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ON COUNTEREXAMPLES IN MULTIVARIATE APPROXIMATION

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In this note we work out several examples in order to indicate how our previous quantitative resonance principles apply to multivariate approximation problems. All processes under consideration will possess some product structure, thus involving limits with regard to several parameters.

Our first example is concerned with error bounds for product cubature formulas. Let $C[0, 1]$ be the space of functions f, continuous on the compact interval $[0, 1]$, endowed with the usual max-norm $\|f\|_C$. For the remainder $R_k := Q - Q_k$ of the compound trapezoidal rule

$$Q_k f := \frac{1}{2k} \sum_{\mu=1}^{k} \left[f\left(\frac{\mu-1}{k}\right) + f\left(\frac{\mu}{k}\right) \right], \quad Qf := \int_0^1 f(u)\, du,$$

one has for any $f \in C[0, 1]$ and $k \in N$ ($:=$ set of natural numbers)

$$(1) \quad |R_k f| \leq \tfrac{1}{2} \sum_{\mu=1}^{k} \int_{(\mu-1)/k}^{\mu/k} \left| \left[f(u) + f\left(\frac{2\mu-1}{k} - u\right) - 2f\left(\frac{2\mu-1}{2k}\right) \right] \right.$$
$$\left. - \left[f\left(\frac{\mu-1}{k}\right) + f\left(\frac{\mu}{k}\right) - 2f\left(\frac{2\mu-1}{2k}\right) \right] \right| du$$
$$\leq \omega_2(f; 1/2k) := \sup_{|h| \leq 1/2k} \|f(u+2h) - 2f(u+h) + f(u)\|_C.$$

This estimate is sharp (see proof of Corollary 1). Introducing the second partial moduli of continuity $\omega_2(f; s, 0)$, $\omega_2(f; 0, t)$ of $f(x, y) \in C[0, 1]^2$, thus (cf. [10], p. 112)

$$(2) \qquad \omega_2(f; s, 0) := \sup_{|h| \leq s} \|f(x+2h, y) - 2f(x+h, y) + f(x, y)\|_C,$$

for the corresponding product cubature rule

$$Q_{m,n}f := (Q_m \times Q_n)f$$
$$:= \frac{1}{4mn}\sum_{\mu=1}^{m}\sum_{\nu=1}^{n}\left[f\left(\frac{\mu-1}{m}, \frac{\nu-1}{n}\right)+f\left(\frac{\mu}{m}, \frac{\nu-1}{n}\right)+f\left(\frac{\mu-1}{m}, \frac{\nu}{n}\right)\right.$$
$$\left.+f\left(\frac{\mu}{m}, \frac{\nu}{n}\right)\right],$$

$$R_{m,n}f := (Q\times Q)f - Q_{m,n}f, \quad (Q\times Q)f := \int_0^1\int_0^1 f(u, v)\,du\,dv,$$

one therefore has for any $f \in C[0, 1]^2$, $(m, n) \in N^2$

$$|R_{m,n}f| \leqslant |((Q-Q_m)\times Q)f| + |(Q_m\times(Q-Q_n))f|$$
$$\leqslant \omega_2(f; 1/2m, 0)+\omega_2(f; 0, 1/2n).$$

Again this error bound is sharp in the following sense. Let ω be a modulus of continuity, thus continuous on $[0, \infty)$ with (cf. [10], p. 96)

$$\omega(0) = 0 < \omega(s) \leqslant \omega(s+t) \leqslant \omega(s)+\omega(t) \quad (0 < s, t), \tag{3}$$

often additionally satisfying

$$\lim_{t\to 0+} \omega(t)/t = \infty. \tag{4}$$

COROLLARY 1. *For any subsequence* $\{(m_k, n_k)\}_{k\in N} \subset N^2$ *with* $m_k, n_k \to \infty$ *for* $k \to \infty$ *and for any moduli* ω, ψ *subject to* (3) *there exists a counterexample* $f_{\omega,\psi} \in C[0, 1]^2$ *satisfying* $(s \to 0+, t \to 0+)$

$$\omega_2(f_{\omega,\psi}; s, 0) = O(\omega(s^2)), \quad \omega_2(f_{\omega,\psi}; 0, t) = O(\psi(t^2))$$

such that for the compound trapezoidal cubature rule $(k \to \infty)$

$$\int_0^1\int_0^1 f_{\omega,\psi}(u, v)\,du\,dv - Q_{m_k,n_k}f_{\omega,\psi} \neq o(\omega(m_k^{-2})+\psi(n_k^{-2})).$$

The proof will be given by an iterative (componentwise) application of the following uniform boundedness principle with small o-rates. Let X be a Banach space and X^* the class of real-valued functionals T on X which are sublinear, i.e.,

$$|T(f+g)| \leqslant |Tf|+|Tg|, \quad |T(af)| = |a|\,|Tf|$$

for all $f, g \in X$ and $a \in \boldsymbol{R}$ (:= set of reals), and which are bounded, i.e.,

$$\|T\|_{X^*} := \sup\{|Tf|: \|f\|_X \leqslant 1\} < \infty.$$

Let $\sigma(r)$ be a strictly positive function on $(0, \infty)$, and let $\{\varphi_k\} \subset \boldsymbol{R}$ be a sequence, strictly decreasing to zero.

THEOREM 1. *Suppose that for linear $T_k \in X^*$ there are elements $g_k \in X$ satisfying*

$$\|g_k\|_X \leqslant C_1 \quad (k \in N), \tag{5}$$

$$\limsup_{k \to \infty} T_k g_k \geqslant C_2 > 0. \tag{6}$$

Furthermore, let $\{U_r\colon r > 0\} \subset X^*$ be such that

$$|U_r g_k| \leqslant M \min\{1, \sigma(r)/\varphi_k\} \quad (k \in N, r > 0). \tag{7}$$

Then for each ω subject to (3), (4) *there exists a counterexample $f_\omega \in X$ such that*

$$|U_r f_\omega| = O(\omega(\sigma(r))) \quad (r \to 0+), \tag{8}$$

$$\limsup_{k \to \infty} T_k f_\omega/\omega(\varphi_k) \geqslant C_2 > 0. \tag{9}$$

For proofs and detailed comments see [3]–[6] in connection with the observation (see [12]) that the linearity of the real-valued functionals T_k ensures $T_k(f+g) \geqslant T_k f - |T_k g|$ (cf. proof of Theorem 3).

Proof of Corollary 1. Proceeding iteratively, let us start with the one-dimensional problem in x for $T_k = R_{m_k}$, $g_k(x) = \sin^2(2\pi m_k x)$, and $U_r f = \omega_2(f; r)$ (cf. (1)) on $X = C[0, 1]$. Obviously, since $T_k g_k = 1/2$, conditions (5), (6) hold true, and (7) follows with $\sigma(r) = r^2$, $\varphi_k = m_k^{-2}$. Consequently, if ω satisfies (4), Theorem 1 ensures the existence of $f_\omega(x) \in C[0, 1]$ satisfying

$$\omega_2(f_\omega; r) = O(\omega(r^2)), \qquad R_{m_k} f_\omega/\omega(m_k^{-2}) \geqslant 1,$$

at least for a subsequence. Concerning the limiting case $\omega(t) = t$ note that for $f_1(x) = -x^2$ one has $T_k f_1 = 1/6m_k^2$. Thus, in any case, for such a subsequence of k the analogous argument in y leads to a corresponding counterexample $f_\psi(y)$. Therefore the assertion follows for $f_{\omega,\psi}(x, y) = f_\omega(x) + f_\psi(y)$ since at least for a subsequence

$$R_{m_k,n_k} f_{\omega,\psi} = R_{m_k} f_\omega + R_{n_k} f_\psi \geqslant \omega(m_k^{-2}) + \psi(n_k^{-2}). \quad \blacksquare$$

A further application of Theorem 1 is concerned with the best approximation $E_{m,n}(f)_p$ of $f(x, y) \in L^p_{2\pi,2\pi}$ by trigonometric polynomials $t_{m,n}(x, y)$ of degree m in x and n in y, i.e., $t_{m,n} \in \Pi_{m,n}$. Here $L^p_{2\pi,2\pi}$, $1 \leqslant p \leqslant \infty$, is the space of functions, 2π-periodic in each variable and, if $1 \leqslant p < \infty$, pth power integrable over $[0, 2\pi]^2$, with the usual norm and the convention $L^\infty_{2\pi,2\pi} = C_{2\pi,2\pi}$, the space of continuous functions. Let $L^p_{2\pi}$ be the corresponding spaces of functions of one variable. Defining second order partial moduli of continuity in $L^p_{2\pi,2\pi}$ as usual (cf. (2)), it is well known (cf. [10], p. 273) that

$$E_{m,n}(f)_p \leqslant M[\omega_2(f; 1/(m+1), 0)_p + \omega_2(f; 0, 1/(n+1))_p]. \tag{10}$$

This estimate is again sharp in the following sense.

COROLLARY 2. *For any subsequence $\{(m_k, n_k)\}_{k\in N} \subset N^2$ with $m_k, n_k \to \infty$ for $k \to \infty$ and for any moduli ω, ψ subject to* (3), (4) *there exists a counterexample $f_{\omega,\psi} \in L^p_{2\pi,2\pi}$ satisfying*

$$\omega_2(f_{\omega,\psi}; s, 0)_p = O(\omega(s^2)), \quad \omega_2(f_{\omega,\psi}; 0, t)_p = O(\psi(t^2)),$$
$$E_{m_k,n_k}(f_{\omega,\psi})_p \neq o(\omega(m_k^{-2}) + \psi(n_k^{-2})) \quad (k \to \infty).$$

Proof. Let us first collect some elementary facts concerning the de La Vallée Poussin means

$$(V_k f)(x) := \sum_{\mu=-2k}^{2k} \lambda(\mu/k) f^\wedge(\mu) e^{i\mu x},$$

$$f^\wedge(\mu) := \frac{1}{2\pi}\int_0^{2\pi} f(u) e^{-i\mu u}\, du, \qquad \lambda(u) := \begin{cases} 1, & |u| \leqslant 1, \\ 2-|u|, & 1 \leqslant |u| \leqslant 2, \\ 0, & 2 \leqslant |u| \end{cases}$$

of $f \in L^p_{2\pi}$. It follows that $V_k f \in \Pi_{2k-1}$ ($:=$ set of trigonometric polynomials of degree $2k-1$ in x) and

$$\|f - V_k f\|_p \leqslant 4E_k(f)_p := 4\inf\{\|f - t_k\|_p :\ t_k \in \Pi_k\}.$$

Correspondingly, for the 2-dimensional analogue

$$(V_{m,n} f)(x, y) := \sum_{\mu=-2m}^{2m} \sum_{\nu=-2n}^{2n} \lambda(\mu/m)\lambda(\nu/n) f^\wedge(\mu, \nu) e^{i(\mu x + \nu y)},$$

(11) $$f^\wedge(\mu, \nu) := \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} f(u, v) e^{-i(\mu u + \nu v)}\, du\, dv,$$

one has for any $f \in L^p_{2\pi,2\pi}$ and $(m, n) \in N^2$

(12) $$\|f - V_{m,n} f\|_p \leqslant 10\, E_{m,n}(f)_p.$$

To start with $p = \infty$, consider the linear functional $T_k f = (V_{m_k} f)(0) - f(0)$ for $f \in C_{2\pi}$. It follows that (cf. (1), (10))

$$|T_k f| \leqslant \|V_{m_k} f - f\|_\infty \leqslant 4E_{m_k}(f)_\infty \leqslant M\omega_2(f; 1/m_k)_\infty.$$

To show that the latter estimate is sharp, choose $g_k(x) = -\cos(2m_k x)$ and $U_r f = \omega_2(f; r)_\infty$ on $X = C_{2\pi}$. Since $T_k g_k = 1$, all the assumptions of Theorem 1 are satisfied, delivering a counterexample $f_\omega \in C_{2\pi}$, which one may normalize by $f_\omega(0) = 0$, such that

$$\omega_2(f_\omega; r)_\infty = O(\omega(r^2)), \quad \limsup_{k\to\infty} (V_{m_k} f_\omega)(0)/\omega(m_k^{-2}) > 1.$$

Upon passing to a suitable subsequence, the analogous argument in y leads to a corresponding $f_\psi(y)$. With $f_{\omega,\psi}(x, y) = f_\omega(x) + f_\psi(y)$ one therefore has by

(12) that at least for a subsequence

$$10E_{m_k,n_k}(f_{\omega,\psi})_\infty \geqslant \|f_{\omega,\psi}-V_{m_k,n_k}f_{\omega,\psi}\|_\infty \geqslant |(V_{m_k,n_k}f_{\omega,\psi})(0)|$$
$$= (V_{m_k}f_\omega)(0)+(V_{n_k}f_\psi)(0) \geqslant \omega(m_k^{-2})+\psi(n_k^{-2}).$$

Concerning the cases $1 \leqslant p < \infty$, choose $X = L_{2\pi}^p$, $g_k(x) = \cos(2m_k x)$, $U_r f = \omega_2(f; r)_p$, and

$$T_{m_k} f = \frac{1}{2\pi} \int_{A_{m_k}} [f(u)-(V_{m_k}f)(u)]\,du, \quad A_{m_k} := \{x \in [0, 2\pi]:\ \cos(2m_k x) \geqslant 1/2\}.$$

In view of

$$T_{m_k} g_k = \frac{1}{2\pi} \int_{A_{m_k}} \cos(2m_k u)\,du \geqslant \frac{1}{4\pi}\operatorname{meas} A_{m_k} = C_2 > 0,$$

all the assumptions of Theorem 1 are satisfied, delivering $f_\omega \in L_{2\pi}^p$ with

$$\omega_2(f_\omega; r)_p = O(\omega(r^2)), \qquad \limsup_{k\to\infty} T_{m_k} f_\omega/\omega(m_k^{-2}) > 1.$$

Proceeding again iteratively, one arrives at some counterexample

$$f_{\omega,\psi}(x, y) = f_\omega(x)+f_\psi(y)$$

for which by (12) and Hölder's inequality

$$10\,E_{m_k,n_k}(f_{\omega,\psi})_p \geqslant \|f_{\omega,\psi}-V_{m_k,n_k}f_{\omega,\psi}\|_1$$
$$\geqslant \left(\frac{1}{2\pi}\right)^2 \int_{A_{n_k}} \int_{A_{m_k}} [f_\omega(x)-(V_{m_k}f_\omega)(x)+f_\psi(y)-(V_{n_k}f_\psi)(y)]\,dx\,dy$$
$$= \frac{1}{2\pi}(\operatorname{meas} A_{n_k})\,T_{m_k}f_\omega + \frac{1}{2\pi}(\operatorname{meas} A_{m_k})\,T_{n_k}f_\psi$$
$$\geqslant 2C_2\,[\omega(m_k^{-2})+\psi(n_k^{-2})],$$

at least for a subsequence of k. ∎

As a consequence of problems, posed by J. Favard in 1963 and by S. B. Stechkin in 1977, Theorem 1 was extended to the following negative result concerning a comparison of two processes (see [5], [6] for details).

THEOREM 2. *Suppose that for $R_k, S_k, T_k \in X^*$ there are elements $g_k \in X$ satisfying* (5), (6) *and*

$$\limsup_{k\to\infty} |R_k g_k| \geqslant C_2 > 0, \tag{13}$$

$$|S_k g_k| \leqslant B_1, \qquad \limsup_{j\to\infty} |S_j g_k|/\varphi_j = B_{2,k} < \infty. \tag{14}$$

Furthermore, let $\{U_r : r > 0\} \subset X^*$ *be such that* (7) *holds true. Then for each* ω *subject to* (3), (4) *there exists a counterexample* $f_\omega \in X$ *satisfying* (8), (9) *as well as*

$$|R_k f_\omega| \neq o(|S_k f_\omega|) \quad (k \to \infty). \tag{15}$$

To indicate multivariate applications also in this case, let the partial best approximation $E_{\infty,n}(f)_p$ (and similarly $E_{m,\infty}(f)_p$) be defined as the best approximation of $f(x, y) \in L^p_{2\pi,2\pi}$ by trigonometric polynomials of degree n in y with coefficients, depending on x (cf. [10], p. 33). Obviously, $E_{m,\infty}(f)_p \leqslant E_{m,n}(f)_p$. In the converse direction one has that for $p = 1, \infty$

$$E_{m,n}(f)_p \leqslant A\,[E_{m,\infty}(f)_p + E_{\infty,n}(f)_p] \log(2+\min\{m, n\}), \tag{16}$$

whereas for $1 < p < \infty$ the log-factor may be replaced by an absolute constant (see [1]; [11]; [10], p. 34). In fact, it was shown in [8], [9] that (16) on the diagonal $m = n$ is sharp for $p = 1, \infty$. Let us illustrate how the latter assertion can be seen in the light of Theorem 2.

To this end, let us start with the familiar inverse results (cf. [10], p. 350)

$$\omega_1(f; 1/m, 0)_p \leqslant (M/m) \sum_{\mu=0}^{m} E_{\mu,\infty}(f)_p, \qquad \omega_1(f; 0, 1/n)_p \leqslant (M/n) \sum_{\nu=0}^{n} E_{\infty,\nu}(f)_p.$$

These together with (10) (with ω_2 replaced by the first moduli ω_1) imply for $m = n$ that the weak-type inequality

$$E_{n,n}(f)_p \leqslant (M/n) \sum_{\mu=0}^{n} [E_{\mu,\infty}(f)_p + E_{\infty,\mu}(f)_p] \tag{17}$$

holds true. Much in the spirit of the problem, posed by S. B. Stechkin in 1977 (cf. [2]), one may then ask whether (17) can even be strengthened to an estimate of (the more direct) type

$$E_{n,n}(f)_p \leqslant M_f\,[E_{n,\infty}(f)_p + E_{\infty,n}(f)_p]$$

for certain classes of not too smooth functions f, in other words, (16) for $m = n$ should hold without any log-factor. As already mentioned, the answer is negative for $p = 1, \infty$ (and positive for $1 < p < \infty$).

COROLLARY 3. *Let* $p = 1$ *or* $p = \infty$. *For each decreasing nullsequence* $\varepsilon = \{\varepsilon_n\}$ *there exists a counterexample* $f_\varepsilon \in L^p_{2\pi,2\pi}$ *such that*

$$E_{n,n}(f_\varepsilon)_p = O(\varepsilon_n), \qquad E_{n,n}(f_\varepsilon)_p \neq o(\varepsilon_{2n}),$$

$$\limsup_{n\to\infty} \frac{E_{n,n}(f_\varepsilon)_p}{[E_{n,\infty}(f_\varepsilon)_p + E_{\infty,n}(f_\varepsilon)_p] \log n} \geqslant 1.$$

Proof. It was shown in [8] that for $p = 1, \infty$ there exist elements $g_n \in \Pi_{n,2n}$, thus $E_{n,\infty}(g_n)_p = 0$, satisfying (5) as well as

$$E_{\infty,n}(g_n)_p \log n \leqslant B_1, \qquad E_{n,n}(g_n)_p \geqslant C_2 > 0. \tag{18}$$

For example, for $p = 1$ one may choose

$$g_n(x, y) = \frac{1}{\log n} e^{i(2n+1)y} \sum_{\mu=1}^{n} \frac{\mu}{n} e^{i\mu(x-y)}.$$

To apply Theorem 2, set $R_n = T_n = U_{1/n} = E_{n,n}$ and

$$S_n f = [E_{n,\infty}(f)_p + E_{\infty,n}(f)_p] \log n.$$

Obviously, (6), (13) follow by (18) as well as (14) for any $\{\varphi_j\}$ since, e.g., $S_j g_k = 0$ for $j \geqslant 2k$. Moreover, (7) holds true with $\sigma(1/n) = \varepsilon_n^2$, $\varphi_k = \varepsilon_{2k}^2$ since

$$U_{1/n} g_k \leqslant \left\{ \begin{matrix} \|g_k\|_p \\ 0 \text{ for } n \geqslant 2k \end{matrix} \right\} \leqslant C_1 \min\{1, \varepsilon_n^2/\varepsilon_{2k}^2\}.$$

Therefore Theorem 2 for $\omega(r) = r^{1/2}$ delivers the assertion. ∎

Let us finally discuss the negative results, given in [7] in connection with double conjugate Fourier series, in the light of the following extension of Theorem 1.

THEOREM 3. *Let σ, τ be strictly positive functions on $(0, \infty)$ and $\{\varphi_k\} \subset \boldsymbol{R}$ be a sequence, strictly decreasing to zero. Let ω, ψ be strictly increasing moduli of continuity satisfying* (3), (4). *Suppose that for linear $T_k \in X^*$ there are elements $g_k \in X$ such that for $N_k \geqslant 1$*

$$\|g_k\|_X \leqslant N_k \quad (k \in \boldsymbol{N}), \tag{19}$$

$$N_k \omega(\varphi_k) = o(1) \quad (k \to \infty), \tag{20}$$

$$\limsup_{k\to\infty} T_k g_k \geqslant C_2 > 0. \tag{21}$$

Furthermore, let $\{U_s\colon s > 0\}$, $\{V_t\colon t > 0\} \subset X^*$ be such that

$$|U_s g_k| \leqslant M \min\{1, \sigma(s)/\varphi_k\} \quad (k \in \boldsymbol{N}, s > 0), \tag{22}$$

$$|V_t g_k| \leqslant M \min\{1, \tau(t)/\alpha_k\} \quad (k \in \boldsymbol{N}, t > 0), \tag{23}$$

where $\alpha_k := \psi^{-1}(\omega(\varphi_k))$ (thus also strictly decreasing to zero). Then there exists a counterexample $f_0 \in X$ such that

$$|U_s f_0| = O(\omega(\sigma(s))) \quad (s \to 0+), \tag{24}$$

$$|V_t f_0| = O(\psi(\tau(t))) \quad (t \to 0+), \tag{25}$$

$$\limsup_{k\to\infty} T_k f_0/\omega(\varphi_k) \geqslant C_2 > 0. \tag{26}$$

Proof. First of all, g_m as well as $-g_m$ satisfy (24), (25) in view of (4), (22), (23). Therefore, if (26) already holds true for $\pm g_m$, there is nothing to prove. But if (26) does not hold true for $\pm g_m$, then

$$\limsup_{k\to\infty} \frac{T_k g_m}{\omega(\varphi_k)} \leqslant 0 \leqslant -\limsup_{k\to\infty} \frac{T_k(-g_m)}{\omega(\varphi_k)} = \liminf_{k\to\infty} \frac{T_k g_m}{\omega(\varphi_k)}.$$

Hence without loss of generality one may assume that

$$(27)\qquad |T_k g| = o(\omega(\varphi_k)) \quad (k \to \infty)$$

for all $g \in G := \operatorname{span}\{g_k \colon k \in N\}$. Now one may successively construct a strictly increasing subsequence $\{k_j\} \subset N$ such that

$$(28)\qquad N_{k_{m+1}}\,\omega(\varphi_{k_{m+1}}) \leqslant \tfrac{1}{2}\omega(\varphi_{k_m}),$$

$$(29)\qquad \sum_{j=1}^{m-1} \omega(\varphi_{k_j})/\varphi_{k_j} \leqslant \omega(\varphi_{k_m})/\varphi_{k_m}, \quad \sum_{j=1}^{m-1} \psi(\alpha_{k_j})/\alpha_{k_j} \leqslant \psi(\alpha_{k_m})/\alpha_{k_m},$$

$$(30)\qquad |T_{k_m} f_{m-1}| \leqslant \omega(\varphi_{k_m})/m \quad \text{for } f_{m-1} := \sum_{j=1}^{m-1} \omega(\varphi_{k_j})\, g_{k_j} \in G,$$

$$(31)\qquad \|T_{k_m}\|_{X^*}\,\omega(\varphi_{k_{m+1}}) \leqslant \omega(\varphi_{k_m})/m,$$

$$(32)\qquad T_{k_m} g_{k_m} \geqslant C_2 - 1/m,$$

upon using (20), (4), (27), (3), (21), respectively. Then by (28)

$$(33)\qquad \sum_{j=m+1}^{\infty} \omega(\varphi_{k_j}) \leqslant \sum_{j=m+1}^{\infty} \omega(\varphi_{k_j})\, N_{k_j} \leqslant 2\omega(\varphi_{k_{m+1}})$$

so that $f_0 := \sum_{j=1}^{\infty} \omega(\varphi_{k_j})\, g_{k_j}$ is well-defined in X. Suppose that $s > 0$ is such that $0 < \sigma(s) \leqslant \varphi_{k_1}$. Then there exists $m \in N$ such that $\varphi_{k_{m+1}} < \sigma(s) \leqslant \varphi_{k_m}$, and therefore by (22), (29), (33)

$$\begin{aligned} |U_s f_0| &\leqslant \sum_{j=1}^{m} M\sigma(s)\,\omega(\varphi_{k_j})/\varphi_{k_j} + M \sum_{j=m+1}^{\infty} \omega(\varphi_{k_j}) \\ &\leqslant 2M\,[\sigma(s)\,\omega(\varphi_{k_m})/\varphi_{k_m} + \omega(\varphi_{k_{m+1}})] \leqslant 6M\omega(\sigma(s)). \end{aligned}$$

If $s > 0$ is such that $\sigma(s) > \varphi_{k_1}$, then

$$|U_s f_0| \leqslant M \sum_{j=1}^{\infty} \omega(\varphi_{k_j}) \leqslant 2M\omega(\varphi_{k_1}) \leqslant 2M\omega(\sigma(s)),$$

thus (24) in any case. Since $\omega(\varphi_k) = \psi(\alpha_k)$ by definition, (25) follows analogously. Finally, since T_k is linear, by (30)–(33)

$$\begin{aligned} T_{k_m} f_0 &\geqslant \omega(\varphi_{k_m})\, T_{k_m} g_{k_m} - |T_{k_m} f_{m-1}| - |T_{k_m}(f_0 - f_m)| \\ &\geqslant \omega(\varphi_{k_m})(C_2 - 1/m) - \omega(\varphi_{k_m})/m - \|T_{k_m}\|_{X^*} \|f_0 - f_m\|_X \\ &\geqslant \omega(\varphi_{k_m})(C_2 - 2/m) - \|T_{k_m}\|_{X^*}\, 2\omega(\varphi_{k_{m+1}}) \\ &\geqslant \omega(\varphi_{k_m})(C_2 - 4/m) \end{aligned}$$

so that all the assertions are established. ∎

Let us deduce one of the typical negative results of [7] as an application to Theorem 3. To this end, consider the Fejér means

$$(\sigma_{m,n}f)(x, y) := \sum_{\mu=-m}^{m} \sum_{\nu=-n}^{n} \left(1-\frac{|\mu|}{m}\right)\left(1-\frac{|\nu|}{n}\right) f^{\wedge}(\mu, \nu)\, e^{i(\mu x+\nu y)}$$

of the double Fourier series of $f \in C_{2\pi,2\pi}$ (cf. (11)) in connection with the (partial) conjugate function

$$f^{\sim}(x, y) := f^{\sim(1,0)}(x, y) := \frac{1}{2\pi}\int_0^{\pi} [f(x-u, y) - f(x+u, y)] \cot(u/2)\, du.$$

Let $C^{\sim}_{2\pi,2\pi} := \{f \in C_{2\pi,2\pi} : f^{\sim} \in C_{2\pi,2\pi}\}$. For $0 < \alpha, \beta < 1$ it was shown in [7] that

$$(34) \qquad \omega_1(f; s, 0)_\infty = O(s^\alpha), \qquad \omega_1(f; 0, t)_\infty = O(t^\beta)$$

imply

$$\|\sigma_{m,n} f^{\sim} - f^{\sim}\|_C = O(m^{-\alpha} + n^{-\beta} \log n) \qquad (m, n \to \infty)$$

and that for any sequence $\{\lambda_m\}$, decreasing to zero, there exists a counterexample $f_{\alpha,\beta} \in C^{\sim}_{2\pi,2\pi}$ satisfying (34) but

$$\|\sigma_{m,n} f^{\sim}_{\alpha,\beta} - f^{\sim}_{\alpha,\beta}\|_C \neq O(\lambda_m) + o(n^{-\beta} \log n).$$

In fact, upon letting $m \to \infty$ the existence of a counterexample $f_{\alpha,\beta} \in C^{\sim}_{2\pi,2\pi}$ was established satisfying (34) but

$$\|\sigma_{\infty,n} f^{\sim}_{\alpha,\beta} - f^{\sim}_{\alpha,\beta}\|_C \neq o(n^{-\beta} \log n),$$

$$(\sigma_{\infty,n} f)(x, y) := \sum_{\nu=-n}^{n} \left(1-\frac{|\nu|}{n}\right) f^{\wedge}(x, \nu)\, e^{i\nu y},$$

$$f^{\wedge}(x, \nu) := \frac{1}{2\pi}\int_0^{2\pi} f(x, v)\, e^{-i\nu v}\, dv.$$

It is the latter result which now may also be obtained as an immediate application to Theorem 3.

COROLLARY 4. *Let ω, ψ be strictly increasing moduli of continuity satisfying* (3), (4), *and suppose that* $\omega(1/k)\,|\log \psi^{-1}(\omega(1/k))| = o(1)$. *Then there exists a counterexample $f_{\omega,\psi} \in C^{\sim}_{2\pi,2\pi}$ such that*

$$\omega_1(f_{\omega,\psi}; t, 0)_\infty = O(\psi(t)), \qquad \omega_1(f_{\omega,\psi}; 0, s)_\infty = O(\omega(s)),$$

$$\|\sigma_{\infty,n} f^{\sim}_{\omega,\psi} - f^{\sim}_{\omega,\psi}\|_C \neq o\big(\omega(n^{-1})\,|\log \psi^{-1}(\omega(1/n))|\big).$$

Proof. Consider the Banach space $X = C^{\sim}_{2\pi,2\pi}$ with norm $\|f\|_{C^{\sim}} := \|f\|_C + \|f^{\sim}\|_C$ and the test elements (with $\alpha_k := \psi^{-1}(\omega(1/k))$ and $p_k \in N$ such that $p_k \leqslant 1/\alpha_k < p_k + 1$)

$$g_k(x, y) = \sum_{\mu=1}^{p_k} \frac{\sin \mu x}{\mu} e^{iky} := h_{1,k}(x)\, h_{2,k}(y).$$

Clearly, $g_k \in \Pi_{p_k,k}$ and

$$\tilde{g_k}(x, y) = \tilde{h_{1,k}}(x)\, h_{2,k}(y) = \sum_{\mu=1}^{p_k} \frac{\cos \mu x}{\mu} e^{iky},$$

$$\|g_k\|_C \leqslant 2\sqrt{\pi}, \qquad \|\tilde{g_k}\|_C \leqslant C\,|\log \alpha_k|$$

so that (19) follows for $N_k = C\,|\log \alpha_k|$. Obviously, the functionals U_s, $V_t \in (\tilde{C}_{2\pi,2\pi})^*$ given by

$$U_s f = \omega_1(f;\, 0, s)_\infty, \qquad V_t f = \omega_1(f;\, t, 0)_\infty$$

satisfy (22), (23) with $\sigma(t) = \tau(t) = t$, $\varphi_k = 1/k$ since, e.g.,

$$V_t g_k = \|h_{2,k}\|_C\, \omega_1(h_{1,k};\, t) \leqslant \min\{2\,\|h_{1,k}\|_C,\, t\,\|h'_{1,k}\|_C\}.$$

Moreover, (21) holds true since for the linear functionals

$$T_k f = (f^{\sim} - \sigma_{\infty,k} f^{\sim})(0, 0)/|\log \alpha_k|$$

one has the estimate

$$T_k g_k = \tilde{h_{1,k}}(0)\,(h_{2,k} - \sigma_k h_{2,k})(0)/|\log \alpha_k| = \frac{1}{|\log \alpha_k|} \sum_{\mu=1}^{p_k} \frac{1}{\mu} \geqslant C_2 > 0.$$

Therefore Theorem 3 may be applied which completes the proof. ∎

Let us finally mention that also the negative results given in [7] for the limiting cases $\alpha = 1$, $\beta = 1$ (thus (4) is violated) may be deduced from a general theorem upon employing the present analysis together with the one elaborated in [12], [13] (see also [6]).

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