ANNALES
POLONICI MATHEMATICI
XXXI (1975)

A note on the class of meromorphic functions, 1

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Abstract. Let $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ be regular in $E\{0 < |z| < 1\}$. Denote this class of functions by Σ. Functions $f \in \Sigma$ and satisfying:

(1) $$-\operatorname{Re}\left\{e^{i\gamma}\frac{zf'(z)}{f(z)}\right\} > 0;$$

(2) $$-\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > 0, \quad \text{where } g \in \Sigma \text{ and satisfies (1) with } \gamma = 0;$$

(3) $$\operatorname{Re}\left[\alpha\left(1+\frac{zf''(z)}{f'(z)}\right) - (\alpha+\beta e^{i\gamma})\frac{zf'(z)}{f(z)}\right] > 0; \quad \alpha, \beta \text{ real and } |\gamma| < \pi/2;$$

(4) $$\operatorname{Re}\left[\alpha\left(1-\frac{zg'(z)}{g(z)}+\frac{zf''(z)}{f'(z)}\right) - \beta\frac{zf'(z)}{g(z)}\right] > 0,$$

where $g \in \Sigma$ and satisfies (1) with $\gamma = 0$;

are said to belong to the class $\Sigma(\gamma)$, Γ, $\Sigma(\alpha, \beta, e^{-i\gamma}f)$ and $\Gamma(\alpha, \beta, f)$ respectively.

We prove:

Theorem A. *If $f \in \Sigma(\alpha, \beta, e^{-i\gamma}f)$ ($f \in \Gamma(\alpha, \beta, f)$), then $f \in \Sigma(\gamma)$ ($f \in \Gamma$). If $\gamma = 0$, then any $f \in \Sigma(\alpha, \beta, f)$ belongs to the class $\Sigma(0)$.*

Theorem B. *If $f \in \Sigma(\alpha, \beta, f)$, then*

$$-K(\alpha, \beta, -r) \leqslant |df^{-\beta/\alpha}(z)| \leqslant K(\alpha, \beta, r),$$

where

$$K(\alpha, \beta, r) = \left|\frac{\beta}{\alpha}\right|(1+r)^{-2\beta/\alpha}r^{-1+\beta/\alpha} \quad \textit{and} \quad \alpha < 0.$$

1. Let $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ be regular in $E\{0 < |z| < 1\}$. Denote this class of functions by Σ. Functions $f \in \Sigma$ and satisfying:

(1.1) $$-\operatorname{Re}\left\{e^{i\gamma}\frac{zf'(z)}{f(z)}\right\} > 0;$$

* This author acknowledges the support of the Council of Scientific and Industrial Research, New Delhi, India under grant. No. 7/92 (300/72 - GAUI).

$$(1.2)\qquad -\mathrm{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > 0, \quad \text{where } g\in\Sigma \text{ and satisfies (1.1) with } \gamma = 0;$$

$$(1.3)\qquad -\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > 0;$$

are respectively called *meromorphically γ-spiral, close-to-convex and convex analytic functions in E.* If $\gamma = 0$ in (1.1), then f is meromorphically starlike and has been extensively studied by Pommerenke [5]. We denote the classes of functions $f\in\Sigma$ and satisfying (1.1), (1.2) or (1.3) respectively by $\Sigma(\gamma)$, Γ and C. If $\gamma = 0$, then $\Sigma(0) = \Sigma^*$ denotes the class of starlike functions. The class Γ was introduced by Libera and Robertson [3]. We find that contrary to regular close-to-convex functions in $|z| < 1$, functions $f\in\Gamma$ need not be univalent. In the present note we extend the definition of α-starlike functions, due to Mocanu and Reade [4], from the regular to the meromorphic case and prove in Theorem 1 that functions of this class are univalent and that this class forms a subclass of the class of starlike functions.

For this purpose we introduce the following:

DEFINITION 1. Let $\Sigma(\alpha, \beta, e^{-i\gamma}f)$ denote the class of functions $f(z) = \frac{1}{z}+\sum_{n=0}^{\infty} a_n z^n$ regular in $E\{0<|z|<1\}$ and satisfying

$$(1.4)\qquad \mathrm{Re}\{M(\alpha, \beta, e^{-i\gamma}f)\} > 0,$$

where

$$(1.5)\qquad M(\alpha, \beta, e^{-i\gamma}f) = \alpha\left(1+\frac{zf''(z)}{f'(z)}\right)-(\alpha+\beta e^{i\gamma})\frac{zf'(z)}{f(z)},$$

α, β real and $|\gamma| < \pi/2$.

DEFINITION 2. Let $\Gamma(\alpha, \beta, f)$ denote the class of functions $f\in\Sigma$ and satisfying

$$(1.6)\qquad \mathrm{Re}\{N(\alpha, \beta, f)\} > 0,$$

where

$$(1.7)\qquad N(\alpha, \beta, f) = \alpha\left(1-\frac{zg'(z)}{g(z)}+\frac{zf''(z)}{f'(z)}\right)-\beta\frac{zf'(z)}{g(z)}$$

for some $g\in\Sigma^*$ and α, β real.

THEOREM 1. *If $f\in\Sigma(\alpha, \beta, e^{-i\gamma}f)$ (or $f\in\Gamma(\alpha, \beta, f)$), then $f\in\Sigma(\gamma)$ (resp. $f\in\Gamma$).*

If $\gamma = 0$, then any $f\in\Sigma(\alpha, \beta, f)$ belongs to the class Σ^*.

Proof. The proof of result is the same as that given in [1]. Let

$$(1.8)\qquad -\frac{zf'(z)}{\varphi(f(z))} = \left\{\frac{1-\omega(z)}{1+\omega(z)}\right\}\psi_1+i\psi_2,$$

where

$$(1.9)\qquad \varphi(f) = \begin{cases} g \in \Sigma^*; & \psi_1 = 1, \qquad \psi_2 = 0; \\ e^{-i\gamma} f; & \psi_1 = \cos\gamma, \quad \psi_2 = \sin\gamma. \end{cases}$$

Differentiating (1.8) with respect to z, we get

$$(1.10)\qquad J(\alpha, \beta, \varphi(f))$$
$$= \beta\left[\frac{\psi_1(1-\omega(z))}{1+\omega(z)} + i\psi_2\right] + \alpha\left[\frac{(i\psi_2 - \psi_1) z\omega'(z)}{(\psi_1 + i\psi_2) + (-\psi_1 + i\psi_2)\omega(z)} - \frac{z\omega'(z)}{1+\omega(z)}\right],$$

where

$$J(\alpha, \beta, \varphi(f)) = \alpha\left[1 - \frac{z(\varphi(f(z)))'}{\varphi(f(z))} + \frac{zf''(z)}{f'(z)}\right] - \beta\frac{zf'(z)}{\varphi(f(z))}.$$

From (1.8) it is clear that $\omega(z)$ is regular in $|z| < 1$ and $\omega(0) = 0$. Let $|\omega(z)| \not< 1$ in $|z| < 1$, then there exists a point $z = z_1$ in $|z| < 1$ for which $|\omega(z)| = \max\limits_{|z| \leqslant |z_1|} |\omega(z)| = 1$ and by a lemma of Jack [2], it follows that $z_1\omega(z_1) = K\omega(z_1)$ for some $K \geqslant 1$. Thus at $z = z_1$, from (1.10) we see that $\operatorname{Re}\{J(\alpha, \beta, \varphi(f))\} = 0$. This completes the proof of the theorem by contradiction and by an appeal to the subordination principle.

Note that if $\gamma = 0$, then $f \in \Sigma(\alpha, \beta, e^{-i\gamma} f)$ is starlike.

THEOREM 2. *If* $f \in \Sigma(\alpha, \beta, f)$, *then*

$$(1.11)\qquad -K(\alpha, \beta, -r) \leqslant |df^{-\beta/\alpha}(z)| \leqslant K(\alpha, \beta, r),$$

where

$$(1.12)\qquad K(\alpha, \beta, r) = \left|\frac{\beta}{\alpha}\right|(1+r)^{-2\beta/\alpha} r^{-1+\beta/\alpha} \quad \text{and } \alpha < 0.$$

Proof. Since $f \in \Sigma(\alpha, \beta, f)$ is starlike, there exists an $h \in \Sigma(0)$ such that

$$(1.13)\qquad \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - (\alpha+\beta)\frac{zf'(z)}{f(z)} = -\beta\frac{zh'(z)}{h(z)},$$

(1.13) yields that

$$(1.14)\qquad \{h(z)\}^{-\beta/\alpha} = zf'(z)[f(z)]^{-(\alpha+\beta)/\alpha}.$$

From (1.14) and the following result of Pommerenke [5]:

$$\frac{(1-r)^2}{r^2} \leqslant \left|\frac{h(z)}{z}\right| \leqslant \frac{(1+r)^2}{r^2},$$

(1.11) is immediate.

If $\alpha > 0$, a similar sharp estimate can be found. For $\alpha = 0$, this coincides with the result of Pommerenke. More precise distortion theorems for $f \in \Sigma(\alpha, \beta, f)$ can be obtained on the lines similar as in [5].

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Reçu par la Rédaction le 7. 4. 1973