as before we get that $T[\Psi_\nu] \to 0$.

Let w be an extension of T:

(20) $$w \in \mathfrak{D}'(\boldsymbol{R}^n), \quad w = T \text{ on } \Omega.$$

Take $\tilde{\varphi} \in C_0^\infty(\boldsymbol{R}^n_+) \subset C_0^\infty(\Omega)$ and put $\varphi = \tilde{\varphi}|_J$. From (19) and from Proposition 1 we get $T[\tilde{\varphi}] = u[\varphi] = v[\tilde{\varphi}]$, thus $T = v$ on $\boldsymbol{R}^n_+$ and by (20) $w = v$ on $\boldsymbol{R}^n_+$. To see that $\operatorname{supp} w \subset \bar{J}$ take $\Psi \in C_0^\infty(\boldsymbol{R}^n \setminus \bar{J})$. Clearly, $\Psi \in C_0^\infty(\Omega)$ and by (20) and (19) $w[\Psi] = T[\Psi] = u[\Psi|_J] = u[0] = 0$.

Note the first version of the theorem on characterization of Mellin transformable distributions due to B. Ziemian in the case $n = 1$([15]).

([13]) Cf. footnote ([4]).

([14]) See [2], subsection 5 and [3], subsection 6 of §3.

([15]) Cf. [6], Proposition 5.

THEOREM 6. *The space* $\mathfrak{M}'(J)$ *of Mellin transformable distributions restricted to* $(0, r)$ *coincides with the space of restrictions to* $(0, r)$ *of distributions from* $\mathfrak{D}'(\boldsymbol{R}^n)$.

Proof. If $u \in \mathfrak{M}'(J)$ then from Theorem 5 there exists $w \in \mathfrak{D}'(\boldsymbol{R}^n)$, $w|_{\boldsymbol{R}^n_+} = u$ and hence $w|_{(0,r)} = u|_{(0,r)}$.

We shall give the proof of the second part of Theorem 6 independently of the analogous part of Theorem 5. It is simpler and does not use Theorem 2.3.10 from [1] based on the Whitney theorem.

Suppose that $v \in \mathfrak{D}'(\boldsymbol{R}^n)$. Write $\Omega = (0, t)$, $v_1 = v|_\Omega$. Thus $v_1 \in \mathfrak{D}'(\Omega)$ is extendable from Ω to $\boldsymbol{R}^n$ and therefore there exist constants $C > 0$ and $m \in \boldsymbol{N}_0$ such that

$$|v_1[\varphi]| \leqslant C \sum_{|\alpha| \leqslant m} \sup|D^\alpha \varphi| \quad \text{for } \varphi \in C_0^\infty(\Omega). \tag{21}$$

Let $a = (-m-1, \ldots, -m-1)$. Take an arbitrary sequence $\{\varphi_\nu\}$, $C_0^\infty(\Omega) \ni \varphi_\nu \to 0$ in $\mathfrak{M}_{(a)}$. This means that for some $\varepsilon > 0$, $C_0^\infty \ni \varphi_\nu \to 0$ in $\mathfrak{M}_{a-\varepsilon}$. By (21)

$$|v_1[\varphi_\nu]| \leqslant C \sum_{|\alpha| \leqslant m} \varrho_{a-\varepsilon,\alpha}(\varphi_\nu) \sup_{x \in J} |x^{-a+\varepsilon-\alpha-1}|$$

and thus $v_1[\varphi_\nu] \to 0$. Hence $v_1 \in \mathfrak{M}'_{(a)} \subset \mathfrak{M}'(J)$.

Remark 5. It follows from Theorem 5 that every Mellin distribution is a distribution of finite order. In subsection 3 of G. Łysik's paper (this volume, pp. 219–229), he constructed a Mellin distribution of Mellin order $+\infty$. Thus the notion of Mellin order is a concept essentially different from that well known from the classical theory of distributions.

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