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On a differential inequality with a lagging argument

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The purpose of this paper is to transfer a result of J. Szarski [2] to the case of differential inequalities with a lagging argument. This note is composed of three parts. In the first part there is a statement of the theorem of J. Szarski for a system of differential inequalities with a discrete lag and parallel to the axis of time. The second part deals with the same theorem in the case of continuous retardement. In the third part we consider a partial differential equation, in which the derivative at a point depends on the values taken by the solution in a plane set.

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1. Let E^{n+1} denote $(n+1)$-dimensional space of points $P(t, x_1, \ldots, x_n)$. Let H be a subset of E^{n+1} given by

$$H = \{t_0 \leqslant t < t_0 + a,\ a_i + N(t-t_0) \leqslant x_i \leqslant b_i - N(t-t_0),\ i = 1, 2, \ldots, n\}$$

where t_0, a, a_i, b_i, N are real constants such that $N > 0$, $a_i < b_i$, $0 < a \leqslant (b_i - a_i)/2N$, $i = 1, 2, \ldots, n$.

Let $a_\nu^\mu(t)$, $\mu = 1, 2, \ldots, m$, $\nu = 1, 2, \ldots, n$, be nonnegative real functions defined in the interval $\langle t_0, t_0 + a)$. Define a subset G of E by

$$G = \{t_0 - \tau \leqslant t \leqslant t_0,\ a_i \leqslant x_i \leqslant b_i,\ i = 1, 2, \ldots, n\},$$

where $\tau = \max\limits_{\mu,\nu}\{\sup\limits_t a_\nu^\mu(t)\}$.

In the sequel we shall use the following notation

$$X = (x_1, \ldots, x_n), \qquad U = (u^1, \ldots, u^m), \qquad Q = (q^1, \ldots, q^n),$$

$$Q(z^\nu) = \left(\frac{\partial z^\nu}{\partial x_1}, \ldots, \frac{\partial z^\nu}{\partial x_n}\right),$$

$$X \leqslant Y \quad \text{if} \quad x_i \leqslant y_i, \quad i = 1, 2, \ldots, n.$$

Let $f^\mu(t, X, U, Q)$, $\mu = 1, 2, \ldots, m$, be a system of functions such that

(i) The function f^μ $(\mu = 1, 2, \ldots, m)$ is defined on a subset Δ_μ of $(2n+m+1)$-dimensional space such that H belongs to $E^{n+1} \cap \Delta_\mu$.

(ii) If $\bar{U} \geqslant \bar{\bar{U}}$ then $f^\mu(t, X, \bar{U}, Q) \geqslant f^\mu(t, X, \bar{\bar{U}}, Q)$ $(\mu = 1, 2, \ldots, m)$.

(iii) The function f^μ $(\mu = 1, 2, \ldots, m)$ satisfies a Lipschitz condition with respect to Q

$$|f^\mu(t, X, U, \bar{Q}) - f^\mu(t, X, U, \bar{\bar{Q}})| \leqslant N \sum_{i=1}^{n} |\bar{q}_i - \bar{\bar{q}}_i|,$$

where N denotes the same constant as in definition of H.

Under these assumptions the following theorem may be proved

THEOREM I. *Assume functions $u^\mu(t, X)$, $v^\mu(t, X)$ $(\mu = 1, 2, \ldots, m)$ to be defined and continuous in $G \cup H$ and to have the differential of Stolz in H. If*

$$u^\mu(t, X) > v^\mu(t, X) \tag{1}$$

for $(t, X) \in G$, $\mu = 1, 2, \ldots, m$,

$$\frac{\partial u^\mu(t, X)}{\partial t} \geqslant f^\mu\big(t, X, U(t - a_\nu^\mu(t), X), Q(u^\mu)\big), \tag{2}$$

$$\frac{\partial v^\mu(t, X)}{\partial t} < f^\mu\big(t, X, V(t - a_\nu^\mu(t), X), Q(v^\mu)\big) \tag{3}$$

for $(t, X) \in H$, $\mu = 1, 2, \ldots, m$, then the inequalities $u^\mu(t, X) > v^\mu(t, X)$, $\mu = 1, 2, \ldots, m$, hold in the whole set H.

The proof of Theorem I is quite analogous to the proof of the theorem of J. Szarski [2] or to the proof of Theorem II in the second part of this paper.

Remark. In the case of $a_\nu^\cdot(t) \equiv 0$, Theorem I remains true under the assumption that the functions f^μ satisfy a weaker condition (W) of [2] instead of (ii).

2. Let G^* be a set of points (t, X) such that $\{t \leqslant t_0,\ a_i \leqslant x_i \leqslant b_i\ (i = 1, 2, \ldots, n)\}$, where a_i and b_i satisfy the same conditions as in the definition of H.

We shall consider a system of differential equations of the form

$$\frac{\partial z^\mu(t, X)}{\partial t} = \int_0^\infty f^\mu\big(t, X, Z(t - s, X), Q(z^\mu)\big)\, dR_s^\mu(s, t, X) + g^\mu(t, X) \tag{a}$$

where

(i′) The functions $R^\mu(s, t, X)$ are defined in $R = \{s \in \langle 0, \infty), (t, X) \in H\}$ and for any fixed $(t, X) \in H$ they are nondecreasing with respect to s.

(ii′) There is a finite number K such that $\bigvee_{s=0}^{\infty} R^\mu(s, t, X) \leqslant K$ for $\mu = 1, 2, \ldots, m$ and for $(t, X) \in H$.

(iii') The function f^{μ} $(\mu = 1, 2, \ldots, m)$ is defined on a subset Δ_{μ} of $(2n+m+1)$-dimensional space such that H belongs to $E^{n+1} \cap \Delta_{\mu}$.

(iv') The function f^{μ} $(\mu = 1, 2, \ldots, m)$ is nondecreasing with respect to Z and satisfies Lipschitz condition with respect to Q

$$|f^{\mu}(t, X, Z, \bar{Q}) - f^{\mu}(t, X, Z, \bar{\bar{Q}})| \leqslant L \sum_{i=1}^{n} |\bar{q}_i - \bar{\bar{q}}_i|,$$

where $L \max_{\mu} \{\sup_{H} \int_{0}^{\infty} dR_s^{\mu}(s, t, X)\} < N$, and N denotes the same number as in the definition of H.

(v') The functions $g^{\mu}(t, X)$ are defined in the set H.

Under all the above assumptions we shall prove the following

THEOREM II. *Suppose that functions $u^{\mu}(t, X)$ and $v^{\mu}(t, X)$, $\mu = 1, 2, \ldots, m$, are defined and continuous in $G^* \cup H$ and that they have the Stolz differential in H. Let*

$$u^{\mu}(t, X) > v^{\mu}(t, X), \tag{4}$$

for $(t, X) \in G^*$, $\mu = 1, 2, \ldots, m$,

$$\frac{\partial u^{\mu}(t, X)}{\partial t} \geqslant \int_{0}^{\infty} f^{\mu}\big(t, X, U(t-s, X), Q(u^{\mu})\big)\, dR_s^{\mu}(s, t, X) + g^{\mu}(t, X), \tag{5}$$

$$\frac{\partial v^{\mu}(t, X)}{\partial t} < \int_{0}^{\infty} f^{\mu}\big(t, X, V(t-s, X), Q(v^{\mu})\big)\, dR_s^{\mu}(s, t, X) + g^{\mu}(t, X) \tag{6}$$

for $(t, X) \in H$, $\mu = 1, 2, \ldots, m$. *Then*

$$u^{\mu}(t, X) > v^{\mu}(t, X) \quad \text{for} \quad (t, X) \in H,\ \mu = 1, 2, \ldots, m.$$

Proof. Since the functions $u^{\mu}(t, X)$ and $v^{\mu}(t, X)$ are continuous and because of (4) there is τ^0 such that $\tau^0 > t_0$ and

$$u^{\mu}(t, X) > v^{\mu}(t, X), \quad \mu = 1, 2, \ldots, m, \quad \text{if} \quad t_0 \leqslant t < \tau^0.$$

Let t^* denote the l.u.b. of all such τ^0. In order to prove our theorem we have to show that $t^* = t_0 + a$.

Suppose $t^* < t_0 + a$. Then, in virtue of the continuity of $u^{\mu}(t, X)$ and $v^{\mu}(t, X)$, we would have

$$u^{\mu}(t, X) \geqslant v^{\mu}(t, X) \quad \text{for} \quad t_0 \leqslant t \leqslant t^*,\ \mu = 1, 2, \ldots, m \tag{7}$$

and there would exist an index σ and a point $(t^*, X^*) \in H$ such that

$$u^{\sigma}(t^*, X^*) = v^{\sigma}(t^*, X^*). \tag{8}$$

We shall consider two cases: (A) the point (t^*, X^*) belongs to the interior of H and (B) the point (t^*, X^*) belongs to the boundary of H.

Case (A). According to the definition of (t^*, X^*) we have

$$\frac{\partial u^\sigma(t^*, X^*)}{\partial x_i} = \frac{\partial v^\sigma(t^*, X^*)}{\partial x_i} \quad \text{for} \quad i = 1, 2, \ldots, n, \tag{9}$$

and

$$\frac{\partial u^\sigma(t^*, X^*)}{\partial t} \leqslant \frac{\partial v^\sigma(t^*, X^*)}{\partial t}. \tag{10}$$

On the other hand, in virtue of (5) and (6), we have

$$\frac{\partial u^\sigma(t^*, X^*)}{\partial t} \geqslant \int_0^\infty f^\sigma\big(t^*, X^*, U(t^*-s, X^*), Q\big(u^\sigma(t^*, X^*)\big)\big) dR_s^\sigma(s, t^*, X^*) + g^\sigma(t^*, X^*), \tag{11}$$

$$\frac{\partial v^\sigma(t^*, X^*)}{\partial t} < \int_0^\infty f^\sigma\big(t^*, X^*, V(t^*-s, X^*), Q\big(v^\sigma(t^*, X^*)\big)\big) dR_s^\sigma(s, t^*, X^*) + g^\sigma(t^*, X^*). \tag{12}$$

By (9) and by the choice of t^* we have

$$Q\big(u^\sigma(t^*, X^*)\big) = Q\big(v^\sigma(t^*, X^*)\big) \quad \text{and} \quad U(t^*-s, X^*) \geqslant V(t^*-s, X^*) \quad \text{for} \quad 0 \leqslant s < \infty. \tag{13}$$

The monotonicity of f^σ and R^σ and the inequalities (11), (12) and (13) give

$$\frac{\partial v^\sigma(t^*, X^*)}{\partial t} < \frac{\partial u^\sigma(t^*, X^*)}{\partial t}, \tag{14}$$

which contradicts (10). So the case (A) does not hold.

Case (B). Here we can assume that the point (t^*, X^*) lies on the intersection of the planes

$$\begin{cases} x_i = b_i - N(t-t_0), & i = 1, 2, \ldots, r, \\ x_i = a_i + N(t-t_0), & i = r+1, r+2, \ldots r+s, \quad r+s \leqslant n. \end{cases} \tag{15}$$

Let us consider onesided partial derivatives of $\{u^\sigma(t, X) - v^\sigma(t, X)\}$ at (t^*, X^*) defined by

$$\frac{\partial(u^\sigma - v^\sigma)}{\partial x_i} = \frac{\partial(u^\sigma - v^\sigma)}{\partial[x_i - 0]}, \quad i = 1, 2, \ldots, r$$

$$\frac{\partial(u^\sigma - v^\sigma)}{\partial x_i} = \frac{\partial(u^\sigma - v^\sigma)}{\partial[x_i+0]}, \quad i = r+1, r+2, \ldots, r+s,$$

$$\frac{\partial(u^\sigma - v^\sigma)}{\partial t} = \frac{\partial(u^\sigma - v^\sigma)}{\partial[t-0]}.$$

Observe that by the definition of (t^*, X^*) we have

$$(16) \quad \begin{cases} \dfrac{\partial[u^\sigma(t^*, X^*) - v^\sigma(t^*, X^*)]}{\partial x_i} \leqslant 0, & i = 1, 2, \ldots, r, \\ \dfrac{\partial[u^\sigma(t^*, X^*) - v^\sigma(t^*, X^*)]}{\partial x_i} \geqslant 0, & i = r+1, r+2, \ldots, r+s, \\ \dfrac{\partial[u^\sigma(t^*, X^*) - v^\sigma(t^*, X^*)]}{\partial x_i} = 0, & i = r+s+1, \ldots, n, \\ \dfrac{\partial[u^\sigma(t^*, X^*) - v^\sigma(t^*, X^*)]}{\partial t} \leqslant 0. \end{cases}$$

Let $h(t)$ denote an auxiliary function defined by

$$h(t) = u^\sigma\big(t, x_1(t), \ldots, x_r(t), x_{r+1}(t), \ldots, x_{r+s}(t), x^*_{r+s+1}, \ldots, x^*_n\big) - \\ - v^\sigma\big(t, x_1(t), \ldots, x_r(t), x_{r+1}(t), \ldots, x_{r+s}(t), x^*_{r+s+1}, \ldots, x^*_n\big),$$

where

$$x_i(t) = \begin{cases} b_i - N(t-t_0), & i = 1, 2, \ldots, r, \\ a_i + N(t-t_0), & i = r+1, r+2, \ldots, r+s. \end{cases}$$

We have $h(t) \geqslant 0$ for $t \leqslant t^*$ and $h(t^*) = 0$. Hence $h'(t^*-0) \leqslant 0$. Since

$$h'(t) = \frac{\partial[u^\sigma - v^\sigma]}{\partial t} - N\sum_{i=1}^{r} \frac{\partial[u^\sigma - v^\sigma]}{\partial x_i} + N\sum_{i=r+1}^{r+s} \frac{\partial[u^\sigma - v^\sigma]}{\partial x_i}$$

and $h'(t^*-0) \leqslant 0$, so according to (16) we get

$$(17) \quad \left|\frac{\partial u^\sigma(t^*, X^*)}{\partial t} - \frac{\partial v^\sigma(t^*, X^*)}{\partial t}\right| \geqslant N\sum_{i=1}^{n}\left|\frac{\partial u^\sigma(t^*, X^*)}{\partial x_i} - \frac{\partial v^\sigma(t^*, X^*)}{\partial x_i}\right|.$$

By (iv′) (Lipschitz condition), (11) and (12) we have

$$\left|\frac{\partial u^\sigma(t^*, X^*)}{\partial t} - \frac{\partial v^\sigma(t^*, X^*)}{\partial t}\right|$$

$$\leqslant \int_0^\infty \left|f^\sigma\big(t^*, X^*, U(t^*-s, X^*), Q(u^\sigma)\big) - f^\sigma\big(t^*, X^*, V(t^*-s, X^*), Q(v^\sigma)\big)\right| dR^\sigma_s(s, t^*, X^*)$$

$$\leqslant L\sum_{i=1}^{n}\left|\frac{\partial u^\sigma(t^*, X^*)}{\partial x_i} - \frac{\partial v^\sigma(t^*, X^*)}{\partial x_i}\right| \int_0^\infty dR^\sigma_s(s, t^*, X^*)$$

$$< N\sum_{i=1}^{n}\left|\frac{\partial u^\sigma(t^*, X^*)}{\partial x_i} - \frac{\partial v^\sigma(t^*, X^*)}{\partial x_i}\right|$$

whence

$$(18)\qquad \left|\frac{\partial u^\sigma(t^*, X^*)}{\partial t}-\frac{\partial v^\sigma(t^*, X^*)}{\partial t}\right| < N\sum_{i=1}^{n}\left|\frac{\partial u^\sigma(t^*, X^*)}{\partial x_i}-\frac{\partial v^\sigma(t^*, X^*)}{\partial x_i}\right|$$

this contradicts (17). So the case (B) does not hold either. The proof is completed.

3. Let us consider the differential equation of the form

$$(\mathrm{b})\qquad \frac{\partial z(t, x)}{\partial t}=\int\limits_0^\infty\left\{\int\limits_{a(t,s)}^{\beta(t,s)} F\left(t, x, z(t-s, \xi), \frac{\partial z(t, x)}{\partial x}\right) dR_\xi(t, x, \xi)\right\} dr_s(s, t, x).$$

Let

$$H^0=\{t_0\leqslant t<t_0+a^0,\ a+N(t-t_0)\leqslant x\leqslant b-N(t-t_0)\},$$

where $N>0$, $a<b$, $0<a^0\leqslant(b-a)/2N$.

Let

$$a(t, s)=\begin{cases}a+N(t-s-t_0), & t-s\geqslant t_0,\\ a, & t-s<t_0;\end{cases}$$

$$\beta(t, s)=\begin{cases}b-N(t-s-t_0), & t-s\geqslant t_0,\\ b, & t-s<t_0.\end{cases}$$

Concerning (b) we shall assume that

(i″) The function $R(t, x, \xi)$ is defined in $H^0\times\langle a, b\rangle$ and nondecreasing with respect to ξ.

(ii″) The function $r(s, t, x)$ is defined in $\langle 0, \infty)\times H^0$ and nondecreasing with respect to s.

(iii″) There is a real number $M>0$ such that

$$\bigvee_{s=0}^{\infty}\{r(s, t, x)\bigvee_{\xi=a(t,s)}^{\beta(t,s)} R(t, x, \xi)\}\leqslant M, \qquad (t, x)\in H^0.$$

(iv″) The function $F(t, x, z, p)$ is defined for $(t, x)\in H^0$ and for all z and p, and is nondecreasing with respect to z and satisfies Lipschitz condition with respect to p

$$|F(t, x, z, \overline{p})-F(t, x, z, \overline{\overline{p}})|\leqslant L|\overline{p}-\overline{\overline{p}}|$$

and moreover $LM<N$.

Under all these assumptions the folloving theorem, analogous to Theorems I and II, may be proved.

THEOREM III. *Let* $G^0 = \{t \leqslant t_0,\ a \leqslant x \leqslant b\}$ *and let functions* $u(t, x)$ *and* $v(t, x)$ *be defined and continuous in* $G^0 \cup H^0$. *Let* $u(t, x)$ *and* $v(t, x)$ *have the differential of Stolz in* H^0.

If $u(t, x) > v(t, x)$ *for* $(t, x) \in G^0$ *and*

$$\frac{\partial u(t,x)}{\partial t} \geqslant \int_0^{\infty} \left\{ \int_{\alpha(t,s)}^{\beta(t,s)} F\left(t, x, u(t-s, \xi), \frac{\partial u(t,x)}{\partial x}\right) dR_{\xi}(t, x, \xi)\right\} dr_s(s, t, x),$$

$$\frac{\partial v(t,x)}{\partial t} < \int_0^{\infty} \left\{ \int_{\alpha(t,s)}^{\beta(t,s)} F\left(t, x, v(t-s, \xi), \frac{\partial v(t,x)}{\partial x}\right) dR_{\xi}(t, x, \xi)\right\} dr_s(s, t, x)$$

for $(t, x) \in H^0$, *then*

$$u(t, x) > v(t, x) \quad \text{for} \quad (t, x) \in H^0.$$

The proof of Theorem III is analogous to the proof of Theorem II.

References

[1] А. Д. Мышкис, *Линейные дифференциальные уравнения с запаздывающим аргументом*, Москва-Ленинград 1951.

[2] J. Szarski, *Systèmes d'inégalités différentielles aux derivées partielles du premier ordre et leurs applications*, Ann. Polon. Math. 1 (1954), pp. 149-165.

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