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A necessary and sufficient condition for normality of linear control system with delays

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Abstract. A control system described by the equation $\dot{x}(t) = Ax(t) + \sum_{j=1}^{m} B_j x(t-j) + Cu(t)$ is considered. We obtain an algebraic necessary and sufficient condition for normality of such system. From this result we derive the well-known condition for normality of linear ordinary control systems.

1. Introduction. Let us consider the control process described by linear differential-difference equation of the form [1]

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{m} B_j x(t-j) + Cu(t), \quad t \in [0, T], \tag{1}$$

$$x(t) = \varphi(t), \quad t \in [-m, 0]. \tag{2}$$

In what follows we shall use matrix notation, where $x(t)$ is an n-vector, $u(t)$ is a measurable r-vector control function, A and B_j are $(n \times n)$ constant matrices, C is an $(n \times r)$ constant matrix and $\varphi(t)$ is a continuous vector function on the interval $t \in [-m, 0]$, that is $\varphi \in C([-m, 0]; R^n)$.

The adjoint equation of the system (1), (2) is given by

$$\dot{y}(t) = -y(t)A - \sum_{j=1}^{m} y(t+j)B_j, \quad t \in [0, T], \tag{3}$$

$$y(T) = y_0, \quad y(t) = 0, \quad t > T, \tag{4}$$

where $y(t)$ is a $(1 \times n)$ vector function.

Denote

$$T = K + \tau, \quad K \in N, \quad \tau \in (0, 1], \quad C = (c_1, c_2, \ldots, c_r). \tag{5}$$

Definition. The system (1), (2) is said to be *s-normal* on $[t_1, t_2]$ $0 \leqslant t_1 < t_2 \leqslant T$, $s = 1, 2, \ldots, r$, if for every $y_0 \neq 0$ the function $\sigma_s(t)$

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[1] We can always transform the equation $\dot{x}(t) = f(t, x(t), x(t-h), x(t-2h), \ldots, x(t-mh))$ with $h \neq 0$ to the case of $h = 1$ applying substitution $t = \tau h$, $y(\tau) = x(\tau h)$.

$= g(t)c_s$ is non-identically zero on any subinterval of $[t_1, t_2]$. The system (1), (2) is said to be *normal* on $[t_1, t_2]$ if it is s-normal for every $s = 1, 2, \ldots, r$.

In this paper we derive an algebraic necessary and sufficient condition for normality of the system (1), (2) on $[0, T]$. From this result we derive the well-known condition for normality of linear ordinary differential control systems. Some methods given in this paper are closely connected with those investigated by Zmood in [4].

The solution of the system (3), (4) can be written (see [1]) in the form

$$y(t) = y_0 X(T, t), \tag{6}$$

where $X(z, t)$ is a unique $(n \times n)$ matrix solution, defined on $[-m, T] \times \times [0, T]$, of

$$\frac{\partial}{\partial z} X(z, t) = AX(z, t) + \sum_{j=1}^{m} B_j X(z-j, t) \tag{7}$$

for $(z, t) \in [t, T] \times [0, T]$, and

$$X(z, t) = \begin{cases} I & \text{for } z = t, \\ 0 & \text{for } (z, t) \in [-m, t) \times [0, T]. \end{cases} \tag{8}$$

The matrix function $X(z, t)$ will be called the *fundamental solution* of (1). In our case we may write (see [1]) the fundamental solution $X(z, t)$ as $X(z-t)$ without any loss of generality because the system (7), (8) is an autonomous system.

We now derive (see [4]) a form of the fundamental solution, which will be useful further on in this paper.

Let us define

$$X_k(\tau) = X(\tau + k), \qquad \tau \in [0, 1], \qquad k = 0, 1, 2, \ldots$$

By a direct substitution in (7) and by assuming $t = 0$, we obtain

$$\begin{aligned}
&\frac{d}{d\tau} X_0(\tau) = AX_0(\tau), && X_0(0) = I, \\
&\frac{d}{d\tau} X_1(\tau) = B_1 X_0(\tau) + AX_1(\tau), && X_1(0) = X_0(1), \\
&\qquad \vdots \qquad\qquad \vdots && \qquad \vdots \\
&\frac{d}{d\tau} X_m(\tau) = \sum_{j=0}^{m-1} B_{m-j} X_j(\tau) + AX_m(\tau), && X_m(0) = X_{m-1}(1), \\
&\qquad\qquad\qquad \vdots && \\
&\frac{d}{d\tau} X_k(\tau) = \sum_{j=0}^{m-1} B_{m-j} X_{j+k-m}(\tau) + AX_k(\tau), && X_k(0) = X_{k-1}(1).
\end{aligned} \tag{9}$$

Thus the solution of (7) and (8) over the interval $t \in [k, k+1]$ is given by $X(t) = X_k(t-k)$.

Denote

$$Z_k(\tau) = [X_0^*(\tau), X_1^*(\tau), \ldots, X_k^*(\tau)]^*.$$

Then we have from (9)

$$\frac{d}{d\tau} Z_k(\tau) = A_k Z_k(\tau), \qquad \tau \in [0, 1], \tag{10}$$

$$X_k(\tau) = E_k Z_k(\tau), \tag{11}$$

where $Z_k(\tau)$ is an $(n(k+1) \times n)$ matrix and

$$E_k = (0, 0, \ldots, 0, I),$$

$$A_k = \begin{pmatrix} A & 0 & 0 & & & 0 \\ B_1 & A & 0 & & \ddots & 0 \\ B_2 & B_1 & A & & & 0 \\ \vdots & & & & & \\ B_m & \ldots & B_1 & A & & \vdots \\ 0 & B_m & \ldots & B_1 & A & \\ \vdots & & \ddots & & & \ddots \\ 0 & \ldots & B_m & \ldots & B_1 & A \end{pmatrix},$$

A_k and E_k are $(n(k+1) \times n(k+1))$ and $(n \times n(k+1))$ matrices, respectively.

As is well known, the unique solution of (10) is given by

$$Z_k(\tau) = e^{A_k \tau} Z_k(0) \tag{12}$$

and so

$$X_k(\tau) = E_k e^{A_k \tau} Z_k(0). \tag{13}$$

It is clear from (9), by the definition of $Z_k(\tau)$, that

$$Z_0(0) = I. \tag{14}$$

From the definition of $Z_k(\tau)$ we also get

$$\begin{aligned} Z_k(0) &= [X_0^*(0), X_1^*(0), \ldots, X_k^*(0)]^* = [I, X_0^*(1), \ldots, X_{k-1}^*(1)]^* \\ &= [I, Z_{k-1}^*(1)]^*. \end{aligned}$$

Hence the recurrent formulas follow:

$$Z_k(0) = \begin{pmatrix} I \\ \cdots\cdots \\ e^{A_{k-1}} Z_{k-1}(0) \end{pmatrix}, \qquad k = 1, 2, \ldots \tag{15}$$

Now we present the other way of computing $Z_k(0)$. If we introduce the notation

$$M_k = \begin{pmatrix} 0 & 0 & & 0 \\ I & 0 & & \vdots \\ \vdots & \ddots & \ddots & \\ 0 & \dots & I & 0 \end{pmatrix}, \quad L_k = \begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where M_k and L_k are $\big(n(k+1)\times n(k+1)\big)$ and $\big(n(k+1)\times n\big)$ matrices, respectively, we can write

$$Z_k(0) = L_k + M_k Z_k(1) = L_k + M_k e^{A_k} Z_k(0). \tag{16}$$

Let us notice some properties of the matrix $M_k e^{A_k}$. We can check by matrix multiplication that

$$\begin{aligned} M_k e^{A_k} &= e^{A_k} M_k, \\ (M_k e^{A_k})^{k+1} &= 0, \\ (I - M_k e^{A_k})^{-1} &= I + M_k e^{A_k} + \dots + (M_k e^{A_k})^k. \end{aligned} \tag{17}$$

The matrix $(I - M_k e^{A_k})$ is non-singular because all non-zero elements of the matrix $M_k e^{A_k}$ lie below the diagonal. From (16) and (17) we obtain by direct substitution that

$$Z_k(0) = \big(I + M_k e^{A_k} + \dots + (M_k e^{A_k})^k\big) L_k. \tag{18}$$

Finally, from (13) and the definition of $X_k(\tau)$ we obtain for $t \in [k, k+1]$ and $k = 0, 1, 2, \dots$

$$X(t) = X_k(t-k) = E_k e^{A_k(t-k)} Z_k(0). \tag{19}$$

2. Main results. We now formulate conditions for matrices A, B_j and C, under which the control system (1), (2) is s-normal on $[T-k-1, T-k]$.

THEOREM. *A necessary and sufficient condition for the control system* (1), (2) *to be s-normal on* $[T-k-1, T-k]$, $k = 0, 1, \dots, K$, $s = 1, 2, \dots, r$, *is that the matrix*

$$Q_k = (E_k C_k, E_k A_k C_k, \dots, E_k A_k^{n(k+1)-1} C_k)$$

has rank n, where $C_k = Z_k(0) c_s$.

Proof. From (6) and (19) we obtain by direct substitution that for $t \in [T-k-1, T-k]$

$$\begin{aligned} \sigma_s(t) = y(t) c_s = y_0 X(T, t) c_s = y_0 X(T-t) c_s &= y_0 E_k e^{A_k(T-t-k)} Z_k(0) c_s \\ &= y_0 E_k e^{A_k(T-t-k)} C_k. \end{aligned}$$

Since $\sigma_s(t)$ is piecewise analytic, the control system (1), (2) is s-normal on $[T-k-1, T-k]$ iff

$$y_0 E_k e^{A_k(T-t-k)} C_k = 0 \quad \text{for } t \in [T-k-1, T-k] \tag{20}$$

implies $y_0 = 0$.

Now we only need to show that (20) implies $y_0 = 0$ if and only if the rank of Q_k is n.

Differentiating equality (20) $n(k+1)-1$ times and writing $t = T-k$, we obtain the following system of equalities:

$$\begin{aligned} y_0 E_k C_k &= 0, \\ y_0 E_k A_k C_k &= 0, \\ &\vdots \\ y_0 E_k A_k^{n(k+1)-1} C_k &= 0. \end{aligned} \tag{21}$$

From the Cayley–Hamilton theorem and by (21) it follows that

$$y_0 E_k A_k^l C_k = 0$$

for $l = 0, 1, 2, \ldots$ Using the power series expansion of the exponential matrix, we find that

$$y_0 E_k e^{A_k(T-t-k)} C_k = 0 \quad \text{for } t \in [T-k-1, T-k].$$

This means that (20) is equivalent to (21). Now it is sufficient to show that $y_0 = 0$ is a consequence of (21) if and only if the rank of $Q_k = n$. But this equivalence is obvious and so the proof is complete.

In the following example we present a control system which is a normal system on $[0, 1]$ and on $[2, 3]$, but is not a normal system on $[1, 2]$.

EXAMPLE. Consider the following control system:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 2e \\ e^2 & 0 \end{pmatrix} \begin{pmatrix} x_1(t-1) \\ x_2(t-1) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t)$$

for $t \in [0, 3]$. We can easily verify that

$$X_1(t) = \begin{pmatrix} e^{t+1} & 2e^{2t+1} - 2e^{t+1} \\ e^{2t+2} - e^{t+2} & e^{2t+2} \end{pmatrix}, \quad t \in [0, 1],$$

$$Z_2^*(0) = \begin{pmatrix} 1 & 0 & e & 0 & e^2 & e^4 - e^3 \\ 0 & 1 & 0 & e^2 & 2e^3 - 2e^2 & e^4 \end{pmatrix},$$

$$Q_0 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} e & 3e & 7e & 15e \\ e^2 & 3e^2 & 7e^2 & 15e^2 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 2e^3 - e^2 & 4e^3 - e^2 & 10e^3 - e^2 & \ldots \\ 2e^4 - e^3 & 4e^4 - e^3 & 8e^4 - e^3 & \ldots \end{pmatrix},$$

$$\operatorname{rank} Q_0 = \operatorname{rank} Q_2 = 2, \quad \operatorname{rank} Q_1 = 1.$$

From our theorem we obtain that this system is normal on $[2, 3]$ and on $[0, 1]$ and is not normal on $[1, 2]$.

The following result is the well-known algebraic criterion for normality of linear ordinary control systems.

COROLLARY 1. *Suppose that in* (1), (2) $B_j = 0$ $j = 1, 2, \ldots, m$. *Then the control system* (1), (2) *is on s-normal system on* $[0, T]$ *if and only if*

$$\operatorname{rank} Q_s = \operatorname{rank}(c_s, Ac_s, \ldots, A^{n-1}c_s) = n.$$

Proof. We know from the theorem that the system (1), (2) is s-normal on $[0, T]$ iff for every $k = 0, 1, \ldots, K$ $\operatorname{rank} Q_k = n$. Hence it is sufficient to show that

$$\operatorname{rank} Q_k = \operatorname{rank} Q_s \quad \text{for } k = 0, 1, \ldots, K.$$

In our case we obtain by matrix multiplication

$$\begin{aligned} E_k A_k^l C_k &= A^l E_k Z_k(0) c_s = A^l X_k(0) c_s = A^l X(k) c_s = A^l e^{Ak} c_s \\ &= e^{Ak} A_0^l Z_0(0) c_s = e^{Ak} A_0^l C_0, \end{aligned}$$

for $l = 0, 1, \ldots,$ $k = 0, 1, \ldots, K$.

Hence for $k = 0, 1, \ldots, K$ we get

$$\operatorname{rank} Q_k = \operatorname{rank} Q_0 = \operatorname{rank} Q_s$$

and the proof is finished.

COROLLARY 2. *Suppose that in* (1) $A = 0$. *Then the control system* (1), (2) *is on s-normal system on* $[T-k-1, T-k]$ *if and only if*

$$\operatorname{rank}(E_k C_k, E_k A_k C_k, \ldots, E_k A_k^k C_k) = n.$$

Proof. The proof is a simple consequence of the equality

$$A_k^l = 0 \quad \text{for } l = k+1, k+2, \ldots$$

COROLLARY 3. *Suppose that in* (1) $A = 0$, $B_1 = B$, $B_j = 0$ *for* $j = 2, 3, \ldots, m$. *Then the control system* (1), (2) *is on s-normal system on* $[T-k-1, T-k]$ *iff*

$$\operatorname{rank} P_k = \operatorname{rank}(c_s, Bc_s, \ldots, B^k c_s) = n.$$

Proof. It is sufficient to prove that $\operatorname{rank} P_k = \operatorname{rank} Q_k$. From the form of A_k we find that

$$(e^{A_k})^l = \begin{pmatrix} I & & & & 0 \\ \frac{l^1 B^1}{1!} & I & & & \\ \frac{l^2 B^2}{2!} & & \ddots & & \\ \vdots & \ddots & & \ddots & \\ \frac{l^k B^k}{k!} & \cdots & \frac{l^2 B^2}{2!} & \frac{l^1 B^1}{1!} & I \end{pmatrix}$$

and hence by (17), (18) we conclude that

$$(22)\qquad Z_k(0) = \begin{bmatrix} I \\ I \\ I+\frac{1^1B^1}{1!} \\ \vdots \\ I+\frac{(k-1)^1B^1}{1!}+\frac{(k-2)^2B^2}{2!}+\ldots+\frac{B^{k-1}}{(k-1)!} \end{bmatrix}.$$

The general term in Q_k is given by $E_kA_k^lC_k$ for $l = 0, 1, \ldots, n(k+1)-1$. Upon substitution for E_k, A_k and $Z_k(0)$, the general term becomes

$$E_kA_k^lC_k = E_AA_k^lZ_k(0)c_s$$

$$= B^l\left(I+\frac{(k-l-1)^1B^1}{1!}+\frac{(k-l-2)^2B^2}{2!}+\ldots+\frac{B^{k-l-1}}{(k-l-1)!}\right)c_s$$

$$(23)\qquad = \left(B^l+\frac{k-l-1}{1!}B^{l+1}+\ldots+\frac{B^{k-1}}{(k-l-1)!}\right)c_s$$

$$\text{for } l = 0, 1, \ldots, k-2,$$

$$E_kA_k^lC_k = B^lc_s \quad \text{for } l = k-1,\ k$$

$$E_kA_k^lC_k = 0 \quad \text{for } l = k+1, \ldots, n(k+1)-1.$$

Using notation (23), we can easily verify that

$$\operatorname{Ker}Q_k = \operatorname{Ker}P_k,$$

but this is equivalent to

$$\operatorname{rank}Q_k = \operatorname{rank}P_k.$$

From Corollary 3 it follows immediately that if in the system (1) $A = 0$, $B_1 = B$, $B_j = 0$, $j = 2, 3, \ldots, m$, then the control system (1), (2) cannot be s-normal on $[T-n+1, T]$.

The Cayley–Hamilton theorem implies

$$\operatorname{rank}P_k = \operatorname{rank}P_{n-1} \quad \text{for } k = n-1, n, \ldots, K.$$

Hence, if the system considered in Corollary 3 is s-normal on $[T-n, T-n+1]$, then it is also s-normal on $[0, T-n+1]$.

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