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ON STABILITY AND GROWTH OF SOLUTIONS FOR CLASSES OF INITIAL AND BOUNDARY VALUE PROBLEMS

L. E. PAYNE

Center for Applied Mathematics, Cornell University,
Ithaca, N.Y. U.S.A.

In the first part of this paper we discuss a number of different methods for stabilizing nonstandard and inherently unstable Cauchy problems for linear second order operator equations. These methods have been used extensively in the literature of the past two decades. A survey of the literature prior to 1975 is given in [26]. In Section 4 we deal with questions of global nonexistence but here much of the work has been done since 1975. Many of these papers are referenced in the bibliography of Knops [13]. Section 5 deals with continuous dependence results for nonlinear operator equations. The final section discusses some results of St. Venant type for classes of second order nonlinear problems in the plane.

The first five sections deal with nonstandard problems for which in general there is no existence theory. We do not deal with the existence question in this paper (except for the global nonexistence results of Section 4) but presuppose the existence of the solution or solutions in question. For the linear problems, results announced in [13] indicate the possibility of establishing existence of solution to at least some classes of nonstandard problems for a dense set of data. Since the data are usually determined by measurement and therefore subject to some error we cannot anyway expect to know the data precisely. For the nonlinear problems of Section 4 for which we establish global nonexistence we really prove that if the solution exists for a sufficiently long time then it must blow up. Whether or not the solution blows up or breaks down in some other way prior to the predicted blow up time is in most cases an open question. Ball [2] and Calderer [6] have established existence up to blow up time in some special classes of problems, but a general theory is yet to be established.

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In deriving continuous dependence results for the nonstandard problems we discuss, it is necessary to restrict solutions to lie in some constraint set. For instance, we may require that the L_2 integral of the solution over the region of definition be uniformly bounded by some constant m over the time interval $[0, T]$. This restriction has the effect of making all such problems nonlinear. In practice one would have to be able to determine the constant m explicitly. Frequently this can be done by observation, particularly since the constant need not be a sharp bound.

Uniqueness of solution will follow directly from our continuous dependence results. As we will see the concavity argument which imply global nonexistence will simultaneously imply uniqueness of solution over the interval of existence.

We do not attempt to give a complete bibliography of numerous papers on the topics discussed in this paper. Investigations of St. Venant type or, more generally, studies of the rate of growth or decay of solutions, have been going on for well over a century. The work on ill posed problems began much later, but both areas of investigation are very active at the present time so that an up-to-date bibliography would be out of the question.

1. Cauchy problems for nonstandard operator equations

In this section we recall several methods that have been proposed in the literature for stabilizing classes of nonstandard linear or quasilinear operator equations. For further information the reader is referred to [26] where these methods are briefly described and original papers referenced. By way of illustration we shall treat reasonably simple examples, primarily those that might arise in elasticity theory. In many cases considerably more general problems have been treated in the literature.

Let H be a real Hilbert space with inner product $(\, , \,)$ and norm $\| \; \|$. Let $D \subset H$ be a dense linear subset of H and let M and N be linear operators which map D into H. We shall be concerned with the following problem:

$$(1.1)\qquad \begin{gathered} M\frac{d^2u}{dt^2} + Nu = 0 \qquad t \in (0, T), \\ u(0) = u_0, \qquad \frac{du}{dt}(0) = v_0. \end{gathered}$$

For simplicity we shall assume throughout that M, N and H are independent of t. The particular case in which M is the identity operator and N is the negative Laplace operator (assumed to act on functions in the appropriate space) would correspond to the Cauchy problem for the ordinary wave equation.

We shall assume further that M and N satisfy

(i) M is symmetric and positive definite,

(ii) N is symmetric,

and that

(iii) $u \in C'([0, T], H)$.

We could formulate our problem in weak setting, but it would unnecessarily complicate the illustrations and the same final result would be obtained in any case. Also the extension to complex Hilbert space can easily be made and results for more general time dependent operators as well as for first order operator equations have actually been dealt with in the literature.

The standard problems for this operator equation (1.1) are those for which N is a positive operator, an assumption usually made in applications to linear elasticity. We shall, however, be dealing with problems for which N is either indefinite or negative. Since we shall be considering this specific problem later on we write down the displacement problem for linear anisotropic elasticity, i.e., solution vector with components u_i satisfying, for $i = 1, 2, 3$

$$\varrho(x)\frac{\partial^2 u_i}{\partial t^2} - \sum_{j,k,l=1}^{3} \frac{\partial}{\partial x_j}\left[c_{ijkl}(x)\frac{\partial u_k}{\partial x_l}\right] = 0 \quad \text{in} \quad \Omega \times (0, T);$$

$$u_i(x, t) = 0 \quad \text{on} \quad \partial\Omega \times [0, T]; \tag{1.2}$$

$$u_i(x, 0) = f_i(x), \qquad \frac{\partial u_i}{\partial t}(x, 0) = g_i(x), \qquad x \in \Omega.$$

Here Ω is a bounded region in R^3 with Lipschitz boundary $\partial\Omega$.

In what follows when we refer to specific examples we shall suppress the summation sign and understand that summation is to be carried out over repeated indices. We shall also use a comma to denote differentiation.

In practice it is usually assumed that

$$c_{ijkl} = c_{klij} \qquad \text{(symmetry)} \tag{1.3}$$

and that

$$c_{ijkl}(x)\psi_{ij}\psi_{kl} \geqslant c_0\psi_{ij}\psi_{ij}, \qquad c_0 > 0 \tag{1.4}$$

for arbitrary second order tensors ψ_{ij}. For the time being we shall retain (1.3) but discard (1.4). This corresponds to the operator N in (1.1) being symmetric, but not necessarily positive; thus conservation of energy (or non-increase in energy in the weak setting) does not imply stability. It is well known, in fact, that if N is negative definite one cannot expect stability in the Hadamard sense.

We now illustrate a few methods that have been introduced to retrieve a form of continous dependence on the Cauchy data. Since the equations are linear it will suffice to investigate the question of stabilizing the solution of (1.1) where u_0 and v_0 are regarded as perturbations of the zero data.

I. The method of logarithmic convexity. This method is based on the following property of smooth functions $F(t)$ of a single variable. Suppose $F(t)$ is a positive function of t for $t \geqslant 0$ and $\log F$ is a convex function of t, then $\log F(t)$ will lie below the chord which joins any two points on the curve and will lie above the tangent line to the curve at any point. Using the standard properties of logarithms and evaluating for specific values of t we may write these two results in inequality form as

$$F(t) \leqslant [F(t_1)]^{(t_2-t)/(t_2-t_1)}[F(t_2)]^{(t-t_1)/(t_2-t_1)}, \qquad t_1 \leqslant t \leqslant t_2, \tag{1.5}$$

and

$$F(t) \geqslant F(0)\exp\{tF'(0)/F(0)\}, \qquad t > 0. \tag{1.6}$$

Our aim then is to try to choose for our function $F(t)$ some appropriate norm of the solution u of (1.1).

Before proceeding we remark that conservation of energy may be expressed as

$$E(t) = K(t) + V(t) \equiv E(0), \tag{1.7}$$

where

$$K(t) = \tfrac{1}{2}(u', Mu'), \qquad V(t) = \tfrac{1}{2}(u, Nu), \tag{1.8}$$

and we have used the prime to denote the time derivative. In the elasticity context K is the kinectic energy and V is the potential (or strain) energy.

To illustrate the method let us choose for some positive constant β

$$F(t) = (u, Mu) + \beta Q, \tag{1.9}$$

where

$$Q = \max(0, E(0)). \tag{1.10}$$

Then

$$F'(t) = 2(u, Mu') \tag{1.11}$$

and

$$F''(t) = 2(u', Mu') - 2(u, Nu). \tag{1.12}$$

Using conservation of energy we obtain

$$F''(t) = 4(u', Mu') - 4E(0). \tag{1.13}$$

Clearly then by Schwarz's inequality and some simple manipulations it follows that

$$(\log F)'' = \frac{FF'' - (F')^2}{F^2} \geqslant -\frac{4}{\beta}. \tag{1.14}$$

If $E(0) < 0$ we could in fact replace the right hand side of (1.14) by 0 and apply (1.5) and (1.6) directly. More generally, however, we may rewrite (1.14) as

$$\{\log[Fe^{2t^2/\beta}]\}'' \geq 0, \tag{1.15}$$

and apply (1.5) and (1.6) with $F(t)$ replaced throughout by $F(t)e^{2t^2/\beta}$. This gives in particular

$$F(t) \leq [F(0)]^{1-t/T}[F(T)]^{t/T}e^{2T(T-t)/\beta}, \tag{1.16}$$

and

$$F(t) \geq F(0)\exp\left[-\left(\frac{2t^2}{\beta} - \frac{F'(0)}{F(0)}t\right)\right]. \tag{1.17}$$

The inequality (1.16) is not a stability inequality since even though $F(0)$ involves only data terms there is no a priori guarantee that if $F(0)$ is small then $F(t)$ must remain small in the measure (u, Mu) for $0 < t < T$. It will be a stability inequality however if we suitably restrict our admissable class of solutions.

We say that function $\phi(t)$ is an element of the set $\mathcal{M}_1$ if $\phi(t) \in H$ and

$$(\phi(T), M\phi(T)) \leq m_1^2 \tag{1.18}$$

for some prescribed constant m_1. Clearly this implies that solutions of (1.1) which belong to $\mathcal{M}_1$ depend continuously on the data in F measure.

By choosing a different form for F one can actually obtain somewhat sharper estimates than those given by (1.16), (1.17) (see e.g. [26]). However, if $E(0) < 0$ we may take $\beta = 0$ in (1.9) and obtain (1.14) with the right hand side equal to zero. Then (1.16) and (1.17) together yield

$$F(0)\exp\left\{\frac{F'(0)}{F(0)}t\right\} \leq F(t) \leq F(0)\left[\frac{F(T)}{F(0)}\right]^{t/T}. \tag{1.16'}$$

Thus (1.16') shows that any solution whose initial data satisfy $E(0) < 0$, $F'(0) > 0$ and which exists for all time must grow at least exponentially (in F measure) as $t \to \infty$. One the other hand the right hand side states that if $E(0) < 0$ and $F(T)$ has at most polynomial growth as $T \to \infty$ then provided the solutions exists for all time, $F(t) \leq F(0)$ for all t.

II. The Lagrange identity method. The basis for this method is the Lagrange identity involving two admissible functions u and v, i.e.,

$$\int_0^t [(u, Mv_{\eta\eta}+Nv)-(v, Mu_{\eta\eta}+Nu)]\,d\eta = [(u, Mv_\eta)-(v, Mu_\eta)]|_0^t. \tag{1.19}$$

We now choose u to be a solution of (1.1) and set

$$v(x, \eta) = u(x, 2t-\eta). \tag{1.20}$$

Thus v satisfies the same equation as u and the left hand side of (1.19) vanishes. This leads to

$$(1.21) \qquad 2(u(t),\ Mu'(t)) = (u(0),\ Mu'(2t)) + (u(2t),\ Mu'(0)).$$

Integrating (1.21), making a change of variable, and using the boundary conditions on u we conclude that

$$(1.22) \quad (u(t),\ Mu(t)) = \tfrac{1}{2}(u_0,\ Mu(2t)) + \tfrac{1}{2}(u_0,\ Mu_0) + \tfrac{1}{2}\int_0^{2t}(v_0,\ Mu(\eta))\,d\eta.$$

Let us now define a set $\mathcal{M}_2$ by the constraint

$$(1.23) \qquad \max_{0 \leqslant \eta \leqslant T} (u,\ Mu) \leqslant m_2^2.$$

Then if $u \in \mathcal{M}_2$ it follows from Schwarz's inequality that

$$(1.24) \quad (u(t),\ Mu(t)) - \tfrac{1}{2}(u_0,\ Mu_0) \leqslant \tfrac{1}{2}\{(u_0,\ Mu_0) + 4t^2(v_0,\ Mv_0)\}^{1/2} m_2$$

for $2t < T$. This clearly implies a stability inequality on the interval $(0,\ T/2)$. One could of course apply the same arguments but now on the interval $(T/2,\ T)$ instead of the interval $(0,\ T)$ and in this way extend the stability inequality to the interval $(0,\ 3T/4)$, etc.

Note that this method did not make use of the energy identity and thus is somewhat more easily adapted to weak solutions. However, it was necessary that the equation be autonomous and linear.

Another interesting identity comes from a slightly different Lagrange identity, i.e., for the some u and v

$$(1.25) \quad 0 = \int_0^t [(u_\eta,\ Mv_{\eta\eta} + Nv) + (v_\eta,\ Mu_{\eta\eta} + Nu)]\,d\eta = [(u_\eta,\ Mv_\eta) + (u,\ Nv)]\Big|_0^t.$$

If we now define

$$(1.26) \quad K(t_1,\ t_2) = \tfrac{1}{2}(u'(t_1),\ Mu'(t_2)); \qquad V(t_1,\ t_2) = \tfrac{1}{2}(u(t_1),\ Nu(t_2))$$

then (1.25) leads to

$$(1.27) \qquad K(t,\ t) - V(t,\ t) = K(0,\ 2t) - V(0,\ 2t).$$

The above identity is quite useful in equipartition of energy arguments [21], and together with the energy identity it leads directly to Hölder type stability estimates (in the appropriate space) for the kinetic energy and/or the absolute value of the potential energy.

The Lagrange identity method is generally attributed to Brun [5] who was the first to use it in the context of elastodynamics.

III. The quasireversibility method. This method was proposed by Lattes and Lions [20] and has been extensively studied by Miller [23]. Let us suppose that not only M and N but also N^2 map D into H. We further suppose w to be the solution of the following problem:

$$(1.28)\qquad \begin{gathered} Mw_{tt}+Nw+\varepsilon N^2 w = 0, \qquad t\in(0, T], \\ w(0) = u_0, \qquad w'(0) = v_0, \end{gathered}$$

where ε is some small positive (but fixed) constant. Since N^2 is a positive operator this problem will in many cases be well posed. The quasireversibility method is not a precisely defined method; its aim is to use the solution w of the well posed problem to construct an approximation to the original problem.

As a specific example suppose we consider the problem

$$(1.29)\qquad \begin{gathered} \frac{\partial^2 u}{\partial t^2}+\Delta u = 0 \quad \text{in} \quad \Omega\times(0, T]; \\ u = 0 \quad \text{on} \quad \partial\Omega\times[0, T]; \\ u(x, 0) = f(x), \qquad \frac{\partial u}{\partial t}(x, 0) = g(x). \end{gathered}$$

The equation for w is

$$(1.30)\qquad \begin{gathered} \frac{\partial^2 w}{\partial t^2}+\Delta w+\varepsilon\Delta^2 w = 0 \quad \text{in} \quad \Omega\times(0, T]; \\ w = 0, \qquad \Delta w = 0 \quad \text{on} \quad \partial\Omega\times[0, T]; \\ w(x, 0) = f(x), \qquad \frac{\partial w}{\partial t}(x, 0) = g(x). \end{gathered}$$

Suppose we can solve for $w(x, T)$. We then define ϕ to be the solution of the well posed problem

$$(1.31)\qquad \begin{gathered} \frac{\partial^2 \phi}{\partial t^2}+\Delta\phi = 0 \quad \text{in} \quad \Omega\times(0, T]; \\ \phi = 0 \quad \text{on} \quad \partial\Omega\times(0, T]; \\ \phi(x, T) = w(x, T); \qquad \phi(x, 0) = f(x). \end{gathered}$$

To check the accuracy of our problem we would form $\int_\Omega \left[\frac{\partial\phi}{\partial t}(x, 0)-g(x)\right]^2 dx$ which should be small if ε is small.

For this particular example everything can be carried out using formal power series, but recent work of Ames [1] provides a direct comparison between the solution of (1.28) (which need not be well posed) and that of

(1.1). Again for functions lying in the appropriate constraint classes she is able to obtain an inequality of the form

$$((u-w),\ M(u-w)) \leqslant K\varepsilon^{(1-t/T)} m_2^{2t/T}, \qquad K = \text{constant}. \tag{1.32}$$

If we constrain u and (if necessary) w to lie in the appropriate constraint sets, instead of the usual type of stability inequality we obtain an inequality of the following type

$$\begin{aligned}(u,\ Mu)^{1/2} \leqslant{} & [K\varepsilon^{(1-t/T)} m_2^{2t/T}]^{1/2} + \\ & + \{K_1(\varepsilon)[(v_0,\ Mv_0) + \|u_0\|^2 + (u_0,\ Mu_0) + \|Nu_0\|^2]\}^{1/2}.\end{aligned} \tag{1.33}$$

For fixed $\varepsilon > 0$ the second term will be small if the initial data is sufficiently small. On the other hand $K_1(\varepsilon)$ will in general tend to infinity as $\varepsilon \to 0$. One can say little more without the explicit representation for M and N.

The quasireversibility method presupposes the ability to find a well posed problem w for which one can extract the necessary explicit information. In fact there will usually be a variety of comparison problems that could be used and one would like to find the "best" one. Even so there is still the difficulty of deciding what explicit value of ε to use. If ε is chosen too large the first term on the right of (1.33) may be intolerably large, while if ε is chosen too small the second term may become too large. For these various reasons the quasireversibility method seems to have practical disadvantages when compared to the two previously discussed methods.

IV. Method of weighted energy. To illustrate this method we shall apply it to a problem which cannot be handled directly by either of the first three methods. For simplicity we shall take M to be the indentity operator, but now we assume N to be time independent and antisymmetric, i.e,

$$(Nu,\ v) = -(u,\ Nv). \tag{1.34}$$

In the context of the equations of anisotropic elasticity this means that

$$c_{ijkl}(x) = -c_{klij}(x). \tag{1.35}$$

The results of this section are due to Murray [24].

We introduce a new function $\psi(x, t)$ defined by

$$u(x, t) = e^{\lambda t}\psi(x, t), \qquad \lambda = \text{constant}, \tag{1.36}$$

and note that ψ satisfies the equation

$$\frac{d^2\psi}{dt^2} + 2\lambda\frac{d\psi}{dt} + \lambda^2\psi + N\psi = 0. \tag{1.37}$$

For the moment the constant λ is left unspecified, but an appropriate choice will be made later on. It follows from the identity

$$\int_0^t \left(\frac{d\psi}{d\eta}+\frac{1}{2\lambda}N\psi\right)\left(\frac{d^2\psi}{d\eta^2}+2\lambda\frac{d\psi}{d\eta}+\lambda^2\psi+N\psi\right)d\eta = 0, \tag{1.38}$$

carrying out an integration by parts and dropping positive terms, that

$$\frac{1}{2}[(\psi'(t), \psi'(t))+\lambda^2(\psi(t), \psi(t))] \leqslant -\frac{1}{2\lambda}(N\psi(t), \psi'(t))+ \tag{1.39}$$
$$+\frac{1}{2\lambda}(N\psi(0), \psi'(0))+\tfrac{1}{2}[(\psi'(0), \psi'(0))+\lambda^2(\psi(0), \psi(0))].$$

Reinserting $u(x, t)$, simplifying and dropping the first term on the left we obtain

$$\|u\|^2 \leqslant -\frac{1}{\lambda^3}(Nu, u')+e^{2\lambda t}\left\{\frac{1}{\lambda^2}\|v_0-\lambda u_0\|^2+\|u_0\|^2+\frac{1}{\lambda^3}(Nu_0, v_0)\right\}. \tag{1.40}$$

We now define a new constraint class $\mathcal{M}_3$ of functions $\phi(x, t)$ satisfying

$$\sup_{0\leqslant t\leqslant T} [\|N\phi\|^2+\|\phi'\|^2] \leqslant m_3^2.$$

Clearly then for solutions of (1.1) (with N antisymmetric) which belong to $\mathcal{M}_3$ we obtain (assuming $\lambda > 1$ in the bracketed term of (1.40)) the inequality

$$\|u\|^2 \leqslant \frac{m_3^2}{2\lambda^3}+e^{2\lambda t}Q, \tag{1.41}$$

where Q is a data term.

We now choose

$$\lambda = \frac{1}{2T}\log[m_3^2/Q] \tag{1.42}$$

in which case

$$\|u\|^2 \leqslant \frac{4T^3 m_3^2}{[\log m_3^2/Q]^3}+m_3^{2t/T}Q^{1-t/T}. \tag{1.43}$$

Inequality (1.43) implies a weak logarithmic continuous dependence on the data for solutions in $\mathcal{M}_3$.

For special classes of problems other method may be employed (see e.g. [26]), but of the four methods introduced, the first has been the most widely used. It is applicable to first and second order operator equations, applied to nonautonomous and nonlinear systems, and has been used to study evolutionary problems backward in time. The Lagrange identity method is pretty well restricted to linear autonomous systems and symmetric operators, but when it is applicable it is usually quite direct and sometimes requires less regularity of the solution and data. We noted that with this method no energy identity had to be employed. It was also observed in [26] that it is possible sometimes to get away from the Hilbert space restrictions.

The weighted energy method appears to be better suited to higher order operator equations and to problems in which some of the operators are not symmetric. John Bell [4] has used this method to study continuous data dependence for classes of noncharacteristic Cauchy problems for parabolic equations and for a nonlinear version of the vibrating beam equation with Cauchy data given on a space like curve. The method has been used to study the asymptotic behavior of solutions to various well-posed problems and has also been used in the proof of nonexistence of global solutions to certain nonlinear problems. It should be observed, however, that in general it leads to a very weak form of continuous dependence.

The method of quasireversibility appears to have few if any theoretical advantages over the other methods; however, in specific problems it may have some practical advantages. We have not up to now discussed the question of actually finding an approximate solution of our problem. We shall make a few remarks in this direction in the next section.

2. Remarks on continuous dependence and other types of data and error bounds

In the context of linear elastodynamics, Knops and Payne [16] derived inequalities which yield continuous dependence on the elastic coefficients, the Dirichlet data, the value of the operator and on the geometry of the spacetime region for solutions lying in the appropriate constraint classes. They used the logarithmic convexity method, but in some cases the same or analogous results could be obtained more easily via the Lagrange identity method. We illustrate with a simple example and show how the method leads to a constructive procedure for obtaining and approximate solution.

Suppose now that instead of (1.1) the following problem is given

$$\begin{gathered} M\frac{d^2u}{dt^2}+Nu=\mathscr{F}(t), \quad 0<t<T, \\ u(0)=u_0, \qquad u'(0)=v_0, \end{gathered} \tag{2.1}$$

and we wish to compare with a solution u_1 of the problem

$$\begin{gathered} M\frac{d^2u_1}{dt^2}+Nu_1=\mathscr{F}_1(t), \qquad 0<t<T, \\ u_1(0)=u_0, \qquad u'(0)=\tilde{v}_0. \end{gathered} \tag{2.2}$$

We could easily treat the more general case in which $u(0) \neq u_1(0)$, but for simplicity of presentation we treat the indicated special case. Set

$$w = u - u_1,$$

and consider the problem

$$M\frac{d^2 w}{dt^2} + Nw = (\mathcal{F} - \mathcal{F}_1), \qquad 0 < t < T,$$

(2.3)

$$w(0) = 0, \qquad w'(0) = v_0 - \tilde{v}_0.$$

Using the Lagrange identity method we obtain in a straightforward way

$$(2.4) \quad 2(w'(t), M(t)) = ((v_0 - \tilde{v}_0), Mw(2t)) + \int_0^t [(w, (\mathcal{F}^* - \mathcal{F}_1^*)) - (w^*, (F - F_1))] \, d\eta,$$

where the argument of the unstarred expressions in the integrand on the right is η and that of the starred expressions is $2t - \eta$. An integration and use of standard inequalities leads to the result

$$(2.5) \quad (w(t), Mw(t)) \leqslant \tfrac{1}{2}\Big[\|M(v_0 - \tilde{v}_0)\|^2 \int_0^{2t} \|w\|^2 \, d\eta\Big]^{1/2} + \\ + \Big[\int_0^{2t} (t - \tfrac{1}{2}\eta)^2 \|\mathcal{F} - \mathcal{F}_1\|^2 \, d\eta \int_0^{2t} \|w\|^2 \, d\eta\Big]^{1/2}.$$

We now define a set $\mathcal{M}_4$ as follows: a function ϕ is said to belong to $\mathcal{M}_4$ if

$$(2.6) \quad \int_0^T \|\phi\|^2 \, d\eta \leqslant m_4^2.$$

Thus if u_1 and u_2 both belong to $\mathcal{M}_4$ we conclude that

$$(2.7) \quad (w, Mw) \leqslant m_4 \Big\{\tfrac{1}{2}\|M(v_0 - \tilde{v}_0)\| + 2\int_0^{2t} (t - \tfrac{1}{2}\eta)^2 \|\mathcal{F} - \mathcal{F}_1\|^2 \, d\eta\Big\}^{1/2}.$$

Clearly this is a stability inequality on the interval $[0, T/2]$ for solutions in $\mathcal{M}_4$.

We indicate now how the use these results to obtain error bounds in the approximation of the solution u of (2.1) by a function $\phi \in C'([0, T), H)$ which also satisfies $\phi(0) = u_0$. If $u \in \mathcal{M}_4$ then clearly

$$(2.8) \quad \int_0^{2t} \|u - \phi\|^2 \, d\eta \leqslant 2\Big[m_4^2 + \int_0^{2t} \|\phi\|^2 \, d\eta\Big].$$

Thus from (2.5) we obtain

$$(2.9)\quad (u-\phi,\ M(u-\phi)) \leqslant \left[m_4^2+\int_0^{2t}\|\phi\|^2\,d\eta\right]^{1/2}\left[\|M(v_0-\phi'(0))\|^2+ +4\int_0^{2t}\left(t-\frac{\eta}{2}\right)^2\left\|\mathcal{F}-M\frac{d^2\phi}{dt^2}-N\phi\right\|^2 d\eta\right]^{1/2}.$$

The right hand side of (2.9) is the product of two terms the second involving only the approximation of data terms. Thus one might think of using a Rayleigh–Ritz procedure to make the data term small. As we select more terms in the Rayleigh–Ritz procedure the second factor will decrease, but the first factor will almost certainly increase. Thus in any Rayleigh–Ritz procedure in which we add a given set of functions one at a time in a definite order the right hand side is likely to be minimized after a finite number of terms and the addition of more terms will only make the error worse. If we choose a different set of functions the same phenomenon will occur but possibly the optimal number of terms will be different and the magnitude of the error terms different. Also if we change the order in which we adjoin terms the optimal number may again be different, but the qualitative behavior will be the same.

3. Linear elasticity in exterior regions

Although we could consider nonlinear hyperelastic problems we shall restrict our attention in this section to the problem defined by (1.2) where, however, the region of definition is $\Omega^*\times(0,\ T]$. We have used the notation

$$(3.1)\qquad \Omega^* = R^3/\Omega.$$

Normally we would expect to have to impose some hypothesis on the behavior of $u_i(x,\ t)$ as $|x|\to\infty$, but we shall merely suppose that the solution does not grow too fast as $|x|\to\infty$. If the c_{ijkl} satisfy conditions (1.3) and (1.4) the problem has been well studied in the literature (see e.g. [13]) but we wish to relax condition (1.4). The full details of the derivation of the results described below will be given in a forthcoming paper by Knops and Payne.

We assume the boundary and initial conditions to be those given by (1.2) with Ω replaced by Ω^*. We assume that c_{ijkl} satisfies the symmetry hypothesis (1.3) and we further impose the following conditions:

(3.2) (a) u_i, $u_{i,j}$, u_i' are $O(e^{k|x|})$ as $|x|\to\infty$ for some positive k;

(b) There exist positive constants R and c_0 such that

$c_{ijkl}\psi_{ij}\psi_{kl}\geqslant c_0\psi_{ij}\psi_{ij}$, $\forall\psi_{ij}$ and $\forall x$ such that $|x|\geqslant R$;

(c) u_i is piecewise smooth in $\Omega^*\times[0,\ T)$.

In order to kill off the possible growth at infinity we introduce a weight function

$$\omega(|x|) = \begin{cases} 1, & |x| \leqslant R_1, \\ e^{-k(|x|-R_1)}, & |x| > R_1 > R, \end{cases} \tag{3.3}$$

and define $F(t)$ as follows:

$$F(t) = \int_0^t \int_{\Omega^c} e^{-\lambda\eta} \omega(|x|) u_i u_i \, dx \, d\eta + k_1 Q, \tag{3.4}$$

where λ is a constant to be determined, k_1 is a positive constant and Q a positive definite data term which vanishes when $f_i = g_i = 0$. The logarithmic convexity method yields (after a careful manipulation of inequalities)

$$F(t) \leqslant \sigma^{\gamma_1} \left[\frac{k_1 Q}{\sigma_0^{\gamma_1}} \right]^{(\sigma-\sigma_1)/(\sigma_0-\sigma_1)} \left[\frac{F(t_1)}{\sigma_1^{\gamma_1}} \right]^{(\sigma_0-\sigma)/(\sigma_0-\sigma_1)}, \tag{3.5}$$

where $\sigma = \gamma_2^{-1} e^{-\gamma_2 t}$, γ_1 and γ_2 are computable constant and t_1 satisfies $0 \leqslant t \leqslant t_1 < T$. This is clearly a stability inequality in the appropriately defined constraint class.

It is still an open question whether some condition of type (3.2b) is necessary to insure uniqueness and continuous dependence.

If the initial data are suitably restricted (and here the role of λ becomes important) then it is possible to obtain sharper upper bounds and meaningful lower bounds for the quantity

$$F_1(t) = \int_{\Omega^c} \omega(|x|) u_i u_i \, dx. \tag{3.6}$$

4. Global nonexistence of solutions

Considerable attention has been given in the recent literature to the question of global nonexistence of solutions of classes of nonlinear problems. One is interested in wether or not a solution *can* exist for all time, whether it *does* exist for all time, and in case it can be shown that the solution cannot exist beyond a time t_1, whether the solution fails to exist by "blowing up" or whether existence fails in some other way. An account of much of the work in this area prior to 1975 is given in [26] and the references cited therein. More recent work is referenced in [13]. We refer also to the very recent results of Glassey [8], John [11], Kato [12] and Sideris [30].

Most of the methods used in the literature to establish global nonexistence merely tell us that if a solution exists for a sufficiently long time then it must in fact blow up. The principle method employed in this study has been the concavity method or some variation of it. It is a method of second order differential inequalities similar to the logarithmic convexity method introduced in the previous section. Since this method is by now reasonably well known we merely sketch it here.

The idea of the concavity method is simple. We merely note that any sufficiently smooth concave function of t which is initially positive and whose initial slope is negative must reach the value zero in a finite time. This means that if we can construct from solutions of nonlinear problems a positive definite function, $F(t)$ satisfying $F'(0) > 0$ and $(F^{-\alpha})'' \leqslant 0$, then the solution cannot exist for all time. (Here α is an arbitrary positive number.)

In [14] (see also [13]) the concavity method was used to investigate the question of global nonexistence of solution of the following problem in nonlinear hyperelasticity

$$\begin{aligned} &\varrho(x)\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial}{\partial x_j}\left(\frac{\partial W}{\partial u_{i,j}}\right) \quad \text{in} \quad \Omega\times(0,\,T); \\ (4.1)\qquad &u_i = 0 \quad \text{on} \quad \partial\Omega\times[0,\,T); \\ &u_i(x,\,0) = f_i, \qquad \frac{\partial u_i}{\partial t}(x,\,0) = g_i, \end{aligned}$$

where the strain energy function W is taken to be a function of the displacement gradients.

For classical solutions energy is conserved, i.e.,

$$E(t) = \tfrac{1}{2}\int_\Omega \varrho u_i' u_i'\,dx + \int_\Omega W\,dx = E(0). \tag{4.2}$$

For weak solutions it is customary to postulate an energy inequality

$$E(t) \leqslant E(0). \tag{4.3}$$

It was shown in [14] that if W satisfies the hypothesis

$$\int_\Omega \left[u_{i,j}\frac{\partial W}{\partial u_{i,j}} - 2(1+2\alpha)\,W\right]dx \leqslant 0 \tag{4.4}$$

for some $\alpha > 0$, then the function $F(t)$ defined on weak solutions of (4.1) by

$$F(t) = \int_\Omega \varrho u_i u_i\,dx + \beta(t+t_0)^2 \tag{4.5}$$

for positive constants β and t_0, satisfies

$$FF'' - (1+\alpha)(F')^2 \geqslant -2(1+2\alpha)(2E(0)+\beta)F(t). \tag{4.6}$$

Thus if $E(0) \leqslant 0$ we may choose $\beta = -2E(0)$ and conclude

$$(F^{-\alpha})'' \leqslant 0 \tag{4.7}$$

for t inside the existence interval. It is now possible to choose t_0 so large that $F'(0) > 0$ and obtain the following two inequalities:

$$F^\alpha(t) \leqslant F^\alpha(0)\left[1 + \frac{t}{T}\left(\frac{F^\alpha(0)}{F^\alpha(T)} - 1\right)\right]^{-1} \tag{4.8}$$

and

$$F^{\alpha}(t) \geq F^{\alpha}(0)\left[1-\frac{\alpha t F'(0)}{F(0)}\right]^{-1}. \tag{4.9}$$

Inequality (4.9) shows that the solution must fail to exist at some $t_1 \leq F(0)/\alpha F'(0)$. On the other hand (4.8) shows that if the solution exists up to time t_1 and blows up at $t = t_1$ then for $t < t_1$

$$F^{\alpha}(t) \leq F^{\alpha}(0)\left[1-\frac{t}{t_1}\right]^{-1}. \tag{4.10}$$

The question of whether or not a solution does exist up to blowup time and actually loses existence through blowup has received little attention in the literature although some special problems have been considered by Ball [2] and others (see references cited in [13]).

An inequality of type (4.7) can also be derived in cases when $E(0) > 0$ (see [14]). Other criteria for nonexistence have been given in the literature (see the references cited in [13] and [26]).

The above result indicates that we should not expect a Hölder type of stability result on arbitrarily large intervals for perturbations of the null solution in the Cauchy problem of hyperelasticity unless it is possible to suitably restrict the form of W. For instance it is known that if the potential energy satisfies a "potential well" property in the appropriate measure then stability is assured, but as illustrated by examples of Ball, Knops, and Marsden [3] and Knops and Payne [17] the existence of a potential well seems to be rather the exceptional situation. For additional results see Knops [13] and papers cited therein.

It should perhaps be pointed out that if at some time t^* it is possible to determined by measurement, observation, or otherwise that $F(t^*) \leq A$ then clearly t^* must be less than t_1 and (4.8) will imply stability of the null solution. In fact we would obtain from (4.8)

$$F^{\alpha}(t) \leq F^{\alpha}(0)\left[1+\frac{t}{t^*}\left(\frac{F^{\alpha}(0)}{M}-1\right)\right]^{-1}. \tag{4.11}$$

In the nex section we shall discuss criteria for establishing continuous dependence on the data for solutions of (4.1).

5. Continuous dependence on the data in nonlinear hyperelasticity

In this section we wish to discuss some of the meager results on continuous dependence on the Cauchy data for solutions of the nonlinear problem (4.1) if we do not impose the hypothesis $W \geq 0$. For the case in which

$$\frac{\partial^2 W}{\partial u_{i,j}\,\partial u_{k,l}}\psi_{ij}\psi_{kl} > c_0\psi_{ij}\psi_{ij}, \qquad \forall\psi_{ij}, \tag{5.1}$$

and W is a sufficiently smooth function of its arguments, it is well known that in the class of C^2 functions in $\Omega\times(0, T)$, the solution depends continuously on the Cauchy data. This follows directly from the energy identity. Let u_i and v_i be solutions of (4.1) corresponding to different initial data and set

$$\phi_i = u_i - v_i. \tag{5.2}$$

Now consider

$$\int_0^t \int_\Omega \frac{\partial \phi_i}{\partial \eta}\left[\varrho \frac{\partial^2 \phi_i}{\partial \eta^2} - \left(\frac{\partial W}{\partial u_{i,j}} - \frac{\partial \tilde{W}}{\partial v_{i,j}}\right)_{,j}\right] dx\, d\eta = 0, \tag{5.3}$$

where W is the strain energy function with argument $u_{i,j}$ and $\tilde{W}$ the strain energy function with argument $v_{i,j}$. Integrating the last term by parts and using the mean value theorem we arrive at the expression

$$\begin{aligned} J(x) &= \tfrac{1}{2} \int_\Omega [\varrho \phi_{i,\eta} \phi_{i,\eta} + B_{ijkl} \phi_{i,j} \phi_{k,l}]\, dx \\ &= J(0) + \tfrac{1}{2} \int_0^t \int_\Omega B'_{ijkl} \phi_{i,j} \phi_{k,l}\, dx\, d\eta, \end{aligned} \tag{5.4}$$

where B_{ijkl} is the value of $\dfrac{\partial^2 W}{\partial u_{i,j}\, \partial u_{k,l}}$ evaluated at some intermediate value between $u_{i,j}$ and $v_{i,j}$. Assuming $\dfrac{\partial}{\partial t} B_{ijkl}$ to be bounded then it follows that for some constant k, (5.4) leads to

$$J(t) \leqslant J(0) + k \int_0^t J(\eta)\, d\eta \tag{5.5}$$

which integrates to give

$$J(t) \leqslant J(0) e^{kt}, \tag{5.6}$$

a continuous dependence inequality.

It is also possible to derive a continuous dependence result for the case in which

$$\frac{\partial^2 W}{\partial u_{i,j}\, \partial u_{k,l}} \psi_{ij} \psi_{kl} < -c_1 \psi_{ij} \psi_{ij}, \qquad \forall \psi_{ij}, \tag{5.7}$$

if we again assume that u_i is a C^2 function in $\Omega\times(0, T)$.

In this case we define

$$F(t) = \int_\Omega \varrho \phi_i \phi_i\, dx + \beta J(0) \tag{5.8}$$

and form

$$F(t) = 2\int_\Omega \varrho\phi_i \frac{\partial\phi_i}{\partial t}\,dx, \tag{5.9}$$

$$\begin{aligned} F''(t) &= 2\int_\Omega \varrho \frac{\partial\phi_i}{\partial t}\frac{\partial\phi_i}{\partial t}\,dx - 2\int_\Omega \frac{\partial\phi_i}{\partial x_j}\left[\frac{\partial W}{\partial u_{i,j}} - \frac{\partial \tilde{W}}{\partial v_{i,j}}\right]dx \\ &= 2\int_\Omega \varrho \frac{\partial\phi_i}{\partial t}\frac{\partial\phi_i}{\partial t}\,dx - 2\int_\Omega B_{ijkl}\,\phi_{i,j}\,\phi_{k,l}\,dx. \end{aligned} \tag{5.10}$$

Using (5.4) we obtain

$$\begin{aligned} F''(t) &= 4\int_\Omega \varrho \frac{\partial\phi_i}{\partial t}\frac{\partial\phi_i}{\partial t}\,dx - 4J(0) - \frac{1}{2}\int_0^t\int_\Omega B'_{ijkl}\,\phi_{i,j}\,\phi_{k,l}\,dx \\ &\geqslant 4\int_\Omega \varrho \frac{\partial\phi_i}{\partial t}\frac{\partial\phi_i}{\partial t}\,dx - 4J(0) + 2k_1\int_0^t\int_\Omega B_{ijkl}\,\phi_{i,j}\,\phi_{k,l}\,dx. \end{aligned} \tag{5.11}$$

To bound the last term on the right we integrate (5.10) and obtain

$$\begin{aligned} F'(t) - F'(0) &= 2\int_0^t\int_\Omega \varrho \frac{\partial\phi_i}{\partial t}\frac{\partial\phi_i}{\partial t}\,dx - 2\int_0^t\int_\Omega B_{ijkl}\,\phi_{i,j}\,\phi_{k,l}\,dx \\ &\geqslant -2\int_0^t\int_\Omega B_{ijkl}\,\phi_{i,j}\,\phi_{k,l}\,dx. \end{aligned} \tag{5.12}$$

Thus

$$F'' \geqslant 4\int_\Omega \phi\frac{\partial\phi_i}{\partial t}\,dx - 4J(0) + 2k_1 F'(0) - 2k_1 F'(t), \tag{5.13}$$

$$FF'' - (F')^2 \geqslant -2k_1 FF' + [2k_1 F'(0) - 4J(0)]\,F(t). \tag{5.14}$$

If the coefficient of $F(t)$ is positive it may be dropped. If it is negative we write

$$FF'' - (F')^2 \geqslant -2k_1 FF' - \frac{[4J(0) - 2k_1 F'(0)]}{\beta J(0)} F^2(t), \tag{5.15}$$

or

$$(\ln F)'' \geqslant -2k_1(\ln F)' - \gamma. \tag{5.16}$$

We now introduce the new variable σ by

$$\sigma = e^{-2k_1 t} \tag{5.17}$$

and rewrite (5.16) as

$$\frac{d^2}{d\sigma^2}\{\log[F(t)\sigma^{-\gamma/4k_1^2}]\} \geqslant 0. \tag{5.18}$$

This implies that

$$F(t) \leqslant \sigma^{\gamma/(4k_1^2)}[F(0)]^{(\sigma-\sigma_1)/(1-\sigma_1)}[F(T)\sigma_1^{-\gamma/4k_1^2}]^{(1-\sigma)/(1-\sigma_1)}, \tag{5.19}$$

where $\sigma_1 = e^{-2k_1 T}$. Inequality (5.19) displays the continuous dependence on the data for solutions which lie in the set $\mathscr{M}_l$.

Of course it is much easier to derive inequalities which will imply stability of the null solution. For instance if we impose the following constitutive hypothesis on W

$$2W - u_{i,j}\frac{\partial W}{\partial u_{i,j}} \geqslant 0, \tag{5.20}$$

then the logarithmic convexity method yields stability of the null solution for solutions in $\mathscr{M}_1$. On the other hand if W satisfies

$$2W + u_{i,j}\frac{\partial W}{\partial u_{i,j}} \geqslant 0 \tag{5.21}$$

we find by direct integration that

$$\int_\Omega \varrho u_i u_i\, dx \leqslant \int_\Omega \varrho f_i f_i\, dx + t\int_\Omega \varrho f_i g_i\, dx + 2t^2 E(0), \tag{5.22}$$

where E is given by (4.2). Results of this type are well known.

There is almost nothing in the literature on the Cauchy problem for the nonlinear equations of elastostatics, i.e.,

$$\begin{aligned} \frac{\partial}{\partial x_j}\left(\frac{\partial W}{\partial u_{i,j}}\right) &= 0 \quad \text{in} \quad \Omega \\ u_i &= f_i \quad \text{on} \quad \Sigma \\ \frac{\partial W}{\partial u_{i,j}} n_j &= g_i \quad \text{on} \quad \Sigma, \end{aligned} \tag{5.23}$$

where Σ is a connected portion of the boundary $\partial\Omega$ of non zero measure and n_j is the jth component of the unit normal. Even for the linearized equations the literature provides little in the way of explicit stability inequalities.

A particular nonlinear case of (5.23) has been considered by Knops and Payne [18], namely

$$\frac{\partial}{\partial x_k}(\varrho(\sigma)\, a_{rk}(x)\, u_{i,r}) = 0, \qquad x \in \Omega$$

(5.24) $$u_i = f_i \qquad \text{on } \Sigma,$$

$$\varrho(\sigma)\, a_{rk}\, u_{i,r}\, n_k = g_i \qquad \text{on } \Sigma,$$

where

(5.25) $$\sigma = a_{rk}\, u_{i,r}\, u_{i,k},$$

and ϱ is assumed to satisfy

i) $\varrho(s) > 0$ for $s > 0$;

ii) $\int\limits_0^s \varrho(t)\, dt \leqslant s\varrho(s)$,

and a_{ij} is a positive definite matrix for $x \in \Omega$.

By introducing a suitable family of smooth surfaces parametrized by a parameter α $(0 \leqslant \alpha \leqslant 1)$, the authors derived an inequality which implied stability of the null solution in a subregion Ω_α for solutions in the appropriate constraint set. They imposed a rather severe constraint set restriction, i.e.,

(5.26) $$\sup_{x \in \Omega_1} \left| \frac{\varrho'(\sigma)}{\varrho(\sigma)} a_{rk}\, u_{i,r}\, u_{i,kj} \right| \leqslant A_j; \qquad \int\limits_{\Omega_1} \varrho(\sigma)\, u_i\, u_i\, dx \leqslant k,$$

where the A_j's and k are constants. The proof used a logarithmic convexity argument with α playing the role of t.

Note that the linearized version of (5.24) does not correspond to classical linear elasticity, and in fact the method of logarithmic convexity cannot be applied directly to linear anisotropic elasticity without making further assumptions on the c_{ijkl}, assumptions that do not appear to be realistic. It is possible that the weighted energy method is applicable to a more general class of problems than that considered in [18], but to the authors knowledge this latter method has not been used in nonlinear elastostatics.

6. Results of St. Venant type

The original St. Venant problem involved the comparison of two elastostatic fields in a cylinder with traction free lateral surface subjected to end loadings that were different but had the same resultant force and moment. St. Venant conjectured that if the cylinder is sufficiently long relative to the diameter of the cross section then away from the ends the two fields will be nearly identical. This conjecture as stated is imprecise, but numerous attempts have been made through the years to attach precise meaning to the conjecture and prove its validity, first in the context of classical elasticity and later in various nonlinear settings. Perhaps the first to establish and prove precise versions of the theorem for classical elasticity were Toupin [31], Knowles [19] and Roseman [28]. The earliest precise versions for nonlinear elasticity were due to Roseman [29]. For more recent results see the review papers of Gurtin [9] and Fichera [7] and for noncylindrical regions see Oleinik and Yosifian [25].

In this section we wish to report on some recent unpublished results of Horgan and Payne on rates of decay of solutions of some two dimensional problems defined in a strip region. For St. Venant type results in related problems see Horgan and Knowles [10] and papers cited therein.

Let Ω now denote the semi infinite plane strip, $x > 0$, $0 < y < b$. We consider the following boundary value problem

$$\begin{gathered} [\varrho(q^2)u_{,i}]_{,i} = 0 \quad \text{in } \Omega, \qquad q^2 \equiv |\text{grad } u|^2, \\ u(x, 0) = u(x, b) = 0, \\ \lim_{x\to\infty} \int_0^b \varrho(q^2) u \frac{\partial u}{\partial x}\, dy = 0. \end{gathered} \tag{6.1}$$

For the moment we do not specify the boundary condition on $x = 0$. Clearly such a problem could arise in two dimensional nonlinear elastostatics.

In what follows we shall not assume that the equation is necessarily elliptic, but merely make the hypotheses that on appropriately defined weak solution of (6.1), the quantity $E(0)$ defined by

$$E(0) \equiv \int_0^\infty \int_0^b q^2 \varrho(q^2)\, dy\, dx \tag{6.2}$$

exists, and that ϱ satisfies either

$$0 < m \leqslant \varrho(q^2) \leqslant M + kq^2 \varrho(q^2), \tag{6.3}$$

or

$$0 < m_1 \leqslant \varrho^{-1} \leqslant M_1 + K_1 q^2 \varrho(q^2). \tag{6.4}$$

Here the quantities m, M and K are positive constants.

We establish the following result.

THEOREM. *If a weak solution of* (6.1) *exists and satisfies* (6.3) *or* (6.4) *then it is possible to compute in explicit constant C such that*

$$E(x) \equiv \int_x^\infty \int_0^b q^2 \varrho(q^2)\, dy\, d\zeta \leqslant CE(0) e^{-2\pi m x/(Mb)}. \tag{6.5}$$

(*If* (6.4) *is satisfied the constants in the exponential are m_1 and M_1*).

In many examples (for instance the minimal surface equation) it is possible to choose $m = M$ (or $m_1 = M_1$). In this case we retrieve the same asymptotic behavior for $E(x)$ that we would obtain if ϱ were constant, i.e., if u is a harmonic function.

It is easily seen that condition (6.3) (or (6.4)) does not imply the ellipticity of the equation. It does, however, imply that the equation becomes elliptic for q sufficiently small.

If (6.3) holds and the boundary condition on $x = 0$ is given by

$$\varrho \frac{\partial u}{\partial x} = f(y) \tag{6.6}$$

then one can easily compute

$$E(0) \leqslant \frac{\sqrt{2}\, b}{\pi m} \int_0^b f^2\, dy, \tag{6.7}$$

so that if (6.7) is used on the right hand side of (6.5) we obtain an explicit inequality for the decay rate of $E(x)$.

If instead of (6.3), inequality (6.4) holds and the boundary condition $x = 0$ is given by

$$u = g(y) \quad \text{on} \quad x = 0, \tag{6.8}$$

then it is possible to obtain the inequality

$$E(0) \leqslant \frac{b}{\pi m_1} \int_0^b \left(\frac{dg}{dy}\right)^2 dy. \tag{6.9}$$

We have considered an example in which the solution decays to the null solution as $z \to \infty$. Let us now consider an equation for which the solution tends to the solution of a one dimensional problem. Let u be the solution of

$$\begin{gathered} [\varrho(q^2) u_{,i}]_{,i} + g(u, y) = 0, \qquad 0 < x < L,\ 0 < y < b; \\ u(x, 0) = u(x, b) = 0; \\ \frac{\partial u}{\partial x}(L, y) = 0, \end{gathered} \tag{6.10}$$

where again $q^2 = |\text{grad}\, u|^2$ and, ϱ and g are assumed to satisfy

$$\frac{\partial g}{\partial u} \leqslant 0, \qquad \forall u, \quad 0 \leqslant y \leqslant b; \tag{6.11a}$$

$$\varrho \geqslant m > 0, \qquad \varrho' \geqslant 0, \qquad \varrho' \leqslant K\varrho. \tag{6.11b}$$

Here K is some positive constant.

We wish to compare the solution of (6.10) with that of

$$[\varrho(p^2)v']' + g(v, y) = 0, \qquad 0 < y < b;$$

(6.12)

$$v(0) = v(b) = 0,$$

where $p^2 = (v')^2$.

Setting

(6.13) $$w = u - v$$

and

(6.14) $$E(x) = \int_x^{\infty} \int_0^b [\varrho(p^2) + \varrho(q^2)]\, w_{,i} w_{,j}\, dy\, d\zeta,$$

it is possible to establish the following result:

(6.15) $$E(x) \leqslant CE(0) \exp\left\{-\pi m z\left(3 \int_0^b \varrho(p^2)\,dy\right)^{-1}\right\},$$

for some constant C which depends on K, m, $E(0)$ and the solution of the one dimensional problem. If u satisfies the initial condition (6.6) then the estimate (6.7) holds for $E(0)$. The assumption $\varrho' > 0$ insures that equation (6.10) is elliptic.

Additional comparison results relating solutions of second order elliptic problems to analogous problems in one dimension are given in [27].

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