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On the tangent bundle and cotangent bundle of a differential space

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Abstract. In the paper the concepts of tangent bundle (t.b.) and cotangent bundle (c.b.) for MacLane–Sikorski's differential space are discussed. There exists a possibility of indexing tangent spaces formally by points; this leads, in some cases, to too large spaces resulting as t.b. or c.b. Here, by a slight modification of Sikorski's concept of tangent space, we obtain a situation in which the tangent spaces at two points are distinct iff the points are functionally distinguishable. Next we introduce a modified concept of t.b. and c.b. and examine the condition of smoothness of vector fields and 1-forms in the terms of t.b. and c.b.

The concept of a differential space is due to R. Sikorski [5] and S. MacLane [1]. These both authors have considered tangent vectors to a differential space (M, C) at a point p of M as $\boldsymbol{R}$-linear mappings $v\colon C \to \boldsymbol{R}$ satisfying the Leibniz rule $v(\alpha\cdot\beta) = \alpha(p)v(\beta)+\beta(p)v(\alpha)$ for $\alpha, \beta \in C$. This gives rise, in a natural way, to the definition of the tangent space $(M, C)_p$ to (M, C) at p.

In the present paper we introduce a slight modification to the original definition of a tangent vector at a point. We regard a tangent vector at a point p as a mapping $v\colon C(p) \to \boldsymbol{R}$ fulfilling the conditions $v(\alpha+\beta) = v(\alpha)+v(\beta)$, $v(c\alpha) = cv(\alpha)$, $v(\alpha\cdot\beta) = \alpha(p)v(\beta)+\beta(p)v(\alpha)$ for $\alpha, \beta \in C(p)$, $c \in \boldsymbol{R}$, where $C(p)$ is the set of all real-valued functions C-smooth at p, i.e. $C(p)$ is the union of all sets C_A, where $p \in A \in \tau_C$ and τ_C stands for the weakest topology on M making all functions in C are continuous. We thus get a vector space $T_p(M, C)$ with the natural definitions of addition and multiplication by real numbers: $(v+w)(\alpha) = v(\alpha)+w(\alpha)$, $(a\cdot v)(\alpha) = a\cdot v(\alpha)$ for $\alpha \in C(p)$, where v, w are arbitrary tangent vectors treated as mappings $C(p) \to \boldsymbol{R}$ and a is any real number. We have a natural isomorphism $i_p\colon T_p(M, C) \to (M, C)_p$ given by the formula $i_p(v)(\alpha) = v(\alpha)$ for v of $T_p(M, C)$ and $\alpha \in C$.

The above modification allows us to connect the Hausdorff condition for the topology τ_C with the following one:

(*) $T_p(M, C) \neq T_q(M, C)$ whenever $p \neq q$ (see Proposition 1).

Condition $(*)$ is of non-topological nature, it concerns the indexed set $(T_p(M, C);\ p \in M)$ of vector spaces. Moreover, there is one more advantage: the tangent spaces $T_p(M, C)$ and $T_q(M, C)$ are disjoint if and only if they are different.

For a linear space V we will denote its (linear) dual by V^*. A mapping π which to every tangent vector v to (M, C) assigns a point $p \in M$ such that v is an element of $T_p(M, C)$ will be called a *projection* of the tangent bundle of the differential space (M, C). It is easy to see that such a mapping need not be uniquely determined by (M, C). For the proof of the following Proposition see [2].

PROPOSITION 1. *The following conditions are equivalent*:

(I) $T_p(M, C) = T_q(M, C)$;

(II) $T_p(M, C)$ *and* $T_q(M, C)$ *have a common element*;

(III) $(T_p(M, C))^*$ *and* $(T_q(M, C))^*$ *have a common element*;

(IV) $C(p) = C(q)$;

(V) *the set of all neighbourhoods of* p *in topology* τ_C *is equal to the set of all neighbourhoods of* q *in* τ_C;

(VI) $\alpha(p) = \alpha(q)$ *for* $\alpha \in C$.

EXAMPLE. Let us take $M = \boldsymbol{R}$ and $C =$ the smallest differential structure containing the set $\{\boldsymbol{R} \ni p \mapsto |p|\}$. Then we have the set of all neighbourhoods of p in topology τ_C equal to the set of all neighbourhoods of $-p$ in τ_C. Thus $T_p(M, C) = T_{-p}(M, C)$ and so the set of all π such that for any tangent vector v we have v in $T_{\pi(v)}(M, C)$ has cardinality 2^c.

As a corollary to Proposition 1 we have

PROPOSITION 2. *For every projection* π *and* π_1 *of the tangent bundle of* (M, C) *and for any* $\alpha \in C$ *we have* $\alpha \circ \pi = \alpha \circ \pi_1$.

Hence it follows that we have a correct definition of a differential structure C' on the set of all tangent vectors to (M, C) as the smallest differential structure containing the set

$$\{\alpha \circ \pi;\ \alpha \in C\} \cup \{\alpha_*;\ \alpha \in C\},$$

where $\alpha_*(v) = v(\alpha)$ for all tangent vectors v (cf. [3]). Thus we obtain a differential space $T(M, C)$ whose underlying space is the set of all tangent vectors to (M, C) and whose differential structure is C'. The differential space $T(M, C)$ will be called the *tangent bundle* of the differential space (M, C).

It has been proved (cf. [4]) that if (M, C) is a differentiable manifold, i.e. a differential space locally diffeomorphic to a Euclidean space, then the tangent bundle of the differential space (M, C) coincides with the tangent bundle of the differentiable manifold.

Another interesting corollary to Proposition 1 is the following one concerning the Hausdorff axiom for the topology induced by the set of real functions on M.

PROPOSITION 3. *For any set D of real functions on M the topological space* (M, τ_D) *is Hausdorff if and only if there exists only one projection of the tangent bundle of the differential space* (M, C), *where C is the smallest of all differential structures including D.*

Proof. We have (cf. [7]) $\tau_D = \tau_C$. Assuming that there exist two distinct projections π and π_1 we find a tangent vector v belonging to $T_{\pi(v)}(M, C)$ and $T_{\pi_1(v)}(M, C)$ simultaneously, where $\pi(v) \neq \pi_1(v)$. Thus by Proposition 1 we have $C(p) = C(p_1)$ with $p \neq p_1$. Hence it follows that every neighbourhood of p in topology τ_C is a neighbourhood of p_1, and vice versa, which ends the proof.

Any mapping X which assigns to each point $p \in M$ a vector $X(p)$ in $T_p(M, C)$ is called a *vector field* on (M, C). A vector field X is said to be *smooth* if and only if for every $\alpha \in C$ the function $\partial_X \alpha$ defined by the formula

$$(\partial_X \alpha)(p) = X(p)(\alpha) \quad \text{for } p \in M$$

belongs to C.

PROPOSITION 4. *Any vector field X on* (M, C) *is smooth iff it defines a smooth mapping* $X\colon (M, C) \to T(M, C)$.

Proof. For any $\alpha \in C$ and any vector field X on (M, C) we have $\alpha_* \circ X = \partial_X \alpha$, and for every point $p \in M$ we have $(\alpha \circ \pi)(X(p)) = \alpha(q)$, where $X(p)$ is in $T_q(M, C)$, $q = \pi(X(p))$. On the other hand, $X(p)$ is in $T_p(M, C)$. Thus $T_p(M, C) = T_q(M, C)$ and $\alpha(p) = \alpha(q)$. Therefore $\alpha \circ \pi \circ X = \alpha$. Hence it follows (cf. [7]) that the smoothness of X viewed as a vector field in the previous sense yields the smoothness of the mapping $X\colon (M, C) \to T(M, C)$. The converse implication is obvious.

The set of all smooth vector fields on (M, C) will be denoted by $\mathscr{X}(M, C)$.

To define the cotangent bundle of a differential space (M, C) we first consider the set of all elements of the spaces $(T_p(M, C))^*$, where $p \in M$. Such elements are said to be *tangent covectors* of (M, C). A mapping π of the set of all tangent covectors w of (M, C) into M such that w is an element of $(T_{\pi(w)}(M, C))^*$ is said to be a *projection* of the cotangent bundle of (M, C).

Similarly to Proposition 2 we can prove

PROPOSITION 5. *For every projection* π *and* π_1 *of the cotangent bundle of* (M, C) *and for any* $\alpha \in C$ *we have* $\alpha \circ \pi = \alpha \circ \pi_1$.

We have a correct definition of the differential structure C'^* on the set of all tangent covectors of (M, C) as the smallest differential structure containing the set

$$\{\alpha \circ \pi;\ \alpha \in C\} \cup \{\tilde{X};\ X \in \mathscr{X}(M, C)\},$$

where π is a projection of the cotangent bundle of (M, C) and for every smooth vector field X on (M, C) and for any tangent covector w we write

$$\tilde{X}(w) = w[X(\pi(w))]. \tag{1}$$

The real-valued function $\tilde{X}$ is, obviously, independent of projection π. The differential space $T^*(M, C)$ whose underlying space is the set of all tangent covectors of the differential space (M, C) and whose differential structure is C'^* will be called the *cotangent bundle* of (M, C).

A real-valued function ω defined on the set of all tangent vectors to (M, C) and linear on all vector spaces $T_p(M, C)$, $p \in M$, is said to be a *differential* 1-*form* on (M, C). A differential 1-form ω is said to be *smooth* if and only if for any smooth vector field $X \in \mathscr{X}(M, C)$ we have $\omega \circ X \in C$.

PROPOSITION 6. *Any* 1-*form* ω *on* (M, C) *is smooth if and only if it defines a smooth mapping*

$$\hat{\omega}\colon (M, C) \to T^*(M, C), \tag{2}$$

where $\hat{\omega}(p)$ *is the restriction of* ω *to the vector space* $T_p(M, C)$ *for* $p \in M$.

Proof. Let ω be a 1-form on (M, C). Suppose that ω is smooth. Let X be a vector field on (M, C). Then for any $p \in M$ we have

$$\tilde{X}(\hat{\omega}(p)) = \hat{\omega}(p)[X(\pi[\hat{\omega}(p)])] = \omega(X(q)),$$

where

$$q = \pi(\hat{\omega}(p)). \tag{3}$$

Then $\hat{\omega}(p)$ is in $(T_p(M, C))^*$. By (3) $\hat{\omega}(p)$ belongs to $(T_q(M, C))^*$. Then $T_p(M, C) = T_q(M, C)$ and by Proposition 1 we have $\alpha(p) = \alpha(q)$ for $\alpha \in C$. Hence it follows that $X(p)(\alpha) = (\partial_X\alpha)(p) = (\partial_X\alpha)(q) = X(q)(\alpha)$ for $\alpha \in C$. Therefore $X(p) = X(q)$ and

$$\tilde{X}(\hat{\omega}(p)) = \omega(X(p)). \tag{4}$$

Thus

$$\tilde{X} \circ \hat{\omega} = \omega \circ X. \tag{5}$$

By the same argument we have $\alpha[\pi(\hat{\omega}(p))] = \alpha(p)$ for $\alpha \in C$. Therefore (cf. [6], [7]) the mapping (2) is smooth.

To prove the inverse assertion assume that the mapping (2) is smooth and take any $p \in M$. Assuming (3) we have $\hat{\omega}(p)$ as an element of $(T_q(M, C))^*$ and of $(T_p(M, C))^*$. Thus by Proposition 1 we have (VI). Therefore we get (4) for any smooth vector field X and any $p \in M$; in other words, (5) is satisfied for any such vector field X. Hence it follows that $\omega \circ X \in C$, because $\tilde{X} \circ \hat{\omega} \in C$. This ends the proof.

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